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EULER-GERGONNE' S THEOREM AND ITS APPLICATIONS

Šefket Arslanagić
University of Sarajevo (Bosnia and Herzegovina)

Abstract. The present paper considers a proof of the Euler-Gergonne's theorem and its application to two assertions.

Keywords: triangle; circumcentre; orthocentre; incentre; centroid; radius of the circumcircle; radius of the incircle of a triangle

The great Swiss mathematician **Leonhard Euler** (1707 – 1783) is well known as the discoverer of many interesting and important facts and theorems in Geometry. We will mention two of them in the sequel.

Theorem 1. The circumcentre O, the centroid T and the orthocentre H of an arbitrary triangle ΔABC are colinear (the common line of the three points is known to be the Euler' s line of the triangle). The distance $|HT|$ between the orthocentre H and the centroid T is two times greater than the distance $|TO|$ between the centroid T and the circumcentre O, i.e. $|HT| = 2|TO|$.

Theorem 2. The distance d between the circumcentre O and the incentre I of a traingle ΔABC is given by the formula:

$$d^2 = |OI|^2 = R^2 - 2Rr\,,$$

where R and r are the radii of these circles, respectively.

The proofs of the above theorems could be found in textbooks and books like: (Arslanagić, 2005), (Lopandić, 1971), (Malcheski, Grozdev & Anevska, 2017) and (Palman, 1994). It is also known in the Geometry of traingle the theorem of Gergonne (**Joseph Diaz Gergonne** (1771 – 1859) is a French astronom and mathematician), which says:

Theorem 3. The lines which connect the vertices A, B and C of a triangle ΔABC with the points of tangency X, Y and Z of the incircle with the opposite sides of the triangle, respectively are conccurent (the commonn point G is known to be the Gergonne' s point).

The proof of this theorem could be found in (Lopandić, 1971) and (Palman, 1994).

We will mention another theorem in the present paper, known in the mathematical literature on the Geometry of triangle as Euler-Gergonne' s theorem. A proof of this theorem will be proposed and some interesting applications of it.

Theorem 4. Given is a triangle ΔABC and let the segments AX, BY and CZ be conccurent, where $X\in BC, Y\in AC$ and $Z\in AB$. If $u=\frac{AK}{KX}, v=\frac{BK}{KY}$ and $w=\frac{CK}{KZ}$, then:

$$(1)\qquad \frac{1}{1+u}+\frac{1}{1+v}+\frac{1}{1+w}=1$$

Proof:

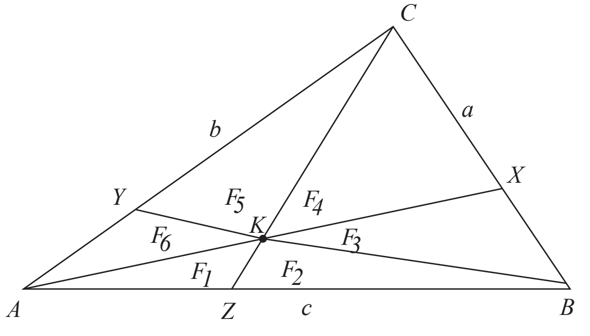

Fig.1

Let F_1, F_2, F_3, F_4, F_5 and F_6 be the areas of the triangles $\Delta AKZ, \Delta BKZ, \Delta BKX, \Delta CKX, \Delta CKY$ and ΔAKY (Figure 1.). Using the formula for the area of a triangle, we get:

$$(2)\qquad u=\frac{AK}{KX}=\frac{F_1+F_2}{F_3}=\frac{F_5+F_6}{F_4},$$

$$(3)\qquad v=\frac{BK}{KY}=\frac{F_3+F_4}{F_5}=\frac{F_1+F_2}{F_6},$$

$$(4)\qquad w=\frac{CK}{KZ}=\frac{F_5+F_6}{F_1}=\frac{F_3+F_4}{F_2}.$$

If F is the area of the triangle ΔABC, then $F=F_1+F_2+F_3+F_4+F_5+F_6$ and it follows that:

$$\frac{1}{1+u}=\frac{F_3}{F_1+F_2+F_3}=\frac{F_4}{F_4+F_5+F_6}=\frac{F_3+F_4}{F_1+F_2+F_3+F_4+F_5+F_6}=\frac{F_3+F_4}{F},$$

$$\frac{1}{1+v}=\frac{F_5}{F_3+F_4+F_5}=\frac{F_6}{F_6+F_1+F_2}=\frac{F_5+F_6}{F_3+F_4+F_5+F_6+F_1+F_2}=\frac{F_5+F_6}{F},$$

$$\frac{1}{1+w}=\frac{F_1}{F_1+F_5+F_6}=\frac{F_2}{F_2+F_3+F_4}=\frac{F_1+F_2}{F_1+F_5+F_6+F_2+F_3+F_4}=\frac{F_1+F_2}{F}.$$

Summing up the above three equations, we obtain (1).

By means of relation (1) we will prove the following two assertions:

Assertion 1. Given is an acute triangle ΔABC. Let the points *D,E* and *F* be the feet of the altitudes from the vertices *A,B* and *C* to the sides *BC,CA* and *AB*, respectively, and let *H* be the orthocentre of the triangle. Then:

$$(5)\qquad \frac{AH}{AD}+\frac{BH}{BE}+\frac{CH}{CF}=2.$$

Proof: We will prove the more general assertion, which states, that:

$$(6)\qquad \frac{AK}{AX}+\frac{BK}{BY}+\frac{CK}{CZ}=2,$$

where the points X,Y,Z and K have the same meaning as in the proof of Theorem 4. If $u=\frac{AK}{KX}, v=\frac{BK}{KY}$ and $w=\frac{CK}{KZ}$, then

$$(7)\qquad \frac{AK}{AX}=\frac{AK}{AK+KX}=\frac{\frac{AK}{KX}}{1+\frac{AK}{KX}}=\frac{u}{1+u},$$

$$(8)\qquad \frac{BK}{BY}=\frac{BK}{BK+KY}=\frac{\frac{BK}{KY}}{1+\frac{BK}{KY}}=\frac{v}{1+v},$$

$$(9)\qquad \frac{CK}{CZ}=\frac{CK}{CK+KZ}=\frac{\frac{CK}{KZ}}{1+\frac{CK}{KZ}}=\frac{w}{1+w}.$$

Further, using (1), we obtain:

$$3-1=\left(\frac{1+u}{1+u}+\frac{1+v}{1+v}+\frac{1+w}{1+w}\right)-\left(\frac{1}{1+u}+\frac{1}{1+v}+\frac{1}{1+w}\right)\Leftrightarrow 2=\frac{u}{1+u}+\frac{v}{1+v}+\frac{w}{1+w}.$$

It follows from (7), (8) and (9), that:

$$\frac{AK}{AX}+\frac{BK}{BY}+\frac{CK}{CZ}=2$$

and we are done. Take now $K\equiv H,\ X\equiv D,\ Y\equiv E$ and $Z\equiv F$. Thus, we get (5).

Assertion 2. Let P be an arbitrary point in the interior of a triangle ΔABC and let the lines AP,BP and CP intersect the sides BC,CA and AB in the points X,Y and Z, respecvtively. Prove that:

$$\text{(10)}\qquad \frac{PA}{PX}\cdot\frac{PB}{PY}+\frac{PB}{PY}\cdot\frac{PC}{PZ}+\frac{PC}{PZ}\cdot\frac{PA}{PX}\geq 12.$$

Proof: Let $u=\frac{PA}{PX}, v=\frac{PB}{PY}$ and $w=\frac{PC}{PZ}$. The inequality is equivalent to:

$$\text{(11)}\qquad uv+vw+wu\geq 12.$$

We have

$$\frac{1}{1+u}+\frac{1}{1+v}+\frac{1}{1+w}=1\Leftrightarrow(1+v)(1+w)+(1+u)(1+w)+(1+u)(1+v)=(1+u)(1+v)(1+w),$$

which is equivalent to

$$\text{(12)}\qquad u+v+w+2=uvw.$$

Apply now the arithmetic-harmonic inequalty:

$$\frac{(1+u)+(1+v)+(1+w)}{3}\geq\frac{3}{\frac{1}{1+u}+\frac{1}{1+v}+\frac{1}{1+w}}\Leftrightarrow(1+u+1+v+1+w)\left(\frac{1}{1+u}+\frac{1}{1+v}+\frac{1}{1+w}\right)\geq 9.$$

By means of (1) we get $3+u+v+w\geq 9$, i.e.

$$\text{(13)}\qquad u+v+w\geq 6.$$

By (12) and (13) we have:

$$\text{(14)}\qquad uvw\geq 8.$$

Finally, we apply the arithmetic-geometric inequalty:

$$\frac{uv+vw+wu}{3}\geq\sqrt[3]{uv\cdot vw\cdot wu}\text{, i.e.}$$

$$uv+vw+wu\geq 3\sqrt[3]{u^2v^2w^2}\Leftrightarrow uv+vw+wu\geq 3\sqrt[3]{64}\Leftrightarrow uv+vw+wu\geq 12,$$

and we are done.

The equality in (11) holds true iff $u=v=w=2$, i.e. iff $\frac{PA}{PX}=\frac{PB}{PY}=\frac{PC}{PZ}=2$, i.e. when the point P is the centroid of the equilateral triangle ΔABC.

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