
$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 1,$$

a, b, c

$$\frac{a^5 + b^5 + c^5}{a^3 + b^3 + c^3} \geq 3.$$

1.
 a, b, c

5 3

$$\sqrt[5]{\frac{a^5 + b^5 + c^5}{3}} \geq \sqrt[3]{\frac{a^3 + b^3 + c^3}{3}}.$$

$$\frac{a^5 + b^5 + c^5}{a^3 + b^3 + c^3} \geq \left(\sqrt[3]{\frac{a^3 + b^3 + c^3}{3}} \right)^2.$$

$$\sqrt[3]{\frac{a^3 + b^3 + c^3}{3}} \geq \sqrt{\frac{a^2 + b^2 + c^2}{3}}.$$

a^2, b^2, c^2

$$\frac{a^2 + b^2 + c^2}{3} \geq \frac{3}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} = 3.$$

2.

$a \leq b \leq c.$

$$a^2 \leq b^2 \leq c^2 \quad a^3 \leq b^3 \leq c^3.$$

$$\frac{a^2 + b^2 + c^2}{3} \cdot \frac{a^3 + b^3 + c^3}{3} \leq \frac{a^5 + b^5 + c^5}{3}.$$

$$\frac{a^5 + b^5 + c^5}{a^3 + b^3 + c^3} \geq \frac{a^2 + b^2 + c^2}{3}.$$

$$\frac{a^2 + b^2 + c^2}{3} \geq 3,$$

1,

3.

$$\frac{5}{a^2} \frac{1}{a^2} + \frac{5}{b^2} \frac{1}{b^2} + \frac{5}{c^2} \frac{1}{c^2} \leq \sqrt{a^5 + b^5 + c^5} \sqrt{a + b + c}$$

$$\frac{a^5 + b^5 + c^5}{a^3 + b^3 + c^3} \geq \frac{a^3 + b^3 + c^3}{a + b + c}.$$

$$\frac{a + b + c}{3} \cdot \frac{a^2 + b^2 + c^2}{3} \leq \frac{a^3 + b^3 + c^3}{3},$$

$$\frac{a^5 + b^5 + c^5}{a^3 + b^3 + c^3} \geq \frac{a^3 + b^3 + c^3}{a + b + c} \geq \frac{a^2 + b^2 + c^2}{3} \geq \frac{3}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} = 3.$$

4.

$$a^5 + b^5 + c^5 \geq a^3 + b^3 + c^3 + 2(a^3 + b^3 + c^3)$$

$$a^5 - a^3 + b^5 - b^3 + c^5 - c^3 \geq 2(a^3 + b^3 + c^3)$$

$$a^5 \left(\frac{1}{b^2} + \frac{1}{c^2} \right) + b^5 \left(\frac{1}{a^2} + \frac{1}{c^2} \right) + c^5 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \geq 2(a^3 + b^3 + c^3)$$

$$\frac{a^5}{b^2} + \frac{b^5}{a^2} + \frac{a^5}{c^2} + \frac{c^5}{a^2} + \frac{b^5}{c^2} + \frac{c^5}{b^2} \geq 2(a^3 + b^3 + c^3).$$

$$\frac{a^5}{b^2} + \frac{b^5}{a^2} \geq a^3 + b^3.$$

$$\frac{a^5}{c^2} + \frac{c^5}{a^2} \geq a^3 + c^3 \quad \frac{b^5}{c^2} + \frac{c^5}{b^2} \geq b^3 + c^3.$$

$$\frac{a^5}{b^2} + \frac{b^5}{a^2} \geq a^3 + b^3 \Leftrightarrow a^7 + b^7 \geq a^2 b^2 (a + b) (a^2 - ab + b^2)$$

$$\begin{aligned}
a^7 + b^7 &= (a+b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6) \\
&= (a+b)[a^5(a-b) + a^2b^2(a^2 - ab + b^2) - b^5(a-b)] \\
&= (a+b)(a-b)(a^5 - b^5) + (a+b)a^2b^2(a^2 - ab + b^2) \\
&= (a+b)(a-b)^2(a^4 + a^3b + a^2b^2 + ab^3 + b^4) + \\
&\quad + (a+b)a^2b^2(a^2 - ab + b^2)
\end{aligned}$$

$$(a-b)^2 \geq 0$$

5.

$$f(x) = x^5 \quad f(x) = x^3, \quad x > 0,$$

$$f\left(\frac{a+b+c}{3}\right) \leq \frac{1}{3}(f(a) + f(b) + f(c)), \quad \dots \left(\frac{a+b+c}{3}\right)^5 \leq \frac{a^5 + b^5 + c^5}{3}.$$

$$\left(\frac{a+b+c}{3}\right)^3 \leq \frac{a^3 + b^3 + c^3}{3}.$$

$$\frac{a^5 + b^5 + c^5}{a^3 + b^3 + c^3} \geq \left(\frac{a+b+c}{3}\right)^2.$$

$$\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3} \cdot \frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3} \leq \frac{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}{3} \quad \dots \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 \leq 3.$$

a, b, c

$$\left(\frac{a+b+c}{3}\right)^2 \geq \frac{9}{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2} \geq \frac{9}{3} = 3.$$

$$\frac{a^5 + b^5 + c^5}{a^3 + b^3 + c^3} \geq \left(\frac{a+b+c}{3}\right)^2 \geq 3.$$