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GEOMETRY OF COMPLEX NUMBERS

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PREFACE

No research of people could be named true science if it is not supported by a mathematical proof.

Reliability of the assertions in different subjects is problematic when application of any mathematical domain is missing, i. e. when there is no connection with mathematics.

Leonardo da Vinci

The present book is devoted to students of the last school grades, university students, teachers, lecturers and all lovers of mathematics who want to enrich their knowledge and skills in complex numbers and their numerous applications in Euclidean Geometry. Few countries in the world include complex numbers in their secondary school curriculum but even if included the volume of the corresponding content is quite insufficient consisting of elementary operations and geometric representation at most. The significance of the complex numbers is far from a real recognition in known textbooks and scholar literature. The applications not only in mathematics but also in many other subjects are considerable and the present book is a strong proof of such a statement.

Mainly, the book will be useful for outstanding students with high potentialities in mathematics preparing themselves for successful participation in mathematical competitions and Olympiads. Other target groups are not excluded, namely those, whose representatives like to meet real challenges, connected with unexpected circumstances in problem solving.

The material in the book is divided into four chapters. The first one contains basic properties of the complex numbers, their algebraic notation, the notion of a conjugate complex number, geometric, trigonometric and exponential presentations, also interesting facts in connection with Reimann interpretation and the set C^n. The second chapter includes various transformations of complex numbers in the Euclidean plane like similarity, homothety, inversion and Mőbius transformation. The third chapter is dedicated to the geometry of circle and triangle on the base of complex numbers. Numerous theorems are proposed, namely: Menelau's theorem, Pascal's and Desargue's theorem, Ceva's and Van Aubel's theorem, Stewart's theorem, Ptolemy's theorem and others. Exercises and problems are included in the Fourth chapter: 122 examples with solutions and 161 solved problems pare proposed. Together with all the 138 theorems, lemmas and corollaries accompanied by 64 examples and 88 figures, the book turns out to be a rather exhaustive collection of the complex number applications in Euclidean Geometry.

A high just appraisal of the book is due to the numerous non-standard problems in it taken from the National Olympiads of Bulgaria, China, Iran, Japan, Korea, Poland, Romania, Russia, Serbia, Turkey, Ukraine and others but also from Several International and Balkan Mathematical Olympiads.

The authors express their sincere thanks to the Editorial House "Archimedes 2" for the decision to accept the manuscript and to support the appearance of the present book. Also, sincere thanks to the reviewers Prof. Dr. Lidia Ilievska and Assoc. Prof. Dr. Veselin Nenkov for their helpful criticism, removal of mistakes and well-wishing advices, which contributed to the final quality of the book. Of course, different lapses are possible and we will be grateful to the readers in case they notice such and bring them to the attention of the Editor.

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Skopje and Sofia

CHAPTER I
COMPLEX NUMBERS

1. THE CONCEPT OF COMPLEX NUMBER, BASIC PROPERTIES

1.1. Definition. *Complex number* $z=(a,b)$ is the ordered pair of real numbers a and b.

The set of complex numbers is denoted by **C**, i.e. $\mathbf{C}=\{(a,b)\,|\,a,b\in \mathbf{R}\}$.

The complex numbers (0,0) and (1,0) are denoted by n and e, respectively.

The definition of a complex number directly implies that two complex numbers $z_1=(a_1,b_1)$ and $z_2=(a_2,b_2)$ are *equal* if $a_1=a_2$ and $b_1=b_2$.

1.2. Definition. *Sum* of two complex numbers $z_1=(a_1,b_1)$ and $z_2=(a_2,b_2)$ is the complex number

$$z_1+z_2=(a_1+a_2,b_1+b_2).$$

1.3. Definition. *Product* of two complex numbers $z_1=(a_1,b_1)$ and $z_2=(a_2,b_2)$ is the complex number

$$z_1\cdot z_2=(a_1a_2-b_1b_2,a_1b_2+a_2b_1).$$

1.4. Theorem. The addition and multiplication of complex numbers, satisfy the already known laws of arithmetic. Namely,:

i) $z_1+z_2=z_2+z_1$, *commutative property of addition,*

ii) $(z_1+z_2)+z_3=z_1+(z_2+z_3)$, *associative property of addition*,

iii) $z_1z_2=z_2z_1$, *commutative property of multiplication*,

iv) $(z_1z_2)z_3=z_1(z_2z_3)$, *associative property of multiplication*, and

v) $(z_1+z_2)z_3=z_1z_3+z_2z_3$, *distributive property*.

hold true for all complex numbers z_1, z_2, z_3.

Proof. *i)* Let $z_1=(a_1,b_1)$ and $z_2=(a_2,b_2)$ be any complex numbers. Thus,

$$(a_1,b_1)+(a_2,b_2)=(a_1+a_2,b_1+b_2)=(a_2+a_1,b_2+b_1)=(a_2,b_2)+(a_1,b_1)$$

i.e., $z_1+z_2=z_2+z_1$.

The properties *ii), iii), iv)* and *v)* can be proven analogously. ■

Let's state that when proving Theorem 1.4 we explicitly used (by coordinates) the commutative, associative and distributive properties of addition and multiplication of real numbers.

1.5. Theorem. Any complex number z satisfies the following equalities: $z+n=z$, $z\cdot n=n$ and $z\cdot e=z.$, where n denotes the additive identity, and e denotes the multiplicative identity.

Proof. Indeed, if $z=(a,b)$ is an arbitrary comblex number, then

$z+n=(a,b)+(0,0)=(a+0,b+0)=(a,b)=z$,

$z\cdot n=(a,b)\cdot(0,0)=(a\cdot 0-b\cdot 0,a\cdot 0+b\cdot 0)=(0,0)=n$, and

$z\cdot e=(a,b)\cdot(1,0)=(a\cdot 1-b\cdot 0,a\cdot 0+1\cdot b)=(a,b)=z.$ ■

1.6. Theorem. If $z_1+z_3=z_2+z_3$, then $z_1=z_2$.

Proof. Let $z_1=(a_1,b_1)$, $z_2=(a_2,b_2)$ and $z_3=(a_3,b_3)$ be complex numbers. Then,

$z_1+z_3=(a_1,b_1)+(a_3,b_3)=(a_1+a_3,b_1+b_3)$ and

$z_2+z_3=(a_2,b_2)+(a_3,b_3)=(a_2+a_3,b_2+b_3)$.

Since the given equality $z_1+z_3=z_2+z_3$ and the Definition 1.1 we get the following

$$a_1+a_3=a_2+a_3 \text{ and } b_1+b_3=b_2+b_3.$$

Furthermore, the properties of real numbersimply that $a_1=a_2$ and $b_1=b_2$, and since Definition 1.1. ■

1.7. Theorem. For each complex number z there exists one and only one complex number w, so that $z+w=n$.

Proof. Let $z=(a,b)$ be an arbitrary complex number, and w be defined as $w=(-a,-b)$. We get,

$$z+w=(a,b)+(-a,-b)=(a+(-a),b+(-b))=(0,0)=n.$$

So, we proved the existence of a complex number w. The uniqueness is directly implied by Theorem 1.6. ■

In our further consideration, the complex number w, so that $z+w=n$, will be denoted by $w=-z$, and w is called to bean *opposite* complex number of z.

Let z and w be arbitrary complex numbers. The complex number $z+(-w)$ is called to be a *substraction* of the numbers z and w, and is denoted by $z-w$.

1.8. Theorem. For all complex numbers z_1 and z_2, the equality

$$(-z_1)\cdot z_2=z_1\cdot(-z_2)=-(z_1z_2)=(-e)\cdot(z_1z_2),$$

holds true and there is no ambiguity in the notation $-z_1z_2$.

Proof. Let $z_1=(a_1,b_1)$ and $z_2=(a_2,b_2)$ be arbitrary complex numbers So,

$$(-z_1)\cdot z_2=(-a_1,-b_1)\cdot(a_2,b_2)=(-a_1a_2+b_1b_2,-a_1b_2-b_1a_2)=$$
$$=-(a_1a_2-b_1b_2,a_1b_2+b_1a_2)=-(z_1z_2).$$

But, the commutativelaw of multiplication holds true, therefore the above equality implies that

$$-(z_1z_2)=-(z_2z_1)=(-z_2)z_1=z_1(-z_2).$$

Finally, since the Theorem 1.5 and the already proved equalities we get that

$$(-e)(z_1z_2)=-(e(z_1z_2))=-(z_1z_2)=(-z_1)z_2=z_1(-z_2)\,.\ \blacksquare$$

1.9. Definition. *The absolute value* of the complex number $z=(a,b)$ is defined by

$$|z|=\sqrt{a^2+b^2}\,.$$

Thus, the absolute value of each complex number z is a non-negative real number.

1.10. Theorem. a) If $z\neq n$, then $|z|>0$ and $|n|=0$.

b) $|z_1\cdot z|=|z_1|\cdot|z|$, for all complex numbers z and z_1.

Proof. Let $z=(a,b)$ and $z_1=(a_1,b_1)$ be any complex numbers

a) It is obvious that

$$|n|=\sqrt{0^2+0^2}=0.$$

If $z\neq n$, then $a\neq 0$ or $b\neq 0$, i.e. $a^2>0$ or $b^2>0$. Thus,

$$|z|^2=a^2+b^2>0.$$

b) Since,

$$\begin{aligned}|z_1z|^2&=|(a_1a-b_1b,a_1b+b_1a)|^2=(a_1a-b_1b)^2+(a_1b+b_1a)^2\\&=a_1^2a^2+b_1^2b^2+a_1^2b^2+b_1^2a^2=(a_1^2+b_1^2)(a^2+b^2)=|z_1|^2|z|^2,\end{aligned}$$

we get that $|z_1\cdot z|=|z_1|\cdot|z|$. $\blacksquare$

1.11. Remark. Theorem 1.10. b) and the principle of mathematical induction directly imply the following:

$$|z_1z_2...z_n|=|z_1|\cdot|z_2|\cdot\dots\cdot|z_n|.\blacksquare \qquad (1)$$

1.12. Theorem. If $zw=n$, then $z=n$ or $w=n$.

Proof. If $zw=n$, then Theorem 1.10 implies

$$|z|\cdot|w|=|zw|=|n|=0.$$

But, $|z|$ and $|w|$ are real numbers, and thus $|z|=0$ and $|w|=0$, i.e. $z=n$ or $w=n$. $\blacksquare$

1.13. Theorem. If $z\neq n$ and $zw=zw_1$, then $w=w_1$.

Proof. The given condition $zw=zw_1$ implies that $-zw=-zw_1$. So,

$$n=zw-zw_1=z(w-w_1)\,.$$

Acording to Theorem 1.12, $z\neq n$ implies that $w-w_1=n$, i.e. $w=w_1$. $\blacksquare$

1.14. Theorem. For each complex number $z \neq n$ there exists one and only one complex number *w*, denoted by $\frac{e}{z}$, so that $zw = e$ holds.

Proof. Firstly, we will prove the existence of the complex number $w = \frac{e}{z}$. Let $z = (a, b) \neq n$ be an arbitrary complex number . Let

$$w = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right),$$

So,

$$zw = (a,b) \cdot \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1,0) = e.$$

The uniqueness is implied immideately by Theorem 1.13. ■

1.15. Theorem. If $z \neq n$, then for each complex number *w* there exists one and only one complex number *u*, so that $zu = w$ holds.

Proof. By Theorem 1.14, for any complex number $z \neq n$ there exists one and only one complex number $\frac{e}{z}$, so that $z \cdot \frac{e}{z} = e$ holds. Let $u = \frac{e}{z} \cdot w$. Thus we get a unique complex number *u* such that satisfies the following

$$zu = z \cdot \frac{e}{z} w = w \text{ . } ■$$

2. ALGEBRAIC NOTATION OF A COMPLEX NUMBER

2.1. In the previous considerations we discussed the arithmetics of complex numbers, but the ususl symbol *i* was not presented, yet. Now, we will prove that the notation (*a*,*b*) is equivalent to the usual notation for a complex number $a + ib$.

The proofs of the stataments in Theorem 2.2 are elementary therefore the ones will not be done.

2.2. Theorem. For all real numbers *a* and *b* the following equalities are satisfied:

a) $(a, 0) + (b, 0) = (a + b, 0)$,

b) $(a, 0)(b, 0) = (ab, 0)$,

c) $|(a, 0)| = |a|$, where $|a|$ is the absolute value of the real number *a*,

d) $\frac{(a,0)}{(b,0)} = \left(\frac{a}{b}, 0\right)$, for $b \neq 0$. ■

2.3. The statements given in Theorem 2.2 immideately imply that the mapping $f : \mathbf{R} \to \mathbf{C}$ defined by $f(a) = (a,0)$ is a bijection between **R** and $A = \{(a,0) \mid a \in \mathbf{R}\} \subseteq \mathbf{C}$, and such that inhire the operations. So, the set of real numbers **R** might be reviewed as a subset of the set of complex numbers **C**.

According to this, the complex numbers *e* and *n* correspond to the notation 1 and 0, respectively. So, they will be used in our further consideration

2.4. Definition. The complex number $i=(0,1)$ is called to be *imaginary unit.* The imaginary unit satisfies the following equality

$$i^2=(0,1)\cdot(0,1)=(0\cdot 0-1\cdot 1,0\cdot 1+0\cdot 1)=(-1,0)=-1.$$

2.5. The obvious equation

$$(0,1)\cdot(b,0)=(0,b)$$

implies that

$$z=(a,b)=(a,0)+(0,b)=(a,0)+(0,1)\cdot(b,0)=a+ib$$

holds true for any complex number $z=(a,b)$.

Definition. The notation $z=a+ib$ is called to be an *algebraic notation* of the complex number $z=(a,b)$.

The addition and multiplication of complex numbers, by using the algebraic notations of complex numbers, are written as following:

$$(a_1+ib_1)+(a_2+ib_2)=(a_1+a_2)+i(b_1+b_2),$$
$$(a_1+ib_1)\cdot(a_2+ib_2)=(a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1).$$

2.6. Definition. The components *a* and *b* of a complex number $z=a+ib$ are called to be *real* and *imaginary part* of *z*, respectively, and we use the following notations $a=\operatorname{Re} z$ and $b=\operatorname{Im} z$ to denote them.

3. A CONJUGATE COMPLEX NUMBER

3.1. Definition. The complex number $a-ib$ is called to be the *complex conjugate* of $z=a+ib$ and is denoted as $\bar{z}$.

3.2. Theorem. a) $\bar{\bar{z}}=z$, for each complex number *z*.

b) $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$, for all complex numbers z_1 and z_2,

c) $\overline{z_1z_2}=\overline{z_1}\cdot\overline{z_2}$, for all complex numbers z_1 and z_2,

d) $z+\bar{z}=2\operatorname{Re} z$, for each complex number *z*,

e) $z-\bar{z}=2\operatorname{Im} z$, for each complex number *z*, and

f) $z\cdot\bar{z}=a^2+b^2=|z|^2\geq 0$, for each complex number *z*.

Proof. The definition of a conjugate complex number, also addition and multilication of complex numbers directly imply the validity of the above theorem. ■

3.3. Remark. The equality $z-\overline{z}=2\operatorname{Im} z$ directly implies that the complex number z is a real number if and only if $z=\overline{z}$.

3.4. Remark. The validity of the equality (1) can be proved since the equality $\overline{z_1 z_2}=\overline{z_1}\cdot\overline{z_2}$ holds and also by applying the principle of mathematical induction,

$$\overline{z_1 z_2 z_3 \ldots z_n}=\overline{z_1}\cdot\overline{z_2}\cdot\ldots\cdot\overline{z_n}, \tag{1}$$

for each $n\in\mathbf{N}$ and any complex numbers $z_1,z_2,\ldots,z_n$. If $z_k=z$ for each $k=1,2,\ldots,n$, then (1) implies that $\overline{z^n}=\overline{z}^{\,n}$.

3.5. Remark Theorem 3.2. f) and a) holds, therefore it is true that

$$|\overline{z}|^2=\overline{z}\cdot\overline{\overline{z}}=\overline{z}\cdot z=|z|^2, \text{ i.e. } |\overline{z}|=|z|.$$

Let $z=x+iy$. The equality $|z|^2=z\overline{z}=x^2+y^2$ directly implies the inequalities

$$-|z|\le \operatorname{Re} z\le |z| \text{ and } -|z|\le \operatorname{Im} z\le |z| \tag{2}$$

3.6. Example. Let

$$P(z)=a_0z^n+a_1z^{n-1}+\ldots+a_{n-1}z+a_n$$

be a real polynomial. If w is a root of $P(z)$, then $\overline{w}$ is also a root of the same polynomial $P(z)$. Prove it!

Solution. Since w is a root of the polynomial $P(z)$, $P(w)=0$. Further,

$$\begin{aligned} P(\overline{w})&=a_0\overline{w}^{\,n}+a_1\overline{w}^{\,n-1}+\ldots+a_{n-1}\overline{w}+a_n\\ &=\overline{a_0w^n+a_1w^{n-1}+\ldots+a_{n-1}w+a_n}=\overline{P(w)}=0. \end{aligned}$$

The latter means that $\overline{w}$ is also a root of the same polynomial $P(z)$. ■

3.7. Example. Let's consider the polynomial

$$P(z)=a_0z^n+a_1z^{n-1}+\ldots+a_{n-1}z+a_n,\ a_0\ne 0,$$

written as the following

$$P(z)=a_0(z-z_1)(z-z_2)\ldots(z-z_n),$$

where z_i, $i=1,2,\ldots,n$ are the roots of $P(z)$.

The identity

$$a_0(z-z_1)(z-z_2)\ldots(z-z_n)=a_0z^n+a_1z^{n-1}+\ldots+a_{n-1}z+a_n$$

implies that

$$a_0\left(z^n-(z_1+z_2+\ldots+z_n)z^{n-1}+(-1)^n z_1z_2\ldots z_n\right)=a_0z^n+a_1z^{n-1}+\ldots+a_{n-1}z+a_n.$$

By equating the coefficients of the corresponding degrees we get the following formulae:

$$z_1 + z_2 + \ldots + z_n = -\frac{a_1}{a_0}$$

$$z_1 z_2 + \ldots + z_1 z_n + z_2 z_3 + \ldots + z_2 z_n + \ldots + z_{n-1} z_n = -\frac{a_2}{a_0}$$

....

$$z_1 z_2 \ldots z_k + \ldots + z_1 \ldots z_{k-1} z_n + \ldots + z_{n-k+1} z_{n-k+2} \ldots z_n = (-1)^k \frac{a_k}{a_0}$$

....

$$z_1 z_2 \ldots z_n = (-1)^n \frac{a_n}{a_0}$$

These formulae are known to be the *Vieta's formulae*. ■

3.8. Remark. In Theorem 1.15 we proved that if $z \neq n$, then for each complex number w there exists a unique complex number u so that $zu = w$ holds. The complex number u is called to be a *quotient of the complex numbers* w and z and is denoted by $u = \frac{w}{z}$.

Hence, if $z = a_2 + ib_2$ and $w = a_1 + ib_1$, then the quotient is expressed as the following

$$u = \frac{w}{z} = \frac{w\overline{z}}{z\overline{z}} = \frac{(a_1+ib_1)(a_2-ib_2)}{(a_2+ib_2)(a_2-ib_2)} = \frac{a_1a_2+b_1b_2}{a_2^2+b_2^2} + i\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}.$$

For the quotient of the complex numbers z_1 and z_2 $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ holds.

3.9. Example. If $|z_1| = |z_2| = 1$ and $z_1 z_2 \neq -1$, then $\frac{z_1+z_2}{1+z_1z_2}$ is a real number. Prove it!

Solution. By multiplying both the numerator and the denominator by the conjugate of $1 + z_1 z_2$ we get

$$\frac{z_1+z_2}{1+z_1z_2} = \frac{z_1+z_2}{1+z_1z_2} \cdot \frac{\overline{1+z_1z_2}}{\overline{1+z_1z_2}} = \frac{(z_1+z_2)(1+\overline{z_1}\,\overline{z_2})}{|1+z_1z_2|^2}$$

$$= \frac{z_1+z_2+z_1\overline{z_1}\,\overline{z_2}+\overline{z_1}z_2\overline{z_2}}{|1+z_1z_2|^2} = \frac{z_1+\overline{z_1}}{|1+z_1z_2|^2} + \frac{z_2+\overline{z_2}}{|1+z_1z_2|^2}.$$

But both, $z_1 + \overline{z_1}$ and $z_2 + \overline{z_2}$ are real numbers, therefore $\frac{z_1+z_2}{1+z_1z_2}$ is also a real number. ■

3.10. Example. Prove that if the absolute value of a complex number is 1, than such a complex number can be expressed as $\frac{c+i}{c-i}$, where c is a real number.

Solution. If $|z| = 1$ and $c = i\frac{z+1}{z-1}$, then $z = \frac{c+i}{c-i}$ and $\left|\frac{c+i}{c-i}\right| = 1$, i.e.

$$\frac{c+i}{c-i} \cdot \frac{\overline{c}-i}{\overline{c}+i} = 1.$$

By reducing the last equality, we get that $c - \overline{c} = 0$. Hence, $c \in \mathbf{R}$ and $z = \frac{c+i}{c-i}$, which actually was supposed to be proven. ■

3.11. Theorem. All complex numbers z and $w \neq 0$ satisfy the equality

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$

Proof. By applying the above stated we get the following,

$$\left|\frac{z}{w}\right|^2 = \frac{z}{w}\cdot\overline{\left(\frac{z}{w}\right)} = \frac{z}{w}\cdot\frac{\overline{z}}{\overline{w}} = \frac{z\overline{z}}{w\overline{w}} = \frac{|z|^2}{|w|^2},$$

that is $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$. ■

3.12. Example. Determine all the complex numbers z such that satisfy the following equalities:

$$\left|\frac{z-12}{z-8i}\right| = \frac{5}{3} \text{ and } \left|\frac{z-4}{z-8}\right| = 1.$$

Solution. Let $z = x + iy$. Since the given condition we get that

$$\left|\frac{z-12}{z-8i}\right|^2 = \frac{|z-12|^2}{|z-8i|^2} = \frac{(x-12)^2+y^2}{x^2+(y-8)^2} = \frac{25}{9} \text{ and } \left|\frac{z-4}{z-8}\right|^2 = \frac{|z-4|^2}{|z-8|^2} = \frac{(x-4)^2+y^2}{(x-8)^2+y^2} = 1.$$

By reducing we get the following system of equations

$$\begin{cases} 2x^2 + 2y^2 + 27x - 50y + 38 = 0 \\ x = 6 \end{cases}$$

whose solutions are $x = 6,\ y = 17$ and $x = 6,\ y = 8$. Hence, the required complex numbers are $z = 6 + 17i$ and $z = 6 + 8i$. ■

3.13. Example. If a, b and c are complex numbers such that

$$|a| = |b| = |c| = r,\ r > 0$$

then

$$|ab + bc + ca| = r\,|a + b + c|.$$

Prove it!

Solution. Since $r^2 = |a|^2 = a\overline{a}$, we get that $\frac{1}{a} = \frac{\overline{a}}{r^2}$. Analogously, $\frac{1}{b} = \frac{\overline{b}}{r^2}$ and $\frac{1}{c} = \frac{\overline{c}}{r^2}$. Therefore,

$$ab + bc + ca = abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = abc\left(\frac{\overline{a}}{r^2} + \frac{\overline{b}}{r^2} + \frac{\overline{c}}{r^2}\right) = \frac{abc}{r^2}\cdot\overline{a + b + c}.$$

The latter implies that,

$$|ab + bc + ca| = \frac{|abc|}{r^2}\left|\overline{a + b + c}\right| = r\,|a + b + c|$$

holds true, which actually was supposed to be proven. ■

3.14. Theorem. All complex numbers z_1 and z_2 satisfy the following

$$|z_1 + z_2| \leq |z_1| + |z_2| \text{ and } |z_1 - z_2| \geq \big||z_1| - |z_2|\big|.$$

Proof. Since Theorem 3.2, the following holds true for all complex numbers z_1 and z_2:

$$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = |z_1|^2 + |z_2|^2 + 2\operatorname{Re} z_1\overline{z_2} \tag{3}$$

$$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)} = |z_1|^2 + |z_2|^2 - 2\operatorname{Re} z_1\overline{z_2}\,. \tag{4}$$

tothe equalities (3) and (4), if we apply the first inequality in (2), imply the following inequalities:

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{5}$$

$$|z_1 - z_2| \ge \big||z_1| - |z_2|\big|\,. \tag{6}$$

The inequality (5) is known to be the *triangle inequality*. ■

3.15. Corollary. For all complex numbers $z_1, z_2, ..., z_n$ the following inequality holds true:

$$|\sum_{i=1}^{n} z_i| \le \sum_{i=1}^{n} |z_i|.$$

Proof. The proof is directly implied by the inequality (5) and the principle of mathematical induction. ■

3.16. Example. For all complex numbers z_1 and z_2

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right). \tag{7}$$

holds true. Prove it!

Solution. If we summarize the equalities (3) and (4) as given in Theorem 3.14 (the ones hold for all complex numbers z_1 and z_2) we get the required identity. This identity is known to be the *parallelogram identity*. ■

3.17. Example. Prove that all complex numbers z_1 and z_2 satisfy

$$|1 - \overline{z_1}z_2|^2 - |z_1 - z_2|^2 = \left(1 + |z_1 z_2|\right)^2 - \left(|z_1| + |z_2|\right)^2.$$

Solution. The identity $z\overline{z} = |z|^2$, directly implies the following

$$\begin{aligned}
|1 - \overline{z_1}z_2|^2 - |z_1 - z_2|^2 &= (1 - \overline{z_1}z_2)\overline{(1 - \overline{z_1}z_2)} - (z_1 - z_2)\overline{(z_1 - z_2)} \\
&= (1 - \overline{z_1}z_2)(1 - \overline{z_2}z_1) - (z_1 - z_2)(\overline{z_1} - \overline{z_2}) \\
&= 1 - \overline{z_1}z_2 - \overline{z_2}z_1 + \overline{z_1}z_1\overline{z_2}z_2 - \overline{z_1}z_1 - \overline{z_2}z_2 + \overline{z_1}z_2 - \overline{z_2}z_1 \\
&= 1 + |z_1 z_2|^2 - |z_1|^2 - |z_2|^2 = \\
&= \left(1 + |z_1 z_2|\right)^2 - \left(|z_1| + |z_2|\right)^2,
\end{aligned}$$

which was suppesed to be proven. ■

3.18. Theorem. Let z_i, w_i, $i = 1, 2, ..., n$ be complex numbers. Thus,

$$|\sum_{i=1}^{n} z_i w_i|^2 \le \sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} |w_i|^2\,. \tag{8}$$

Proof. Let

$$C = \sum_{i=1}^{n} w_i z_i \,,\ A = \sum_{i=1}^{n} |\, z_i \,|^2 \ \text{ and } \ B = \sum_{i=1}^{n} |\, w_i \,|^2 \,.$$

If $A = 0$, then $|\, z_i \,| = 0$, for all $i = 1,2,...,n$. that is, $z_i = 0$, for all $i = 1,2,...,n$. So, also $C = 0$. Thus, the inequlity (8) is satisfied.

If $A \neq 0$, then the inequality (8) is implied by the obvious equality

$$\sum_{i=1}^{n} \left| C\overline{z_i} - Aw_i \right|^2 = \sum_{i=1}^{n} (C\overline{z_i} - Aw_i)(\overline{C}z_i - \overline{Aw_i}) = A(AB - |\, C\,|^2)$$

and the inequalities

$$\sum_{i=1}^{n} \left| C\overline{z_i} - Aw_i \right|^2 \geq 0 \,,\ A > 0 \,.$$

The inequality (8) is known to be the *Cauchy-Schwarz-Bunyakovsky inequality* for complex numbers. ■

4. GEOMETRIC PRESENTATION OF A COMPLEX NUMBER

The Eucledian plane with Cartesian coordinates is denoted by $\mathbf{R}^2$. Each complex number $z = x + iy$ is an ordered pair of real numbers (x,y). Since it exists a one-to-one correspondence between the set of the ordered pairs of real numbers (x,y) and $\mathbf{R}^2$, we get that to each point $A \in \mathbf{R}^2$ may be adjoined a complex number $z = x + iy$, and conversely (figure 1). The complex number z such that it corresponds to the point A is called to be the *affix* of the point A. This correspondence between the complex numbers and the points of the Euclidean plane is bijection. Thereby, the real part of the complex numbers mapps onto the points of the x-axis (abscissa), while the imaginary part of the complex numbers mapps onto the points of the y-axis (ordinate). So, the real numbers map onto the points onto the abscissa, and the pure imaginary numbers map onto the points of the ordinate. Thus, the abscissa is called to be the *real axis*, and the ordinate is called to be the *imaginary axis*. Thus, the Euclidean plane $\mathbf{R}^2$, is naturaly called to be the *complex plane*, and the complex numbers to be points in this plane.

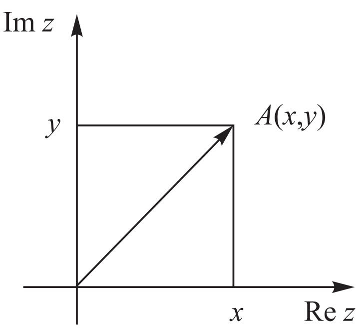

Figure 1

Clearly, the points z and $-z$ are symmetric with respect to the origin, and $\overline{z}$ and z are symmetric with respect to the real axis. Namely, if $z = x + iy$ then

$$-z=(-x)+i(-y) \text{ and } \bar{z}=x+(-y)i.$$

Obviously, the complex number z corresponds to the vector with tail in the origin point O and head in the point z. Clearly, this correspondence between the complex numbers and the vectors in the complex plane with tails in O is bijective. Thus, the vector which determines a complex number z, will be denoted by the same letter z.

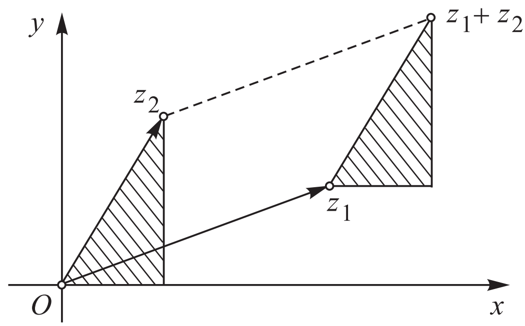

Figure 2

By using the vector interpretation of the complex numbers we can demonstrate the addition and the subtraction of complex numbers. Since 1.2 we get that the number z_1+z_2 corresponds to the vector obtained by adding the vectors z_1 and z_2 (figure 2). The vector z_1-z_2 is constructed as the sum of the vectors z_1 and $-z_2$ (figure 3).

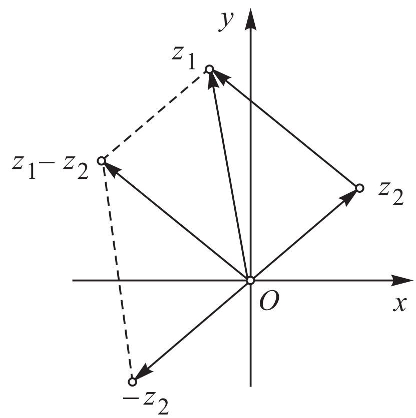

Figure 3

The already stated and also using figure 3 implies that the distance between the points z_1 and z_2 is equal to the length of the vector z_1-z_2, i.e. it is equal to $|z_1-z_2|$. Clearly, the absolute value $|z|$ is equal to the length of the corresponding vector of the point z If we consider the triangles whose vertices are

$$O,\ z_1,\ z_1+z_2 \text{ and } O,\ z_1,\ z_1-z_2,$$

then the geometric sence of the inequalities (5) and (6) from paragraph 1 is obvious.

4.2. Dividing a line segment in a given ratio. Let A and B be given points with affixes z_1 and, z_2 respectively, and C be a point on the line segment AB, such that C divides AB in a given ratio $\lambda:\mu\neq-1$, i.e. $\mu\overrightarrow{AC}=\lambda\overrightarrow{CB}$. Since $\overrightarrow{AC}=z-z_1$ and $\overrightarrow{CB}=z_2-z$, we get that $\mu(z-z_1)=\lambda(z_2-z)$. Hence, the affix of C is $z=\frac{\lambda z_2+\mu z_1}{\lambda+\mu}$. If $\lambda:\mu=1$ then $z=\frac{z_2+z_1}{2}$ is an affix of C, the midpoint of the line segment AB.

Example A. Let $a=1+i$ and $b=3+5i$ be the affixes of the endpoints of the line segment AB. Then

$$c=\frac{1\cdot a+1\cdot b}{1+1}=\frac{(1+i)+(3+5i)}{2}=1+3i.$$

is the affix of C, the midpoint of the line segment AB. ■

Example B. Let $ABCD$ be a quadrilateral and M, N, P, Q, K, L be the midpoints of the line segments AB, BC, CD, DA, BD, CA, respectively. Show that the line segments MN, NQ, KL concur at such a point T that bisects each of them.

Solution. Let the affixes of the points A, B, C, D, M¸ N, P, Q, K, L are denoted by lowcase letters $a, b, c, d, m, n, p, q, k, l$, respectively, then

$$m = \frac{a+b}{2},\ n = \frac{b+c}{2},\ p = \frac{c+d}{2},\ q = \frac{a+d}{2},\ k = \frac{b+d}{2},\ l = \frac{a+c}{2}.$$

The affixes of the midpoints of MP, NQ, KL are

$$t = \frac{a+b+c+d}{4},\ t_1 = \frac{b+c+a+d}{4},\ t_2 = \frac{b+d+a+c}{4},$$

respectively and since $t = t_1 = t_2$, we get that MP, NQ, KL are concurrent at the point which bisects each of these segments. ■

4.3. Example. a) The set of points z, such that satisfy the equation $|z - z_0| = R$, is a circle centered at z_0 and radius R. Namely, $|z - z_0|$ is the distance between the points A and B with affixes z and z_0, respectively.

b) The equation

$$\big||z - z_1| - |z - z_2|\big| = 2a,$$

where

$$a < \tfrac{1}{2}|z_1 - z_2|,$$

is a hyperbola, whose foci are at points whose affixes are z_1 and z_2 and a real semi-axis whose length is a, (why?).

c) The set of points z, such that satisfy the equation

$$|z - z_1| = |z - z_2|,$$

is a set of points equidistanced form z_1 and z_2. Thus,

$$|z - z_1| = |z - z_2|$$

is an equation of the bisector of the segment, whose extremities have affixes z_1 and z_2.

d) The set of points z, such that satisfy the equation

$$|z - z_1| + |z - z_2| = 2a,$$

where $a > \frac{1}{2}|z_1 - z_2|$, is an ellipse with focal points z_1, z_2, and major semi-axis a, thereby $|z - z_1| + |z - z_2|$ is the sum of the distances between the point M with affix z and the points A and B with affixes z_1 and z_2, respectively. ■

4.4. Example. Determine the set of points, which corresponds to such complex numbers z, that satisfy the following condition

$$|2z| \geq |1 + z^2|.$$

Solution. Let $z = x + iy$. Thus,

$$\left|1 + z^2\right| = \left|1 + (x + iy)^2\right| = \sqrt{(1 + x^2 - y^2)^2 + 4x^2y^2}.$$

Hence,

$$4(x^2+y^2)\ge 1+x^4+y^4+2x^2-2y^2-2x^2y^2+4x^2y^2,$$
$$1+x^4+y^4-2x^2-2y^2+2x^2y^2-4y^2\le 0,$$
$$(x^2+y^2-1)^2-4y^2\le 0,$$
$$(x^2+y^2-1-2y)(x^2+y^2-1+2y)\le 0$$

or

$$\left(x^2+(y-1)^2-2\right)\left(x^2+(y+1)^2-2\right)\le 0.$$

The last inequality is satisfied if and only if

$$x^2+(y-1)^2-2\ge 0 \text{ and}$$
$$x^2+(y+1)^2-2\le 0$$

or

$$x^2+(y-1)^2-2\le 0 \text{ and}$$
$$x^2+(y+1)^2-2\ge 0.$$

The required set of points is shown in figure 4. ■

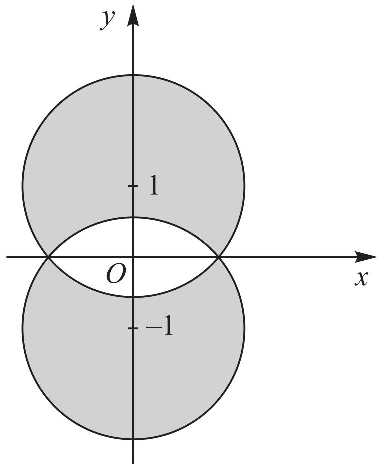

Figure 4

5. EXTENDED COMPLEX PLANE. REIMANN INTERPRETATION OF COMPLEX NUMBERS

5.1. Stereographic projection. In Euclidean space $\mathbf{R}^3$ with Cartesian coordinates ξ, η, ζ, consider the sphere *S* which is centered at $\left(0,0,\frac{1}{2}\right)$ and its radius is $\frac{1}{2}$:

$$\xi^2+\eta^2+\zeta^2-\zeta=0. \qquad (1)$$

The plane $\zeta=0$ coincides with the complex plane **C**, the real axis $\operatorname{Im} z=0$ coincides with the axis $\eta=0$, $\zeta=0$, and the imaginary axis $\operatorname{Re} z=0$ coincides with the axis $\zeta=0$, $\xi=0$.

We draw a line through *P*(0,0,1), such that it meets the sphere *S* at a point *M*(ξ,η,ζ), which differs from *P*. The intersection of the line *PM* and the complex plane is denoted by $z=x+iy$. The point *M*(ξ,η,ζ) is called to be the *stereographic projection* of the complex number (point) *z* onto the sphere *S* with pole *P* (figure 5).

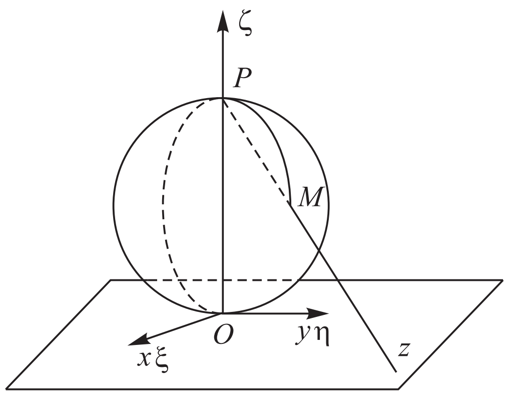

Figure 5

The stereographic projection defines a bijection between the points in the complex plane **C** and the points on the sphere *S*, except the pole *P*. Therefore, each point of the sphere *S*, except the pole

P, may be considered as a point in the complex plane. Such the interpretation of complex numbers is called to be *Reimann interpretation* of complex numbers, and the sphere S is called to be *Reimann sphere*.

The points $P(0,0,1)$, $M(\xi,\eta,\zeta)$ and z are collinear, so

$$\frac{\xi}{x}=\frac{\eta}{y}=\frac{\zeta-1}{-1}$$

or

$$x=\frac{\xi}{1-\zeta},\ y=\frac{\eta}{1-\zeta},\ z=\frac{\xi+i\eta}{1-\zeta}. \tag{2}$$

Hence, $|z|^2=\frac{\xi^2+\eta^2}{(1-\zeta)^2}$. If we substitute in (1) we get $|z|^2=\frac{\zeta}{1-\zeta}$, i.e.

$$\zeta=\frac{|z|^2}{1+|z|^2}. \tag{3}$$

If we substitute in (2) and also have on mind the fact that $x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}$ we get

$$\xi=\frac{z+\bar{z}}{2(1+|z|^2)},\ \eta=\frac{z-\bar{z}}{2i(1+|z|^2)}. \tag{4}$$

The formulae (3) and (4) are known to be *formulae of the stereographic projection*.

5.2. Extended complex plane. In 5.1 we defined a bijection between the complex plane and the Reimann sphere S without the pole P. If we add the "ideal complex number" $z=\infty$ to the set of complex numbers **C**, and complete the complex plane by adjoining the unique infinity point, denoted by ∞, then it exists a bijection between the Reimann sphere S and the set $\mathbf{C}\cup\{\infty\}$, whereby the pole P corresponds to the infinity point ∞.

The complex plane, together with the infinity point, is called to be an *extended complex plane* and is denoted by $\mathbf{C}_\infty=\mathbf{C}\cup\{\infty\}$. Let us state that the infinity point is not involved in the algebraic operations with complex numbers.

5.3. Distance in the extended complex plane. In the complex plane **C** the distance between the points z and z' is defined by $|z-z'|$. In $\mathbf{C}_\infty$ we define the distance between the points z and z', $d(z,z')$ as a distance between appropriate stereographic projections of the points z and z'. Namely, if $M(\xi,\eta,\zeta)$ and $M'(\xi',\eta',\zeta')$ are the stereographic projections of the points $z\neq\infty$ and $z'\neq\infty$, respectively, then

$$d(z,z')=\sqrt{(\xi-\xi')^2+(\eta-\eta')^2+(\zeta-\zeta')^2}=\frac{|z-z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}}$$

and if $z'=\infty$, then

$$d(z,\infty)=\frac{1}{\sqrt{1+|z|^2}}.$$

5.4. Theorem. Under the stereographic projection each circle in the complex plane maps to a circle on the Reimann sphere,such that it does not pass through the pole, and conversely.

Proof. Let

$$x^2 + y^2 + Ax + By + C = 0, \quad A, B, C \in \mathbf{R} \tag{5}$$

be any circle in the complex plane xOy. Since (2) and (5) we get that

$\frac{\zeta}{1-\zeta} + A\frac{\xi}{1-\zeta} + B\frac{\eta}{1-\zeta} + C = 0$, i.e.

$$A\xi + B\eta + (1-C)\zeta + C = 0\,. \tag{6}$$

The equation (6) is the equation of a plane which does not pass through the pole $P(0,0,1)$. So, the coordinates ξ, η, ζ satisfy the equalities (1) and (6). Thus, the points (ξ,η,ζ) lie on the sphere (1) and the plane (6), i.e. they create a circle on the Reimann sphere such that it does not pass through the pole.

Conversely, each circle on the Reimann sphere (1) such that it does not pass through the pole by the stereographic projection is mapped to a circle in the complex plane,thereby, by using arbitrary numbers A, B, C the equation of the intersecting plane may be expressed as (6). ■

5.5. Theorem. Each line in the complex plane under the stereographic projection maps to a circle through the pole and vice versa.

Proof. Let

$$Ax + By + C = 0, \quad A, B, C \in \mathbf{R} \tag{7}$$

be any line in the complex plane xOy. Since the identities (2) and (7), we get that

$$A\frac{\xi}{1-\zeta} + B\frac{\eta}{1-\zeta} + C = 0$$

i.e.

$$A\xi + B\eta + -C\zeta + C = 0\,. \tag{8}$$

The equation (8) is the equation of a plane such that it passes through the pole $P(0,0,1)$. Therefore, the coordinates ξ, η, ζ satisfy the equalities (1) and (8). So, the points (ξ,η,ζ) lie on the sphere (1) and in the plane (8), i.e. they create a circle on the Reimann sphere which passes through the pole.

Conversely, each circle on the Reimann sphere (1) which passes through the pole by the stereographic projection is mapped to a circle in the complex plane, thereby the arbitrariness of numbers A, B, C allows the equation of the intersecting plane to be represented as (8). ■

5.6. Let l and q be two distinct curves on the Reimann sphere (1) such that they meet at a point M. Through such a point it can be drawn tangents to the curves l and q. Let α be the angle formed by the tangents. Let l', q' and M' be the images of l, q and M, respectively, under the stereograph projectionon the complex plane. It is easy to prove that the angle create by the tangents to the curves l' and q' through M' is congruent to α. The proof of this statement will not be elaborated, thereby, it is beyond our main considerations.

6. TRIGONOMETRIC ENTRY OF A COMPLEX NUMBER

6.1. Argument of a complex number. The angle φ created by the positive part of the real axis and the position vector of the point z, is called to be the *argument of the complex number* z, and is denoted as $\varphi = \operatorname{Arg} z$, (figure 6). The argument is either positive or negative, depending on the orientation of the angle φ. The argument is called to be positive if it is oriented from the positive direction of the real axis to the positive direction of the imaginary axis or negative if the one is oriented from the positive direction of the real axis to the negative direction of the imaginary axis.

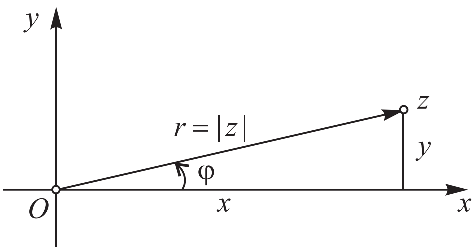

Figure 6

For $z = 0$ the argument is not specified. So, in our further discussion about arguments, we assume that $z \neq 0$.

The position of the point z in the complex plane is uniquely determined by its Cartesian coordinates x, y and by its polar coordinates $r = |z|$ and $\varphi = \operatorname{Arg} z$. The relation between these two types of coordinates is given by the following formulae:

$$x = r\cos\varphi\,,\;\; y = r\sin\varphi\,. \tag{1}$$

For a given point z, its absolute value is uniquely determined, while the argument is determined by accuracy of up to a summand $2k\pi$, $k = 0, \pm1, \pm2, \ldots$. The value of the argument, such that it satisfies the condition $0 < \operatorname{Arg} z \leq 2\pi$ is called to be the *main value of the argument* and is denoted by $\arg z$. Most commonly, in our further considerations we will use the main value of the argument.

6.2. Trigonometric entry of a complex number. Using formulae (1), which refer to the Cartesian and polar coordinates of z, we get the so called *trigonometric representation of a complex number*

$$z = |z|\big(\cos(\arg z) + i\sin(\arg z)\big)\,. \tag{2}$$

By the notation (2) for the product of two complex numbers

$$z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1) \;\text{ and }\; z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2)$$

we get

$$z_1 z_2 = |z_1|\cdot|z_2|\big(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\big)\,. \tag{3}$$

Further, by 1.10 and the definition for the argument of a complex numbers it is true that

$$z_1 z_2 = |z_1 z_2|\big(\cos\big(\arg(z_1 z_2)\big) + i\sin\big(\arg(z_1 z_2)\big)\big)\,.$$

The above statement implies that

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 + 2k\pi\,,\;\; k = 0, \pm1, \pm2, \ldots \tag{4}$$

Analogously, by the equality $z_1 = z_2 z_3$, for $z_2 \neq 0$, using 3.11 we get

$$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 + 2k\pi,\ k = 0, \pm 1, \pm 2, \ldots$$

6.3. De Moivre's formula. Using mathematical induction, the formulaes (3) and (4) could be easily generalized for a finite number of multiples $z_1, z_2, \ldots, z_n$. Namely,

$$\arg(z_1 z_2 \ldots z_n) = \arg z_1 + \arg z_2 + \ldots + \arg z_n + 2k\pi,\ k = 0, \pm 1, \pm 2, \ldots \tag{5}$$

Particularly, for $z_1 = z_2 = \ldots = z_n$ we get

$$|z^n| = |z|^n \text{ and } \arg z^n = n \arg z + 2k\pi,\ k = 0, \pm 1, \pm 2, \ldots$$

i.e.

$$z^n = |z|^n \left(\cos(n \arg z) + i \sin(n \arg z)\right). \tag{6}$$

The formula (6) is known to be the *De Moivre's formula*.

6.4. Example. Compute the difference

$$\left(-1 + i\sqrt{3}\right)^9 - \left(1 + i\sqrt{3}\right)^9.$$

Solution. Since,

$$\left|-1 + i\sqrt{3}\right| = 2 \text{ and } \arg\left(-1 + i\sqrt{3}\right) = \tfrac{2\pi}{3} + 2k\pi,\ k = 0, \pm 1, \pm 2, \ldots$$

we get

$$-1 + i\sqrt{3} = 2\left(\cos\tfrac{2\pi}{3} + i \sin\tfrac{2\pi}{3}\right).$$

Using De Moivre's formula, we obtain:

$$\left(-1 + i\sqrt{3}\right)^9 = 2^9\left(\cos\tfrac{2\pi}{3}\cdot 9 + i \sin\tfrac{2\pi}{3}\cdot 9\right) = 2^9.$$

Analogously,

$$\left(1 + i\sqrt{3}\right)^9 = 2^9\left(\cos\tfrac{9\pi}{3} + i \sin\tfrac{9\pi}{3}\right) = -2^9.$$

Hence,

$$\left(-1 + i\sqrt{3}\right)^9 - \left(1 + i\sqrt{3}\right)^9 = 2^9 - \left(-2^9\right) = 2^{10}. \blacksquare$$

6.5. Example. a) Find the exact value of the expression:

$$\left(1 - i\sqrt{3}\right)^3 (1 + i)^{10}.$$

b) Let

$$f(n) = \left(\tfrac{1+i}{\sqrt{2}}\right)^n + \left(\tfrac{1-i}{\sqrt{2}}\right)^n.$$

Determine the sum $f(1990) + f(1994)$.

Solution. a) We have that:

$$1 - i\sqrt{3} = 2\left(\cos\tfrac{\pi}{3} - i \sin\tfrac{\pi}{3}\right) \text{ and } 1 + i = \sqrt{2}\left(\cos\tfrac{\pi}{4} + i \sin\tfrac{\pi}{4}\right).$$

Therefore,

$$\begin{aligned}\left(1 - i\sqrt{3}\right)^3 (1 + i)^{10} &= 2^3(\cos\pi - i \sin\pi)\cdot 2^5\left(\cos\tfrac{5\pi}{2} + i \sin\tfrac{5\pi}{2}\right)\\ &= 2^8(-1 - i\cdot 0)(0 + i) = -256i.\end{aligned}$$

b) Since,

$$\frac{1+i}{\sqrt{2}}=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4} \text{ and } \frac{1-i}{\sqrt{2}}=\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}$$

we express the function f in terms of cos as follows

$$f(n)=2\cos\frac{n\pi}{4}.$$

Hence,

$$f(1990)+f(1994)=2\cos\frac{1990\pi}{4}+2\cos\frac{1994\pi}{4}=2\cos\frac{995\pi}{2}+2\cos\frac{997\pi}{2}=0. \blacksquare$$

6.6. Example. If $z+\frac{1}{z}=1$, then find the exact value of

$$z^{158}+z^{152}+\frac{2}{z^{122}}.$$

Solution. The given condition $z+\frac{1}{z}=1$ implies $z^2-z+1=0$, i.e.

$$z=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}=\cos\left(\pm\frac{\pi}{3}\right)+i\sin\left(\pm\frac{\pi}{3}\right).$$

Thus,

$$z^{158}+z^{152}+\frac{2}{z^{122}}=\cos\frac{158\pi}{3}\pm i\sin\frac{158\pi}{3}+\cos\frac{152\pi}{3}\pm i\sin\frac{158\pi}{3}-2\left(-\cos\frac{122\pi}{3}\pm i\sin\frac{122\pi}{3}\right)$$

$$=\cos\frac{2\pi}{3}\pm i\sin\frac{2\pi}{3}+\cos\frac{2\pi}{3}\pm i\sin\frac{2\pi}{3}-2\left(-\cos\frac{2\pi}{3}\pm i\sin\frac{2\pi}{3}\right)$$

$$=4\cos\frac{2\pi}{3}=-2. \blacksquare$$

7. ROOTS OF A COMPLEX NUMBER

7.1. Definition. Let $z\neq 0$ be given complex number and n be a positive integer. The *n-th root of z* is defined as such a complex number w that

$$w^n=z. \tag{1}$$

We denote $w=\sqrt[n]{z}$.

Since applying the De Moivre formula

$$z^n=|z|^n\left(\cos(n\arg z)+i\sin(n\arg z)\right)$$

and the trigonometric notations

$$z=|z|\left(\cos(\arg z)+i\sin(\arg z)\right) \text{ and } w=|w|\left(\cos(\arg w)+i\sin(\arg w)\right)$$

we get

$$|w|^n\left(\cos n(\arg w)+i\sin n(\arg w)\right)=|z|\left(\cos(\arg z)+i\sin(\arg z)\right)$$

i.e.

$$|w|^n=|z| \text{ and } n(\arg w)=\arg z+2k\pi,\ k=0,\pm1,\pm2,... \tag{2}$$

Therefore,

$$|w|=\sqrt[n]{|z|},\ \arg w=\frac{\arg z+2k\pi}{n},\ k=0,\pm1,\pm2,...,$$

i.e.

$$w=\sqrt[n]{z}=\sqrt[n]{|z|}\left(\cos\frac{\arg z+2k\pi}{n}+i\sin\frac{\arg z+2k\pi}{n}\right), \tag{3}$$

By letting $k=0,1,2,\ldots,n-1$ in (3), we obtain n distinct complex numbers $w_0, w_1, \ldots, w_{n-1}$ as w for $k=n$, and also thereby the periodicity of the trigonometric functions we obtain w_0, etc. Thus, the n-th root of the complex number z has exactly n distinct values, obtained by the formula (3) for $k=0,1,2,\ldots,n-1$.

7.2. Example. Find $\sqrt[3]{27i^5}$.

Solution. Since,

$$i^5 = i^4 \cdot i = i = \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}$$

we get

$$\sqrt[3]{27i^5} = \sqrt[3]{27i} = \sqrt[3]{27\left(\cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2}\right)} = 3\left(\cos\frac{2k\pi+\frac{\pi}{2}}{3} + i\sin\frac{2k\pi+\frac{\pi}{2}}{3}\right)$$

$$= 3\left(\cos\frac{(4k+1)\pi}{6} + i\sin\frac{(4k+1)\pi}{6}\right),$$

for $k=0,1,2$. ∎

7.3. Example. Prove that all complex numbers a and b satisfy

$$2(|a|+|b|) = \left|a+b-2\sqrt{ab}\right| + \left|a+b+2\sqrt{ab}\right|,$$

where $\sqrt{ab}$ denotes one of the two roots of ab.

Solution. Since,

$$2(|a|+|b|) = 2\left(\left|\sqrt{a}\right|^2 + \left|\sqrt{b}\right|^2\right)$$

we obtain

$$2(|a|+|b|) = \left|\sqrt{a}+\sqrt{b}\right|^2 + \left|\sqrt{a}-\sqrt{b}\right|^2 =$$

$$= \left|\left(\sqrt{a}+\sqrt{b}\right)^2\right| + \left|\left(\sqrt{a}-\sqrt{b}\right)^2\right|$$

$$= \left|a+b+2\sqrt{ab}\right| + \left|a+b-2\sqrt{ab}\right|.$$

So, the required equality is proved. ∎

7.4. Example. Prove that all complex numbers a and b satisfy

$$|a+b|+|a-b| = \left|a+\sqrt{a^2-b^2}\right| + \left|a-\sqrt{a^2-b^2}\right|,$$

where $\sqrt{a^2-b^2}$ denotes one of the two roots of a^2-b^2.

Solution. By example 3.16, we get

$$\left(\left|a+\sqrt{a^2-b^2}\right| + \left|a-\sqrt{a^2-b^2}\right|\right)^2 =$$

$$= \left|a+\sqrt{a^2-b^2}\right|^2 + \left|a-\sqrt{a^2-b^2}\right|^2 + 2\left|a+\sqrt{a^2-b^2}\right| \cdot \left|a-\sqrt{a^2-b^2}\right|$$

$$= \left|a+\sqrt{a^2-b^2}\right|^2 + \left|a-\sqrt{a^2-b^2}\right|^2 + 2\left|a^2-(a^2-b^2)\right|$$

$$=2\left(\left|a^2\right|+\left|\sqrt{a^2-b^2}^{\,2}\right|\right)+2\left|b^2\right|=2\left(\left|a^2\right|+\left|b^2\right|\right)+2\left|a^2-b^2\right|$$

$$=|a-b|^2+|a+b|^2+2|a-b|\cdot|a+b|=(|a-b|+|a+b|)^2$$

i.e.

$$|a+b|+|a-b|=\left|a+\sqrt{a^2-b^2}\right|+\left|a-\sqrt{a^2-b^2}\right|. \blacksquare$$

7.5. Example. Solve the equation

$$(x+i)^n+(x-i)^n=0,\ n\in\mathbf{N},\ n>1.$$

Solution. Since, $x\neq i$, the given equation is equivalent to

$$\left(\frac{x+i}{x-i}\right)^n=-1.$$

Therefore,

$$\frac{x+i}{x-i}=\sqrt[n]{-1}=\sqrt[n]{\cos\pi+i\sin\pi}=\cos\frac{\pi+2k\pi}{n}+i\sin\frac{\pi+2k\pi}{n},\ k=0,1,\dots,n-1.$$

By using the last equation we get that

$$\frac{x+i}{x-i}-1=\cos\frac{\pi+2k\pi}{n}+i\sin\frac{\pi+2k\pi}{n}-1,$$

$$\frac{2i}{x-i}=2i\sin\frac{\pi+2k\pi}{2n}\left(\cos\frac{\pi+2k\pi}{2n}-\frac{1}{i}\sin\frac{\pi+2k\pi}{n}\right),$$

$$x-i=\frac{1}{\sin\frac{\pi+2k\pi}{2n}\left(\cos\frac{\pi+2k\pi}{2n}+i\sin\frac{\pi+2k\pi}{n}\right)},$$

$$x-i=\frac{\cos\frac{\pi+2k\pi}{2n}-i\sin\frac{\pi+2k\pi}{n}}{\sin\frac{\pi+2k\pi}{2n}},$$

$$x-i=\operatorname{ctg}\frac{\pi+2k\pi}{2n}-i.$$

that is,

$$x=\operatorname{ctg}\frac{\pi+2k\pi}{2n},\ k=0,1,\dots,n-1. \blacksquare$$

7.6. The *n*-th roots of the unity. In 7.1 we discussed roots of complex numbers. If $z=1$, then $\arg z=0$ and by using (3) the n different roots of the unity are expessed as

$$u_k=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n},\quad k=0,1,2,\dots,n-1. \qquad (4)$$

If

$$u=u_1=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n},$$

then, since the DeMoivre's formula

$$u_k=u^k,\quad k=0,1,\dots,n-1.$$

Let us consider that in geometric terms, for $n\geq 3$, the points in the of complex plane with affixes the n-th roots of unity, form a regular n-gon inscribed in the unit circle and one of the polygon's vertices coincides to the point with affix $z=1$.

7.8. Example. Let $S_p = \sum_{k=0}^{n-1} u_k^p$ be the sum of the p-th exponents of the n-th roots of the unity and $n \in \mathbf{N}$. Prove that

$$S_p = \begin{cases} n, & \text{for } n \mid p \\ 0, & \text{for } n \nmid p. \end{cases}$$

Solution. Since,

$$u_k = u^k, \quad k = 0,1,\ldots,n-1$$

and

$$u = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},$$

we get

$$S_p = 1 + u^p + u^{2p} + \ldots + u^{(n-1)p}. \tag{5}$$

If $n \mid p$ and $\frac{p}{n} = m$, then

$$u^p = u^{mn} = (u^n)^m = 1^m = 1$$

and moreover, thereby (5) we deduce that $S_p = n.$

Let $n \nmid p$. Thus

$$u^{np} = (u^n)^p = 1^p = 1 \text{ holds.}$$

Since $n \nmid p$, it follows that $u^p - 1 \neq 0.$ Therefore,

$$S_p = 1 + u^p + u^{2p} + \ldots + u^{(n-1)p} = \frac{u^{np}-1}{u^p-1} = 0. \ \blacksquare$$

7.9. Example. Prove the following identities

a) $\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots + \cos\frac{2(n-1)\pi}{n} = -1$, for $n = 2,3,\ldots$ and

b) $\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \ldots + \sin\frac{2(n-1)\pi}{n} = 0$ for $n = 2,3,\ldots$

Solution. The equation $z^n - 1 = 0$ has n roots. The before stated roots are the n-th roots of the unity

$$u_k = u^k, \quad k = 0,1,\ldots,n-1 \text{ and } u = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

Example 7.8 implies that their sum is equal to zero. Hence,

$$\sum_{k=0}^{n-1} u_k = 0,$$

i.e.

$$\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots + \cos\frac{2(n-1)\pi}{n} + i\left(\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \ldots + \sin\frac{2(n-1)\pi}{n}\right) = -1.$$

The latter is equivalent to a) and b). So, the required identities are proven. ■

7.10. Definition. The complex number u is called to be *primitive n-th root of the unity,* if $u^n = 1$ and there is no any lower exponent of u, which is equal to 1.

7.11. Example. Let

$$u_k = u^k, \quad k = 0,1,\ldots,n-1$$

be the n-th roots of the unity. Prove that u_k is the primitive n-th root of unity if and only if n and k are co-prime numbers.

Solution. Let n and k be co-prime numbers and let there exists $r < n$ such that satisfies $u_k^r = 1$. According to the De Moivre's formula we get

$$1 = u_k^r = \cos\tfrac{2kr\pi}{n} + i\sin\tfrac{2kr\pi}{n}.$$

According to the last equality, we get

$$\cos\tfrac{2kr\pi}{n} = 1, \quad \sin\tfrac{2kr\pi}{n} = 0.$$

Hence, $\frac{kr}{n} \in \mathbf{Z}$ and since, n and k are co-prime numbers, we get that $n \mid r$, which is contradiction, thereby $r < n$. Thus, u_k is the primitive n-the root of the unity.

Conversely, let u_k be the primitive n-th root of the unity. Let assump the greatest common divisor of n and k is d, $d > 1$. Let $k = k_1 d$, $n = n_1 d$. Then

$$u_k^{n_1} = (u_1^k)^{n_1} = u_1^{n_1 k} = u_1^{k_1 d n_1} = u_1^{k_1 n} = (u_1^n)^{k_1} = 1^{k_1} = 1.$$

The latter contradicts to the fact that u_k is a primitive n-the root of unity, thereby $n_1 < n$. ■

8. EXPONENTIAL ENTRY OF A COMPLEX NUMBER

8.1. In our previous discussion we have presented the algebraic and trigonometric representations of complex numbers. In this section we will focus on the so called *exponential entry* of complex numbers.

Theorem. Let the function $f : \mathbf{R} \to \mathbf{C}$ be defined by

$$f(\alpha) = \cos\alpha + i\sin\alpha, \text{ for each } \alpha \in \mathbf{R}.$$

Then,

a) $f(0) = 1$ and $f(\alpha) \neq 0$, for each $\alpha \in \mathbf{R}$.
b) $f(\alpha+\beta) = f(\alpha)f(\beta)$, for all $\alpha, \beta \in \mathbf{R}$.
c) $f(-\alpha) = \frac{1}{f(\alpha)}$, for each $\alpha \in \mathbf{R}$.

Proof. a) Clearly, $f(0) = \cos 0 + i\sin 0 = 1$. Let there exist $\alpha \in \mathbf{R}$, such that $f(\alpha) = 0$. Thus, it exists $\alpha \in \mathbf{R}$, so that $\cos\alpha + i\sin\alpha = 0$, i.e. $\cos\alpha = \sin\alpha = 0$, which contradicts to the basic trigonometric identity

$$\cos^2\alpha + \sin^2\alpha = 1.$$

b) For all $\alpha, \beta \in \mathbf{R}$

$$f(\alpha+\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$
$$= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = f(\alpha)f(\beta).$$

c) For each $\alpha \in \mathbf{R}$

$$f(-\alpha) = \cos(-\alpha) + i\sin(-\alpha) = \cos\alpha - i\sin\alpha$$
$$= \frac{(\cos\alpha - i\sin\alpha)(\cos\alpha + i\sin\alpha)}{\cos\alpha + i\sin\alpha} = \frac{1}{\cos\alpha + i\sin\alpha} = \frac{1}{f(\alpha)}. \blacksquare$$

8.2. In the previous theorem we proved that the function *f* satisfies the ordinary properties of the exponential function, so it is natural to introduce the notation $f(\alpha) = e^{i\alpha}$, for each $\alpha \in \mathbf{R}$. Therefore, the properties b) and c) of the theorem could be stated as the following:

$$e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)} \tag{1}$$

$$e^{-i\alpha} = \frac{1}{e^{i\alpha}}. \tag{2}$$

By using the identities (1) and (2) and also the principle of mathematical induction, we obtain that

$$(e^{i\alpha})^n = e^{in\alpha}, \text{ for } n = 0, \pm1, \pm2, \dots. \tag{3}$$

8.3. Euler's formulae. The above stated, implies that each complex number *z*, such that $|z| = 1$ and $\varphi = \arg z$ may be denoted as

$$z = \cos\varphi + i\sin\varphi = e^{i\varphi}. \tag{4}$$

Thus, $e^{2\pi i} = 1$, $e^{\pi i} = -1$, $e^{\frac{\pi i}{2}} = i$, $e^{\frac{3\pi i}{2}} = -i$. If we change φ by $-\varphi$ we obtain that

$$\cos\varphi - i\sin\varphi = e^{-i\varphi}. \tag{5}$$

By using the dentities (4) and (5) we obtain the well known *Euler's formulae*:

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \quad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2}. \tag{6}$$

These formulae allow the trigonometric functions cos and sin to be expessed in terms of the exponential function.

At the present moment we shall state that in Theorem 1 we did not give the proof of the formula (4)., we only gave its "acceptable" explanation.

8.4. The formula (4) and the trigonometric form of complex numbers imply that each complex number $z \neq 0$ may be written as

$$z = re^{i\varphi}, \tag{7}$$

where $r = |z|$ and $\varphi = \arg z$. The notation (7) of a complex number $z \neq 0$ is called to be *exponential representation* of *z*.

Using formulae (1) and (2), we obtain the exponential forms of the formulae for multiplication and division of complex numbers, i.e.

$$z_1 z_2 = r_1 e^{i\varphi_1} r_2 e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1+\varphi_2)}, \tag{8}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1-\varphi_2)}. \tag{9}$$

Let $z = re^{i\varphi}$. Since (4) and (5) the expression for z_ is the following $\bar{z} = re^{-i\varphi}$. Hence, if $\varphi = \arg z$, then $-\varphi = \arg \bar{z}$.

8.5. Example. Find the sums:

a) $A = \cos x + \cos(x+\alpha) + \cos(x+2\alpha) + \ldots + \cos(x+n\alpha)$, and

b) $B = \sin x + \sin(x+\alpha) + \sin(x+2\alpha) + \ldots + \sin(x+n\alpha)$.

Solution. Let $S = A + iB$. Thus

$$S = e^{ix} + e^{i(x+\alpha)} + e^{i(x+2\alpha)} + \ldots + e^{i(x+n\alpha)}$$

$$= e^{ix}\left(1 + e^{i\alpha} + e^{i2\alpha} + \ldots + e^{in\alpha}\right) = \frac{e^{ix}\left(e^{i(n+1)\alpha} - 1\right)}{e^{i\alpha} - 1}.$$

Since, $A = \operatorname{Re} S$ and $B = \operatorname{Im} S$. by applying the last formula we express *A* and *B*. If the numerator and the denominator be divided by $e^{i\frac{\alpha}{2}}$, then thereby the Euler's formulae we get that the denominator is $2i\sin\frac{\alpha}{2}$, and the numerator is

$$\cos\left(x + \left(n+\tfrac{1}{2}\right)\alpha\right) - \cos\left(x - \tfrac{\alpha}{2}\right) + i\left(\sin\left(x + \left(n+\tfrac{1}{2}\right)\alpha\right) - \sin\left(x - \tfrac{\alpha}{2}\right)\right) =$$

$$= 2\sin\tfrac{(n+1)\alpha}{2}\left(-\sin\left(x + \tfrac{n\alpha}{2}\right) + i\cos\left(x + \tfrac{n\alpha}{2}\right)\right).$$

Hence,

$$A = \frac{\sin\frac{(n+1)\alpha}{2}\cos\left(x+\frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}} \text{ and } B = \frac{\sin\frac{(n+1)\alpha}{2}\sin\left(x+\frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}. \ \blacksquare$$

8.6. Remark. If $x = 0$ in example 8.5 we get

$$1 + \cos\alpha + \cos 2\alpha + \ldots + \cos n\alpha = \frac{\sin\frac{(n+1)\alpha}{2}\cos\frac{n+\alpha}{2}}{\sin\frac{\alpha}{2}}, \text{ and}$$

$$\sin\alpha + \sin 2\alpha + \ldots + \sin n\alpha = \frac{\sin\frac{(n+1)\alpha}{2}\sin\frac{n+\alpha}{2}}{\sin\frac{\alpha}{2}}. \ \blacksquare$$

8.7. Let *E* be the point with affix 1. Consider the points *A* and A' with affixes

$$a = \rho e^{i\theta} \text{ and } a' = \rho' e^{i\theta'},$$

respectively (figure 7). The product $b = aa'$ corresponds to a point *B*, obtained as the third vertex of the triangle $OA'B$, if this triangle is constructed as a similar one to the triangle *OEA*.

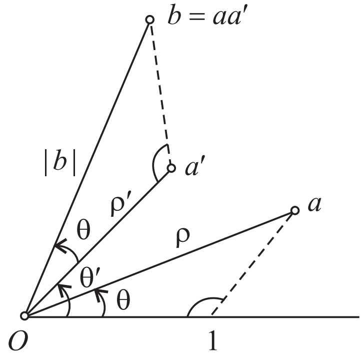

Figure 7

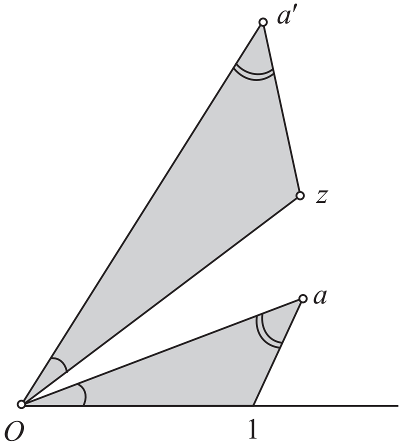

Figure 8

Indeed, the similarity of these triangles implies $\angle EOA = \angle A'OB$, i.e. $\arg b = \theta + \theta'$. For the same reason $\rho : 1 = |b| : \rho''$ holds, i.e. $b = \rho\rho'$ holds. Therefore, $b = aa'$.

The point Z whose affix is the complex number $z = \frac{a'}{a}$ is obtained by construction the triangle OZA' similar to the triangle OEA.

Indeed, the similarity of these triangles implies $az = a'$. Therefore, $z = \frac{a'}{a}$ (figure 8).

By using the relation $a^n = a^{n-1}a$ and consecutively applying the procedures for constructing the affixes of the product and the quotient of two complex numbers, we obtain the points

$$\ldots, A_{-3}, A_{-2}, A_{-1}, E, A_1, A_2, A_3, \ldots$$

whose affixes are the complex numbers

$$\ldots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \ldots,$$

respectively.

Let be $r > 1$ and $0 < \alpha < \pi$. The points $A_2, A_3, \ldots$ (figure 9), with affixes $a^2, a^3, \ldots$ are obtained by a consecutive construction of the similar triangles

$$OEA_1, OA_1A_2, OA_2A_3, \ldots.$$

Constructing the similar triangles

$$OEA_1, OA_{-1}E, OA_{-2}A_{-1}, OA_{-3}A_{-2}, \ldots$$

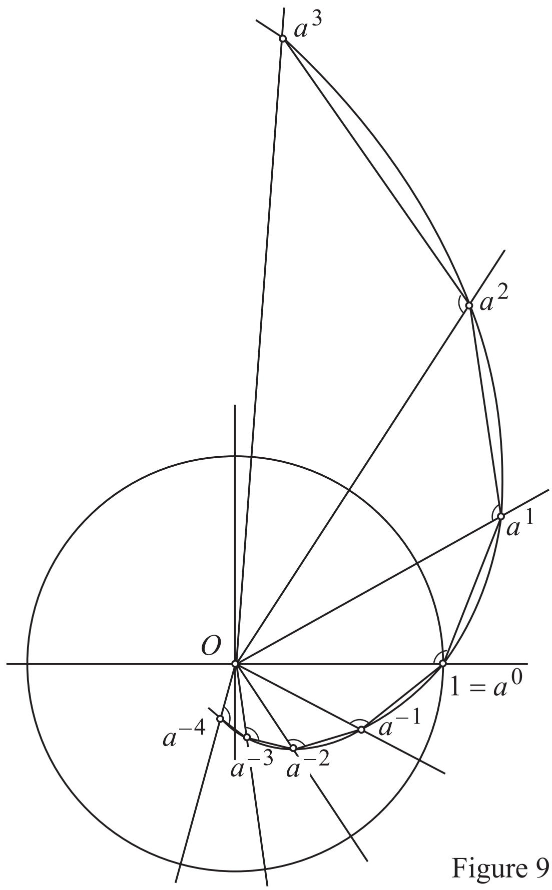

Figure 9

by applying the same procedure, but in the opposite direction, we get the points

$$A_{-1}, A_{-2}, A_{-3}, \ldots$$

with affixes

$$a^{-1}, a^{-2}, a^{-3}, \ldots.$$

For $\rho = r^n$, $\theta = n\alpha$ and after eliminating n in the above equations we obtain that $\rho = r^{\frac{\theta}{\alpha}}$. Thus, each powers a^n lie on the curve, which is presented in polar form by the previous relation. This curve is known to be the *logarithmic (Bernoulli) spiral*. Clearly, in the previous considerations the absolute values of the powers increase or decrease like a geometric progression, while the arguments – like an arithmetic progression.

Obviously, for $r < 1$ and $0 < \alpha < \pi$, or $r > 1$ and $-\pi < \alpha < 0$, the logarithmic spiral is in opposite direction of the spiral given in figure 9 and it wraps around the origin when θ decreases. Likewise if $r < 1$ and $-\pi < \alpha < 0$, then the logarithmic spiral has the same appearance as shown in figure 9.

9. THE SET $\mathbf{C}^n$

9.1. Definition. The notation $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, for $a_i \in \mathbf{C}$, and $i = 1, 2, \ldots, n$ is called to be an ordered *n-tuple of complex numbers.*

Thereby, we accept the notation

$$\mathbf{C}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbf{C}, i = 1, 2, \ldots, n\}\cdot$$

9.2. Definition. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two ordered *n*-tuples of complex numbers, i.e. $\mathbf{a}, \mathbf{b} \in \mathbf{C}^n$. *The sum of* **a** and **b** is the ordered *n*-tuple $\mathbf{c} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$.

Thereby, we accept the notation $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

9.3. Theorem. a) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$, for all $\mathbf{a}, \mathbf{b} \in \mathbf{C}^n$.

b) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$, for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{C}^n$.

c) There exists $\mathbf{o} \in \mathbf{C}^n$ such that $\mathbf{a} + \mathbf{o} = \mathbf{a}$, for each $\mathbf{a} \in \mathbf{C}^n$.

d) For each $\mathbf{a} \in \mathbf{C}^n$ there exists $\mathbf{b} \in \mathbf{C}^n$ sothat $\mathbf{a} + \mathbf{b} = \mathbf{o}$.

Proof. a) Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$. By applying the commutative and the associative lows of the addition of complex numbers, the definition 9.2 implies

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)\\ &= (b_1 + a_1, b_2 + a_2, \ldots, b_n + a_n)\\ &= \mathbf{b} + \mathbf{a}.\end{aligned}$$

b) The proof is analogical to the previous one. We should use the associative law of addition of complex numbers.

c) For an ordered n-tuple $\mathbf{o}=(0,0,...,0)$ and for each $\mathbf{a}\in \mathbf{C}^n$

$$\begin{aligned}\mathbf{a}+\mathbf{o}&=(a_1,a_2,...,a_n)+(0,0,...,0)\\&=(a_1+0,a_2+0,...,a_n+0)\\&=(a_1,a_2,...,a_n)=\mathbf{a}.\end{aligned}$$

holds true.

d) Let $\mathbf{a}=(a_1,a_2,...,a_n)$ be a given ordered n-tuple. By letting $\mathbf{b}=(-a_1,-a_2,...,-a_n)$, we get

$$\begin{aligned}\mathbf{a}+\mathbf{b}&=(a_1-a_1,a_2-a_2,...,a_n-a_n)\\&=(0,0,...,0)=\mathbf{o}.\end{aligned}$$

Hence, for each $\mathbf{a}\in \mathbf{C}^n$ ther exists $\mathbf{b}\in \mathbf{C}^n$ so that $\mathbf{a}+\mathbf{b}=\mathbf{o}$. Moreover, the ordered n-tuple **b** is called to be an *opposite* n-tuple of **a** and is denoted by $\mathbf{b}=-\mathbf{a}$. ■

9.4. Definition. Let be an ordered n-tuple and $\lambda\in \mathbf{C}$. The *product* of the ordered n-tuple **a** and the complex number λ is the ordered n-tuple $\mathbf{c}=(\lambda a_1,\lambda a_2,...,\lambda a_n)$.

Therefore we accept the notation $\mathbf{c}=\lambda\mathbf{a}$.

9.5. Theorem. a) $\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}$, for all $\mathbf{a},\mathbf{b}\in \mathbf{C}^n$ and $\lambda\in \mathbf{C}$.

b) $(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}$, for all $\lambda,\mu\in \mathbf{C}$ and $\mathbf{a}\in \mathbf{C}^n$.

c) $(\lambda\mu)\mathbf{a}=\lambda(\mu\mathbf{a})$, for all $\lambda,\mu\in \mathbf{C}$ and $\mathbf{a}\in \mathbf{C}^n$.

d) $1\cdot\mathbf{a}=\mathbf{a}$ for each $\mathbf{a}\in \mathbf{C}^n$.

Proof. a) Let $\mathbf{a}=(a_1,a_2,...,a_n)$, $\mathbf{b}=(b_1,b_2,...,b_n)$ and $\lambda\in \mathbf{C}$. Since the definition 9.4 and by applying the distributive laws of addition and the multiplication to the coordinates it follows that

$$\begin{aligned}\lambda(\mathbf{a}+\mathbf{b})&=\lambda\big((a_1,a_2,...,a_n)+(b_1,b_2,...,b_n)\big)\\&=\lambda(a_1+b_1,a_2+b_2,...,a_n+b_n)\\&=\big(\lambda(a_1+b_1),\lambda(a_2+b_2),...,\lambda(a_n+b_n)\big)\\&=(\lambda a_1+\lambda b_1,\lambda a_2+\lambda b_2,...,\lambda a_n+\lambda b_n)\\&=(\lambda a_1,\lambda a_2,...,\lambda a_n)+(\lambda b_1,\lambda b_2,...,\lambda b_n)\\&=\lambda(a_1,a_2,...,a_n)+\lambda(b_1,b_2,...,b_n)=\lambda\mathbf{a}+\lambda\mathbf{b}.\end{aligned}$$

The proofs for the other statements are analogous. While proving them it is necessary to apply the distributive and the associative lows and also the fact that $1\cdot z=z$, for each $z\in \mathbf{C}$. ■

9.6. Definition. *Scalar (inner) product* of an ordered n-tuples $\mathbf{a}=(a_1,a_2,...,a_n)$ and $\mathbf{b}=(b_1,b_2,...,b_n)$ is the complex number

$$f=a_1\overline{b_1}+a_2\overline{b_2}+...+a_n\overline{b_n}.$$

Therefore we accept the notation $f=(\mathbf{a},\mathbf{b})=\mathbf{a}\cdot\mathbf{b}$.

9.7. Theorem. a) $(\mathbf{a},\mathbf{a})\in\mathbf{R}^+$ for each $\mathbf{a}\in\mathbf{C}^n$.

b) $(\mathbf{a},\mathbf{b})=\overline{(\mathbf{b},\mathbf{a})}$, for all $\mathbf{a},\mathbf{b}\in\mathbf{C}^n$.

c) $(\mathbf{a}+\mathbf{b},\mathbf{c})=(\mathbf{a},\mathbf{c})+(\mathbf{b},\mathbf{c})$ for all $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbf{C}^n$.

d) $(\mathbf{a},\mathbf{b}+\mathbf{c})=(\mathbf{a},\mathbf{b})+(\mathbf{a},\mathbf{c})$ for all $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbf{C}^n$.

e) $(\lambda\mathbf{a},\mathbf{b})=\lambda(\mathbf{a},\mathbf{b})$ and $(\mathbf{a},\lambda\mathbf{b})=\overline{\lambda}(\mathbf{a},\mathbf{b})$ for all $\mathbf{a},\mathbf{b}\in\mathbf{C}^n$ and $\lambda\in\mathbf{C}$.

Proof. a) Let $\mathbf{a}=(a_1,a_2,...,a_n)$. Then

$$(\mathbf{a},\mathbf{a})=a_1\overline{a_1}+a_2\overline{a_2}+...+a_n\overline{a_n}=|a_1|^2+|a_2|^2+...+|a_n|^2\in\mathbf{R}^+.$$

b) By using the properties of conjugate complex numbers we obtain

$$\begin{aligned}(\mathbf{a},\mathbf{b})&=a_1\overline{b_1}+a_2\overline{b_2}+...+a_n\overline{b_n}\\&=\overline{\overline{a_1}b_1+\overline{a_2}b_2+...+\overline{a_n}b_n}=\overline{(\mathbf{b},\mathbf{a})}.\end{aligned}$$

The proofs of the other statements are direct implications of the scalar product definition, the distributive and the associative laws of the operations in the set of comlex numbers and the statement b). ■

9.8. Remark. The sum of ordered n-tuples and also the product of an ordered n-tuple and real numbers are ordered n-tuples. On the other hand the scalar product of two ordered n-tupes is a complex number.

9.9. We define the mapping T: $\mathbf{C}^n\to\mathbf{C}^n$ by

$$T\mathbf{a}=(a_2,a_3,...,a_n,a_1),\text{ for each }\mathbf{a}=(a_1,a_2,...,a_n).$$

By induction we determine $T^m\mathbf{a}=TT^{m-1}\mathbf{a}$, for $m\geq 2$. For example,

$$T^2\mathbf{a}=T(a_2,a_3,...,a_n,a_1)=(a_3,a_4,...,a_n,a_1,a_2).$$

We define a mapping T_1 from $\mathbf{C}^n$ to $\mathbf{C}^n$, such that each ordered n-tuple $\mathbf{a}=(a_1,a_2,...,a_n)$ maps to an ordered n-tuple $T_1\mathbf{a}=(a_n,a_1,...,a_{n-1})$. Obviously, $T_1T\mathbf{a}=TT_1\mathbf{a}=\mathbf{a}$, for each $\mathbf{a}\in\mathbf{C}^n$, i.e. the mappings T and T_1 are inverse to each other. Therefore, $T_1=T^{-1}$ and by induction we determine

$$T^{-m}\mathbf{a}=T^{-1}T^{-(m-1)}\mathbf{a},\text{ for }m\geq 2.$$

9.10. Theorem. a) $T(\mathbf{a}+\mathbf{b})=T\mathbf{a}+T\mathbf{b}$ for all $\mathbf{a},\mathbf{b}\in\mathbf{C}^n$.

b) $T(\lambda\mathbf{a})=\lambda T\mathbf{a}$ for each and each $\lambda\in\mathbf{C}$.

c) If $\mathbf{a}=(\alpha,\alpha,...,\alpha)$, $\alpha\in\mathbf{C}$, then $T\mathbf{a}=\mathbf{a}$.

d) $T^n\mathbf{a} = T^{-n}\mathbf{a} = \mathbf{a}$, for each $\mathbf{a} \in \mathbf{C}^n$, (or more generally $T^{nk}\mathbf{a} = T^{-nk}\mathbf{a} = \mathbf{a}$ for each $\mathbf{a} \in \mathbf{C}^n$ and each $k \in \mathbf{N}$).

e) $(T^m\mathbf{a}, T^s\mathbf{b}) = (T^{m+k}\mathbf{a}, T^{s+k}\mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{C}^n$ and all $m, k, s \in \mathbf{N}$.

Proof a) Let $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_n)$ be arbitrary elements of $\mathbf{C}^n$. Then,

$$\begin{aligned} T(\mathbf{a}+\mathbf{b}) &= T\left((a_1, a_2, ..., a_n) + (b_1, b_2, ..., b_n)\right) \\ &= T(a_1 + b_1, a_2 + b_2, ..., a_n + b_n) \\ &= (a_2 + b_2, a_3 + b_3, ..., a_n + b_n, a_1 + b_1) \\ &= (a_2, a_3, ..., a_n, a_1) + (b_2, b_3, ..., b_n, b_1) \\ &= T\mathbf{a} + T\mathbf{b}. \end{aligned}$$

The other statements can by proved analogously, by applying the definition of the mapping T and the properties of the arithmetic operations in $\mathbf{C}^n$. ■

9.11. Corollary. For each $\mathbf{a} \in \mathbf{C}^n$

$$(T\mathbf{a}, \mathbf{a}) = (T^2\mathbf{a}, T\mathbf{a}) = \ldots \text{ holds.}$$

Proof. The proof is directly implied by Theorem 9.10. e). ■

CHAPTER II
TRANSFORMATION IN EUCLEDIAN PLANE

In this chapter we will firstly discuss few elementary transformations in the complex plane.Special attention will be paid to the similarities, their group propertiesand classification. Furthermore, the inversion and the Möbius transformation, treated as the most important elementary transformation of the complex plane, will be elaborated in separate paragraphs.

1. LINE EQUATION. PARALLEL AND PERPEDNICULAR LINES

1.1. Let the line (p) do not pass through the origin and let the point A, with affix a, be symmetric to the origin O with respect to (p). Then, a point B, with affix z, is on the line (p) if and only if $\overline{OB} = \overline{AB}$, i.e. $|z| = |z-a|$, that is

$$z\overline{z} = (z-a)(\overline{z}-\overline{a}).$$

The last equality may be transformed and rewritten as the following

$$\overline{a}z + a\overline{z} = a\overline{a}. \qquad (1)$$

If (p) passes through the origin and the points A and A', with affixes a and a', respectively, are symmetric to each other with respect to the origin O and to the line (p), then any arbitrary point B with affix z, such that B lies on (p) satisfies the following relation $\overline{AB} = \overline{A'B}$, i.e. $|z+a| = |z-a|$, that is

$$(z+a)(\overline{z}+\overline{a}) = (z-a)(\overline{z}-\overline{a}).$$

The last equality may be transformed and rewritten as the following

$$\overline{a}z + a\overline{z} = 0. \qquad (2)$$

If $a = re^{i\varphi}$, then $\overline{a} = re^{-i\varphi}$. Hence, if we divide the equalities (1) and (2) by $\overline{a}$ we obtain the following equations

$$z = \eta\overline{z} + a \qquad (3)$$

and

$$z = \eta\overline{z}, \qquad (4)$$

where $\eta = -\frac{a}{\overline{a}} = -e^{2i\varphi}$. The number η is called to be *complex gradient* of the line (p), and the point A is called to be *mirror point* of the line (p). Obviously, each line (p), which does not pass through the origin, is determined by the mirror point A, with affix $a = re^{i\varphi}$ and the complex gradient $\eta = -e^{2i\varphi}$. Each line (p), which passes through the origin, is uniquely determined by its complex gradient. It is easy to prove that in both cases, the angle between the line (p) and the positive part of the real axis is $\alpha = \varphi - \frac{\pi}{2}$.

The above stated implies the validity of the following theorem.

Theorem. If *A*, with affix *a*, is a symmetric point to the origin with respect to a given line (*p*), such that it does not pass through the origin and if φ is the oriented (directed) angle between the real axis and the line through the origin and perpendicular to (*p*), then the equation of (*p*) is given by (3), where $\eta = -e^{2i\varphi}$. If (*p*) passes through the origin, then its equation is given by (4). ■

1.2. Theorem. The equation of a line (*p*), which passes through two distinct points *A* and *B* with affixes z_0 and z_1, respectively, is

$$z - z_0 = \frac{z_1 - z_0}{\overline{z_1} - \overline{z_0}}(\overline{z} - \overline{z_0}) \qquad (5)$$

and its complex gradient is

$$\eta = \frac{z_1 - z_0}{\overline{z_1} - \overline{z_0}}. \qquad (6)$$

Proof. Let z_0 and z_1 be the affixes of *A* and *B*, respectively. By substituting these affixes into the equation (3) we get that $z_0 = \eta\overline{z_0} + a$ and $z_1 = \eta\overline{z_1} + a$. Further, by subtracting the last two equalities we obtain the complex gradient of the line as following $\eta = \frac{z_1 - z_0}{\overline{z_1} - \overline{z_0}}$, i.e. the equality (6) holds true. If the so obtained expression for complex gradient η is substitute $z_0 = \eta\overline{z_0} + a$ we obtain the following *:*

$$a = z_0 - \frac{z_1 - z_0}{\overline{z_1} - \overline{z_0}}\overline{z_0},$$

Moreover, if the above determined values for η and *a*, we substitute into the equation (3) we get the equation (5). ■

1.3. Corollary. The points z_0, z_1 and z_2 are collinear if and only if

$$\frac{z_2 - z_0}{z_1 - z_0} = \frac{\overline{z_2} - \overline{z_0}}{\overline{z_1} - \overline{z_0}}. \qquad (7)$$

Proof. According to Theorem 1.2, the equation of a line (*p*), such that it passes through z_0 and z_1, is given by (5). The points z_0, z_1 and z_2 are collinear if and only if z_2 satisfies the equality (5), that is if and only if the equality which is equivalent to (7).

$$z_2 - z_0 = \frac{z_1 - z_0}{\overline{z_1} - \overline{z_0}}(\overline{z_2} - \overline{z_0}),$$

is satisfied. ■

1.4. Corollary. The points z_0, z_1 and z_2 are collinear if and only if $\frac{z_2 - z_0}{z_1 - z_0}$ is a real number.

Proof. The proof is directly implicated by Corollary 1.3 and the properties of the complex numbers.■

1.5. Remark. Since $|\eta| = \left|\frac{z_1 - z_0}{\overline{z_1 - z_0}}\right|$, the equation (5) can be rewritten as

$$z - z_0 = \eta(\bar{z} - \overline{z_0}), \ |\eta| = 1 \tag{8}$$

Conversely, each equation of type (8) is a line equation.

Indeed, thereby $|\eta| = 1$, it follows that it exists $\varphi \in [0, \pi)$ so that $\eta = e^{2i\varphi}$. Thus, the equation of the line through z_0 and $z_1 = z_0 + e^{i\varphi}$, shall be as (8).

1.6. Remark. According to theorem 1.1 the equation of a line which passes through the origin and the point $z_0 \neq 0$ is the following

$$z = \eta\bar{z}, \ \eta = \frac{z_0}{\overline{z_0}} = e^{2i\varphi} \tag{9}$$

The line (9) passes through points whose affixes are the square roots of the complex gradient η.

Indeed, by substituting one of the two values of the square root of η into (9) we obtain the following

$$\eta\overline{\sqrt{\eta}} = \sqrt{\eta}^2\overline{\sqrt{\eta}} = \sqrt{\eta} \cdot \left|\sqrt{\eta}\right|^2 = \sqrt{\eta},$$

i.e. the points with affixes $\sqrt{\eta}$ satisfy the equation (9).

1.7. Theorem. The oriented angle φ between the lines (p) and (q) with complex gradients $\eta_1 = -e^{2i\varphi_1}$ and $\eta_2 = -e^{2i\varphi_2}$, respectively, is given by the formula $e^{2i\varphi} = \frac{\eta_1}{\eta_2}$.

Proof. Let (p') and (q') be perpendicular to (p) and (q), respectively. Then, according to theorem 1.1 the lines (p') and (q') and the positive part of the real axis create oriented angles φ_1 and φ_2, respectively. Thus, the oriented angle of the above lines is $\varphi = \varphi_2 - \varphi_1$ and it is congruent to the angle of (p) and (q) (as angles with perpendicular rays). Thus, the statement given in the theorem is implicated by the relation

$$\frac{\eta_1}{\eta_2} = \frac{-e^{2i\varphi_1}}{-e^{2i\varphi_2}} = e^{2i(\varphi_2 - \varphi_1)} = e^{2i\varphi}. \ \blacksquare$$

1.8. The equality $\varphi = 0$ is equivalent to the equality $\eta_1 = \eta_2$, and $\varphi = \frac{\pi}{2}$ is equivalent to $\eta_1 = -\eta_2$. Hence, the following corollary holds true.

Corollary A. a) Two lines are parallel if and only if their complex gradients are equal.

b) Two lines are perpendicular if and only if their complex gradients are opposite numbers. ■

Corollary B. a) Let the points $M_i,\ i=1,2,3,4$ have affixes $z_i,\ \ i=1,2,3,4$. The lines M_1M_2 and M_3M_4 are perpendicular to each other if and only if $\frac{z_1-z_2}{z_3-z_4}\in i\mathbf{R}^*$, where $\mathbf{R}^*=\mathbf{R}\setminus\{0\}$.

b) The lines M_1M_2 and M_3M_4 are perpendicular to each other if and only if $\frac{z_1-z_2}{z_3-z_4}\in i\mathbf{R}^*$.

Proof. a) The complex gradients of the lines M_1M_2 and M_3M_4 are $\frac{z_1-z_2}{\overline{z_1}-\overline{z_2}}$ and $\frac{z_3-z_4}{\overline{z_3}-\overline{z_4}}$, respectively. Since Corollary A, the lines M_1M_2 and M_3M_4 are perpendicular to each other if and only if $\frac{z_1-z_2}{\overline{z_1}-\overline{z_2}}=-\frac{z_3-z_4}{\overline{z_3}-\overline{z_4}}$, i.e. if and only if $\frac{z_1-z_2}{z_3-z_4}=-\overline{\frac{z_1-z_2}{z_3-z_4}}$, that is if and only if $\frac{z_1-z_2}{z_3-z_4}\in i\mathbf{R}^*$.

b) The proof is a direct implication of the statement a). ■

Corollary C. a) The equation of a line (p') through a point *M* with affix *m* and is such that it is parallel to (p): $z=\eta\overline{z}+a$ is the following $z-m=\eta(\overline{z}-\overline{m})$.

b) The equation of a line (p') through a point *M* with affix *m* and is such that it is perpendicular to (p): $z=\eta\overline{z}+a$ is the following $z-m=-\eta(\overline{z}-\overline{m})$.

Proof. The proof is a direct implication of Remark 1.5 and Corollary A. ■

1.9. Example. Let *A* and *B* be two distinct points in the complex plane with affixes z_1 and z_2, respectively. Determine the affix p' of the point P', symmetric to *P* with affix *p*, with respect to the line *AB*.

Solution. Through the point *P* we draw the line *l*, perpendicular to the line *AB* and we find P_1, the point of intersection () of *l* and *AB*. Thus, $P_1(p_1)$ is the midpoint of the line segment PP', i.e. $p_1=\frac{p+p'}{2}$ that is $p'=2p_1-p$, and the point P_1 is the projection of *P* onto the line *AB*.

The equation of the line through the points *A* and *B* is the following

$$z-z_1=\frac{z_2-z_1}{\overline{z_2}-\overline{z_1}}(\overline{z}-\overline{z_1}). \tag{10}$$

The complex gradient of the line *l* is $\eta_1=-\frac{z_2-z_1}{\overline{z_2}-\overline{z_1}}$. Thus, its equation is

$$z-p=-\frac{z_2-z_1}{\overline{z_2}-\overline{z_1}}(\overline{z}-\overline{p}). \tag{11}$$

If we add the last two equations (10) and (11) we obtain the affix p_1 of the point P_1:

$$p_1=\frac{(\overline{p}-\overline{z_1})(z_2-z_1)+(\overline{z_2}-\overline{z_1})(p+z_1)}{2(\overline{z_2}-\overline{z_1})}. \tag{12}$$

By substituting the so obtained expression for p_1 in $p' = 2p_1 - p$, we find the affix p' of the point P':

$$p' = \frac{\overline{p}(z_2 - z_1) + \overline{z_2} z_1 - z_2 \overline{z_1}}{\overline{z_2} - \overline{z_1}} . \blacksquare$$

1.10. Example. Determine the locus of the points which are equidistant from two given points *A* and *B*.

Solution. Let *a* and *b* be the affixes of the points *A* and *B*, respectively, and let the point *M* with affix *z* belong to the required locus. Then $|z-a|^2 = |z-b|^2$ thereby $\overline{MA} = \overline{MB}$. The last equation is equivalent to

$$z - \frac{a+b}{2} = -\frac{b-a}{\overline{b}-\overline{a}}\left(\overline{z} - \frac{\overline{a}+\overline{b}}{2}\right).$$

Thus, the required locus is a line which passes through the midpoint of the line segment *AB* and is perpendicular to *AB*. ■

1.11. Example. Let *ABC* be a given triangle and let *K* and *H* be such points on the sides *AB* and *AC*, that $\overline{AK} = \frac{1}{p}\overline{AB}$ and $\overline{AH} = \frac{1}{p+1}\overline{AC}$, respectively. Prove the following statement: for each *p*, $p > 0$ the lines *KH* passthrough a unique point.

Solution. Let 0, *b*, *c*, *k*, *h* be the affixes of *A*, *B*, *C*, *K*, *H*, respectively. Then $k = \frac{b}{p}$, $h = \frac{c}{p+1}$. Since Corollary 1.4, the point *M* with affix *z* lies on the line *KH* if and only if $\frac{z-k}{h-k} = t \in \mathbf{R}$. Hence,

$$z = \frac{1}{p}\left(b + \frac{t}{p+1}\big((c-b)p - b\big)\right).$$

If $t = p+1$, then we get that $z = c-b$. Therefore, each line *KH* consists of the point *X* with affix $c-b$. ■

2. DISTANCE FROM A POINT TO A LINE

2.1. Lemma. The line equation

$$z - z_0 = \frac{z_1 - z_0}{\overline{z_1} - \overline{z_0}}(\overline{z} - \overline{z_0})$$

can be written as

$$Az + B\overline{z} + C = 0, \text{ where } C \in \mathbf{R} \text{ and } B = \overline{A} \neq 0. \qquad (1)$$

Conversely, each equation as (1) is a line equation.

Proof. Let be given the equation

$$z - z_0 = \frac{z_1 - z_0}{\overline{z_1} - \overline{z_0}}(\overline{z} - \overline{z_0}) .$$

So,

$$z(\overline{z_1}-\overline{z_0})-\overline{z}(z_1-z_0)+\overline{z_0}z_1-z_0\overline{z_1}=0\,.$$

If we multiply the latter by i, we get that

$$i(\overline{z_1}-\overline{z_0})z-i(z_1-z_0)\overline{z}+i(\overline{z_0}z_1-z_0\overline{z_1})=0\,.$$

Let

$$A=i(\overline{z_1}-\overline{z_0}),\ \ B=-i(z_1-z_0),\ \ C=i(\overline{z_0}z_1-z_0\overline{z_1}),$$

the equation of a line through the points M and N with affixes z_0 and z_1, respectively is as (1).

Conversely, let the equation (1) be given. If (1) is divided by A and and then set that $\eta=-\frac{B}{A},\ a=-\frac{C}{A}$, we obtain the equation as $z=\eta\overline{z}+a,\ |\eta|=1$. Since Theorem 1.1, the letter is a line equation. ■

2.2. Definition. The line equation (1) is called to be a *self-conjugate line equation*.

2.3. Let line (p) be given by its self-conjugated equation (1) and the point z_0. If (1) is rewritten as

$$z=-\frac{B}{A}\overline{z}-\frac{C}{A},$$

then the complex gradient of an arbitrary line perpendicular to (p) is $\eta'=\frac{B}{A}$. Thus, the equation of (q) such that it passes through the point z_0 and is perpendicular to (p) is the following

$$z-z_0=\frac{B}{A}(\overline{z}-\overline{z_0}),$$

i.e.

$$Az-B\overline{z}-z_0A+\overline{z_0}B=0\,. \tag{2}$$

By adding the equations (1) and (2), we obtain that $2Az'=Az_0-B\overline{z_0}-C$ is the intersection of (p) and (q), i.e. the projection z_0 of onto the line (p), that is

$$z'=\frac{Az_0-B\overline{z_0}-C}{2A}\,.$$

Thus,

$$z_0-z'=\frac{Az_0+B\overline{z_0}+C}{2A},$$

so, the *distance* from a point z_0 to a line (p), given by its self-conjugate equation (1) is

$$d\left(z_0,(p)\right)=\frac{|Az_0+B\overline{z_0}+C|}{|2A|}\,.$$

3. CIRCLE EQUATION

3.1. As already stated, $|z-z_0|=R$ is the equation of a circle centered at S (with affix z_0) and radius R. In this section we will discuss circles in the complex plane.

3.2. Example. Let P_1 and P_2 be arbitrary points in the complex plane with affixes z_1 and z_2, respectively. Prove that the circumcircle of the line segment P_1P_2, viewed as its diameter, has the following equation

$$|2z-z_1-z_2|=|z_1-z_2|. \tag{1}$$

Solution. Since, the radius of a circumscircle of the line segment P_1P_2, viewed as its diameter, is $R=\frac{|z_1-z_2|}{2}$ and P_0 (the midpoint of the line segment P_1P_2, with affix $z_0=\frac{z_1+z_2}{2}$) is its center, we get that the equation of the considered circle is $\left|z-\frac{z_1+z_2}{2}\right|=\frac{|z_1-z_2|}{2}$. If we multiply the last equation by 2, then we obtain the equation which is equivalent to the latter, that is we obtain the equation (1). ■

3.3. Example. Let A, B and C be three distinct points in a plane. Determine the locus of the points equidistant to points A, B and C.

Solution. Let a, b and c be the affixes of the points A, B and C, respectively. Since Example 1.10, the locus of the points equidistant to the points A and B, B and C, A and C, are the bisectors of the line segments AB, BC and CA and their equations are

$$z-\frac{a+b}{2}=-\frac{b-a}{\overline{b}-\overline{a}}\left(\overline{z}-\frac{\overline{a}+\overline{b}}{2}\right) \tag{2}$$

$$z-\frac{b+c}{2}=-\frac{b-c}{\overline{b}-\overline{c}}\left(\overline{z}-\frac{\overline{b}+\overline{c}}{2}\right) \tag{3}$$

$$z-\frac{a+c}{2}=-\frac{a-c}{\overline{a}-\overline{c}}\left(\overline{z}-\frac{\overline{a}+\overline{c}}{2}\right) \tag{4}$$

respectively. We will discuss two cases:

a) If the points A, B and C are collinear, then Corollary 1.3 implies that the bisectors of the line segments AB, BC and CA have equal complex gradients, and since Corollary 1.8 it implies that the lines are parallel. But, the fact that the points A, B and C differ from each other, implies that the midpoints of the line segments AB, BC and CA, also differ from each other. The latter means that it does not exist any point such that it satisfies the given conditions.

b) If the points A, B and C are non-collinear, then the bisectors of the line segments AB, BC and CA parwise intersect . If we subtract the equations (3) from (2), we obtain the affix o of O, the point of intersection of the bisectors of the segments AB and BC,

$$o=\frac{a\overline{a}(c-b)+b\overline{b}(a-c)+c\overline{c}(b-a)}{a\overline{b}+b\overline{c}+c\overline{a}-\overline{a}b-\overline{b}c-\overline{c}a}.$$

By direct checking we prove that the point O lies on the line CA. Therefore, the required locus is the point O with affix o. ■

3.4. Remark. In the previous example, we actually proved that through three distinct non-collinear points A, B and C it passes exactly one i.e. a unique circle which is centered at O and whose radius is $R = |a - o|$. That is we proved that for each triangle there exists a unique circumcircle, and the center of the such circle is the point of intersection of the side bisectors of the given triangle.

3.5. Example. Prove that all complex numbers so that $|z-1| = 2|z+1|$ is satisfied, lie on a same circle. Determine the centre and the radius of that circle.

Solution. For any complex number $z = x + iy$, its absolute value is given by $|z| = \sqrt{x^2 + y^2}$.

This fact applied to the given identity $|z-1| = 2|z+1|$ implies that

$$\sqrt{(x-1)^2 + y^2} = 2\sqrt{(x+1)^2 + y^2}\ .$$

After reducing, we obtain the following expression: $\left(x + \frac{5}{3}\right)^2 + y^2 = \left(\frac{4}{3}\right)^2$, which obviously is the equation of a circle centred at $\left(-\frac{5}{3}, 0\right)$ and whose radius is $\frac{4}{3}$. ■

3.6. As above stated, the equation of a circle with centre z_0 and radius R is $|z - z_0| = R$. But, it is useful also to emphasize the circle equation similar to the self-conjugate line equation. Therefore, we will show that

$$z\overline{z} + \overline{A}z + A\overline{z} + B = 0,\ \ B \in \mathbf{R},\ \ A \in \mathbf{C},\ \ |A|^2 - B > 0 \tag{5}$$

is a circle equation.

Indeed, if

$$z_0 = -A \ \text{ and } \ R^2 = z_0\overline{z_0} - B = |A|^2 - B > 0\,,$$

and we substitute in (5) we obtain the following

$$z\overline{z} - z_0\overline{z} - z\overline{z_0} + z_0\overline{z_0} = R^2\,,$$

i.e. the equation $|z - z_0| = R$, which is the equation of a circle with centre z_0 and radius R. Hence, the equation (5) is the equation of a circle with centre z_0 and radius $R = \sqrt{|A|^2 - B}$, which is called to be a *self-conjugate circle equation*.

3.7. Remark. In the above discussions , in chapter 1, we proved that the stereographic projections of a line and a circle in the extended complex plane, i.e. their Reimann interpretation, are circles which consist of or do not consist of the pole, respectively. This is one of the reasons the lines and the circles in the extended complex plane are called as circles, and the circles in the complex plane are called as true circles. In the following example we will give one more argument which enforces this terminology.

3.8. Example. (Apollonius circle). Let A and B be arbitrary points in the plane. The locus of the point M so that $\overline{MA} : \overline{MB} = k,\ (k > 0,\ k \neq 1)$ is a circle. Prove it!

Solution. We will consider the case where $k>1$. Firstly, let set a coordinate system xOy such that the x-axis coincides with the line AB, and the origin with the midpoint of the line segment AB. Hence, $A(a,0)$ and $B(-a,0)$, i.e. the affixes of the points A and B are $z_1 = a$ and $z_2 = -a$, respectively. If the point M, which belongs to the considered locus, has affix z, then the given condition implies that $k = \frac{|z-a|}{|z+a|}$, i.e.

$$z\bar{z} + a\frac{k^2+1}{k^2-1}(z+\bar{z}) + a^2 = 0. \qquad (6)$$

The constants

$$A = a\frac{k^2+1}{k^2-1},\ B = a^2$$

satisfy the condition

$$|A|^2 - B > 0.$$

Therefore, (6) is the equation of a circle with centre

$$z_0 = -a\frac{k^2+1}{k^2-1},$$

and radius

$$R = \sqrt{|A|^2 - B} = \frac{2ak}{k^2-1}.$$

The case where $0<k<1$ can be considered analogously. ■

3.9. Relationship between a line and a circle. Let $z - z_0 = \eta(\bar{z} - \overline{z_0})$ and $|z - z_1| = R$ be the equations of a given line (p) and a circle (K), respectively. Through the centre of the circle with affix we draw the line (p') perpendicular to (p). The line equation of (p') is $z - z_1 = -\eta(\bar{z} - \overline{z_1})$. If we add the equations of the lines (p) and (p') then we find the affix of the point of intersection of these two lines

$$z^* = \frac{\eta(\overline{z_1} - \overline{z_0}) + z_1 + z_0}{2}.$$

So, the distance between the centre of the circle and the line (p) is the following

$$d(z^*, z_0) = |z^* - z_0| = \frac{\left|\eta(\overline{z_1} - \overline{z_0}) + z_1 - z_0\right|}{2}.$$

The above stated implies that:

- if $\frac{\left|\eta(\overline{z_1} - \overline{z_0}) + z_1 - z_0\right|}{2} = R$, then the line (p) is tangent to the circle (K) and the point of touching has affix z^*;

- if $\frac{\left|\eta(\overline{z_1} - \overline{z_0}) + z_1 - z_0\right|}{2} < R$, then the line (p) and the circle (K) have two points of intersection;

- if $\frac{\left|\eta(\overline{z_1} - \overline{z_0}) + z_1 - z_0\right|}{2} > R$, then the line (p) and the circle (K) do not have any common points.

3.10. Example. Determine the relationship between the line (p) and the circle (K) whose equations are $z=\bar{z}+3i$ and $|z+4-2i|=3$, respectively.

Solution. Since the line equation $z=\bar{z}+3i$, we get that $z_0=\frac{3i}{2}$. Further, since the circle equation

$$|z+4-2i|=3,$$

we obtain that $z_1=-4+2i$ and $R=3$. Therefore,

$$d=\frac{|\eta(\bar{z}_1-\bar{z}_0)+z_1-z_0|}{2}=\frac{\left|\left(-4-2i+\frac{3i}{2}\right)+\left(-4+2i-\frac{3i}{2}\right)\right|}{2}=\frac{|-8|}{2}=4>3=R,$$

and thereby 3.9, the line (p) and the circle (K) do not have any common points i.e. they do not have any points of inetrsection.■

3.11. Example. Let (K): $|z-z_0|=R$ be a given circle and z_1 be a point which is placed on the circle. Find the equation of the tangent to the circle (K) at the point z_1.

Solution. The equation of a line (p), such that it passes through points z_0 and z_1, is the following

$$z-z_0=\frac{z_1-z_0}{\bar{z}_1-\bar{z}_0}(\bar{z}-\bar{z}_0).$$

So, the equation of the tangent (p') to (K) at the point z_1 is the following

$$z-z_1=-\frac{z_1-z_0}{\bar{z}_1-\bar{z}_0}(\bar{z}-\bar{z}_1). \blacksquare$$

3.12. Remark. a) If (K): $|z|=1$ is the unit circle and z_1 is a point on (K), then the equation of the tangent to (K) at the point z_1 is the following $z+z_1^2\bar{z}=2z_1$.

b) If A, B, C and D with affixes a, b, c and d, respectively, are on the unit circle (K): $|z|=1$, then $\bar{a}=a^{-1}$, $\bar{b}=b^{-1}$, $\bar{c}=c^{-1}$ and $\bar{d}=d^{-1}$. Thereby Corollary 1.8, the chords AB and CD are parallel if and only if

$$(b-a)(\bar{d}-\bar{c})=(\bar{b}-\bar{a})(d-c)$$

holds true, that is if and only if $ab=cd$. Analogously, the chords AB and CD are perpendicular to each other if and only if $ab+cd=0$.

Apparently, if A and B with affixes a and b, respectively are points on the unit circle, then $\bar{a}=a^{-1}$, $\bar{b}=b^{-1}$, and therefore

$$\frac{a-b}{\bar{a}-\bar{b}}=\frac{a-b}{a^{-1}-b^{-1}}=-ab.$$

Further, if M with affix m is a point on the chord AB, then by Corollary 1.3 it is true that

$$\frac{m-a}{b-a}=\frac{\bar{m}-\bar{a}}{\bar{b}-\bar{a}}=\frac{\bar{m}-\bar{a}}{a-b}ab=-\frac{\bar{m}ab-b}{b-a},$$

and by equivalent transformations, we express $\bar{m}$ as the following $\bar{m}=\frac{a+b-m}{ab}$.

c) Let A, B, C and D with affixes a, b, c and d, respectively, be points on the unit circle (K): $|z|=1$, and $AB\cap CD=\{S\}$. The equations of the lines AB and CD are

$z + ab\bar{z} = a + b$ and $z + cd\bar{z} = c + d$, respectively. By eliminating $\bar{z}$ from the last two equations, we obtain that the affix of S is $s = \frac{(a+b)cd-(c+d)ab}{cd-ab}$.

d) Let A and B, with affixes a and b, respectively, be points on the unit circle, so that the line segment AB is not its diameter. According to the statement a), the equations of tangents (t_A) and (t_B) are $z + a^2\bar{z} = 2a$ and $z + b^2\bar{z} = 2b$, respectively. If we eliminate $\bar{z}$ from the last two equations, we get that $s = \frac{2ab}{a+b}$, is the affix of the intersection point $S = (t_A) \cap (t_B)$.

e) Let the line (p) meet the unit circle at two points A and B, whose affixes are a and b, respectively, and let M, whose affix is m, be an arbitrary point on the plane. It is easy to prove that the affix of the orthogonal projection of M onto the line (p), is given by $c = \frac{a+b+m-ab\bar{m}}{2}$.

4. DIRECT SIMILARITIES

4.1. Definition. The mapping $S : \mathbf{C} \to \mathbf{C}$ defined by

$$w = S(z) = az + b,\ a,b \in \mathbf{C},\ a \neq 0 \tag{1}$$

is called to be a *direct similarity*.

4.2. Theorem. The set of the direct similarities $\boldsymbol{DS}$ under the operation composition of mappings is a noncommutative group.

Proof. If $S_1, S_2 \in \boldsymbol{DS}$, then

$$S_1(z) = az + b,\ a,b \in \mathbf{C},\ a \neq 0 \text{ and } S_2(z) = cz + d,\ c,d \in \mathbf{C},\ d \neq 0.$$

Hence,

$$S_1(S_2(z)) = S_1(cz + d) = a(cz + d) + b = (ac)z + (ad + b),\ ac, ad + b \in \mathbf{C},\ ac \neq 0$$

i.e. $S_1 \circ S_2 \in \boldsymbol{DS}$. Thus, the set $\boldsymbol{DS}$ is closed with respect to the composition of mappings and in general the following holds true

$$S_1(S_2(z)) = (ac)z + (ad + b) \neq (ac)z + (bc + d) = S_2(S_1(z)).$$

Let $S_1, S_2, S_3 \in \boldsymbol{DS}$. With direct checking the following could be proved

$$S_1 \circ (S_2 \circ S_3)(z) = (S_1 \circ S_2) \circ S_3)(z), \text{ for each } z \in \mathbf{C}.$$

Therefore, $S_1 \circ (S_2 \circ S_3) = (S_1 \circ S_2) \circ S_3$, i.e. the associative law holds true.

The mapping $E(z) = z$, for each $z \in \mathbf{C}$ is an element of $\boldsymbol{DS}$ and further $E \circ S = S \circ E = S$, for each $S \in \boldsymbol{DS}$.

Let $S(z) = az + b,\ a,b \in \mathbf{C},\ a \neq 0$ be an arbitrary direct similarity. The mapping defined by $S_1(z) = \frac{1}{a}z - \frac{b}{a}$ is a direct similarity and furthermore, the following holds true

$$S(S_1(z)) = S_1(S(z)), \text{ for } z \in \mathbf{C}, \text{ i.e. } S^{-1} = S_1 \in \boldsymbol{DS}. \ \blacksquare$$

4.3. Theorem. Each direct similarity is uniquely determined by two pairs of corresponding points.

Proof. Let S be an arbitrary direct similarity so that $S(z_1)=w_1$ and $S(z_2)=w_2$. Then $S(z)=az+b$, where $a,b\in \mathbf{C}$, $a\neq 0$ are coefficients which should be obtained. Since Theorem 4.2 each direct similarity is bijection, and therefore $z_1\neq z_2$ implies that $w_1\neq w_1$. By substituting in $S(z)=az+b$ we get the following system of linear equations of the variables a and b

$$\begin{cases} w_1=az_1+b \\ w_2=az_2+b \end{cases} \qquad (2)$$

By solving the system (2) with respect the variables a and b, we obtain

$$a=\frac{w_1-w_2}{z_1-z_2}\,,\ b=\frac{z_1w_2-z_2w_1}{z_1-z_2}$$

and $a\neq 0$, i.e. the coefficients a and b of the direct similarity $S(z)=az+b$ are completely determined by two pairs of corresponding points $(z_1,S(z_1))$ and $(z_2,S(z_2))$. ■

4.4. Theorem. a) The image of a line (p) under a direct similarity is a line (p').

b) Two parallel lines under direct similarity map to parallel lines.

c) Two perpendicular lines under direct similarity map to perpendicular lines.

Proof. a) Let be given the direct similarity (1) and a equation of (p) as $z=\eta\overline{z}+c$. According to (1), we get that $z=\frac{w-b}{a}$ and if we substitute in the line equation we get

$$w=\left(\frac{a}{\overline{a}}\eta\right)\overline{w}+ac+b-\frac{a\overline{b}}{\overline{a}}\eta\,.$$

Further, $\left|\frac{a}{\overline{a}}\eta\right|=1$ implies that an image of line (p) under a direct similarity is the line (p') with complex gradient $\frac{a}{\overline{a}}\eta$.

The proofs of the statements b) and c) are direct implications of the statement a) and Corallary 1.8 A. ■

4.5. Theorem. The image of a circle (K) under direct similarity is a circle (K').

Proof. Let be the given direct similarity (1) and a circle (K) with equation $|z-c|=R$. According to (1), we get that $z=\frac{w-b}{a}$ and if we substitute in the circle equation we obtain $|w-(ac+b)|=|a|R$. The latter actually means that the image of the circle (K) under the given similarity (1) is the circle (K'). The centre of the image circle has an affix $ac+b$, and the length of its radius is $|a|R$. ■

4.6. Theorem. If A, B are arbitrary distinct points, A', B' are their images under the direct similarity (1), respectively, and if $a=re^{i\varphi}$, then $\overline{A'B'}=r\overline{AB}$, and the lines AB and $A'B'$ form an oriented angle φ.

Proof. Let z_1, z_2, w_1, w_2 be the affixes of the points A, B, A', B', respectively. Hence, $z_2 - z_1 = \overline{AB}e^{i\alpha}$ and $w_2 - w_1 = \overline{A'B'}e^{i\alpha_1}$, where α and α_1 are the angles formed by the real axis and the vectors $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$, respectively. By the equalities $w_1 = az_1 + b$ and $w_2 = az_2 + b$, we get the following equality

$$w_2 - w_1 = a(z_1 - z_2),$$

i.e. the equality

$$\overline{A'B'}e^{i\alpha_1} = r\overline{AB}e^{i(\alpha+\varphi)},$$

which implies that $\overline{A'B'} = r\overline{AB}$ and $\varphi = \alpha_1 - \alpha$. ■

The real number r is called to be the *ratio of the direct similarity* (stretching factor or direct similarity coefficient) (1), and the angle φ is called to be an *angle of the direct similarity* (1).

4.7. Definition. Two figures are said to be *directly similar* if there exists a direct similarity under which one of the figures is mapped to the other one.

4.8. Corollary. If ABC and $A'B'C'$ are directly similar triangles, then $\overline{A'B'} : \overline{A'C'} = \overline{AB} : \overline{AC}$ and $\angle A'B'C' = \angle ABC$.

Proof. The proof is directly implicated by Theorem 4.6.. ■

4.9. Theorem. Let $z_1, z_2, z_3,\ w_1, w_2, w_3,$ be the affixes of $A, B, C,\ A', B', C'$ respectively. The triangles ABC and $A'B'C'$ are directly similar if and only if

$$z_1(w_2 - w_3) + z_2(w_3 - w_1) + z_3(w_1 - w_2) = 0 \tag{3}$$

holds.

Proof. The triangles ABC and $A'B'C'$ are directly similar if and only if there exists a direct similarity (1) so that $w_i = az_i + b$, for $i = 1,2,3$. The last equalities imply the following

$$w_1 - w_2 = a(z_1 - z_2) \text{ and } w_1 - w_3 = a(z_1 - z_3).$$

If we divide the first equality by the other one, we actually obtain the following equality

$$\frac{w_1 - w_2}{w_1 - w_3} = \frac{z_1 - z_2}{z_1 - z_3}. \tag{3'}$$

The latter is equivalent to (3). ■

4.10. Remark. The condition (3), i.e. the condition (3') of the previous Theorem is equivalent to the following condition

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

Indeed, the condition (3') and the properties of the determinants imply

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} z_1 - z_2 & z_1 - z_3 \\ w_1 - w_2 & w_1 - w_3 \end{vmatrix}$$

$$= (z_1 - z_2)(w_1 - w_3) - (z_1 - z_3)(w_1 - w_2) = 0.$$

4.11. Definition. A point z is said to be a *fixed point* under the direct similarity (1) if it satisfies the condition $z = az + b$.

Apparently, for $a \neq 1$, the direct similarity (1) has a unique fixed point and its affix is $z_1 = \frac{b}{1-a}$. If $a = 1$, then $b = 0$, i.e. the direct similarity (1) is the identity mapping and each point of the complex plane is a fixed point.

The point C with affix $c = \frac{b}{1-a}$ is said to be *center of the direct similarity* $S(z) = az + b$.

4.12. Let (p): $z - z_0 = \eta(\overline{z} - \overline{z_0})$ be a tangent to the circle (K): $|z - z_1| = R$ and let $w = S(z) = az + b,\ a, b \in \mathbf{C},\ a \neq 0$ be a direct similarity. Since 3.9 ,

$$\frac{\left|\eta(\overline{z_1} - \overline{z_0}) + z_1 - z_0\right|}{2} = R \tag{4}$$

holds. Further, according to Theorem 4.5 the image of the circle (K) is a circle (K') with equation $|w - (az_1 + b)| = |a| R$. Analogously to the proof of Theorem 4.4 a) if we substitute that $z = \frac{w-b}{a}$ in the equation of (p) and we get that the image of a line (p) is a line (p') with the following equation

$$w - (az_0 + b) = \frac{\eta a}{\overline{a}}(\overline{w} - \overline{az_0 + b}).$$

By using the equation (4) we find that the circle (K') and the line (p') satisfy the following

$$\frac{\left|\frac{\eta a}{\overline{a}}(\overline{az_1 + b} - \overline{az_0 + b}) + az_1 + b - (az_0 + b)\right|}{2} = \frac{\left|\frac{\eta a}{\overline{a}}\overline{a}(\overline{z_1} - \overline{z_0}) + a(z_1 - z_0)\right|}{2}$$

$$= |a| \frac{\left|\eta(\overline{z_1} - \overline{z_0}) + (z_1 - z_0)\right|}{2} = |a| R$$

Hence, 3.9 implies that the line (p') is tangent to the circle (K').

Thus, we proved the following theorem.

Theorem. Let (p) be a tangent line to the circle (K) and (p') and (K') be their images under the direct similarity (1), respectively. Then (p') is a tangent line to the circle (K'). ■

4.13. Example. Let *ABCD* be a given parallelogram. On its sides *CD* and *CB*, similar and same oriented triangles (directly similar) *CDE* and *FBC*, are constructed.

Prove that the triangle *FAE* is similar and is also the same oriented to the triangles *CDE* and *FBC*.

Solution. Let the origin be the point of intersection of the parallelogram diagonals. Then $c=-a$ and $d=-b$. The triangles *CDE* and *FBC* are similar and a same oriented. Thus, Theorem 4.9 implies that $\frac{c-b}{b-f}=\frac{e-d}{d-c}$. Therefore,

$$f=\frac{be+c^2-bc-cd}{e-d}=\frac{be+a^2}{e+b}.$$

Hence,

$$f-a=\frac{(b-a)(e-a)}{e+b},\ c-d=c+b,\ d-e=-(b+e) \text{ and } b-a=c-d,$$

therefore

$$\frac{f-a}{a-e}=\frac{\frac{(b-a)(e-a)}{e+b}}{a-e}=\frac{b-a}{-(e+b)}=\frac{c-d}{d-e},$$

The latter according to Theorem 4.9, means that the triangles *FAE* and *CDE* are directly similar.■

Example 4.14. On the sides *AB*, *BC* and *CA* of a triangle *ABC*, such pairwise similar triangles *ABK*, *BCL* and *ACM* are constructed, that the first two are out of the triangle *ABC* and the third one is in (see figure 1). Prove that the quadrilateral *BLMK* is parallelogram.

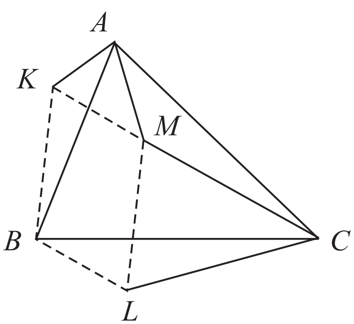

Figure 1

Solution. The triangles *AKB* and *BLC* are directly similar. Therefore,

$$\frac{k-a}{b-a}=\frac{l-b}{c-b},\ \text{i.e. } l=b+(k-a)\frac{c-b}{b-a}.$$

The triangles *AKB* and *AMC* are directly similar, and therefore $\frac{k-a}{b-a}=\frac{m-a}{c-a}$, i.e.

$$m=a+(k-a)\frac{c-a}{b-a}.$$

Thus,

$$\overrightarrow{BL}=l-b=(k-a)\frac{c-b}{b-a}$$

and

$$\overrightarrow{KM}=m-k=a+(k-a)\frac{c-a}{b-a}-k=(k-a)\left(\frac{c-a}{b-a}-1\right)=(k-a)\frac{c-b}{b-a},$$

i.e. $\overrightarrow{BL}=\overrightarrow{KM}$. The latter means that the quadrilateral *BLMK* is parallelogram. ■

5. MOTIONS

5.1. In the previous section we considered and discussed the direct similarities and we also proved several properties about them. In this section the focus of our interest is one of the most important classes of direct similaritiesand their classification.

5.2. Definition. The direct similarity $S(z)=az+b$, for $|a|=1$, for $|a|=1$ is said to be *motion*.

5.3. Theorem. The set of motions $\boldsymbol{D}$ under the operation composition of mappings is a subgroup of the direct similarity group $\boldsymbol{DS}$.

Proof. If $S_1,S_2 \in \boldsymbol{D}$, then

$$S_1(z)=az+b,\ \ S_2(z)=cz+d,\ |a|=|d|=1.$$

Thus,

$$S_1\left(S_2(z)\right)=S_1(cz+d)=a(cz+d)+b=(ac)z+(ad+b),\ |ac|=1,$$

therefore, $S_1 \circ S_2 \in \boldsymbol{D}$. The latter means that the set $\boldsymbol{D}$ is closed under the composition of mappings.

If $S_1,S_2,S_3 \in \boldsymbol{D}$, then $S_1,S_2,S_3 \in \boldsymbol{DS}$, therefore

$$S_1 \circ (S_2 \circ S_3)=(S_1 \circ S_2)\circ S_3,$$

i.e. the associative law holds true.

If $a=1,\ b=0$ then $1\cdot z+0=E(z)\in \boldsymbol{D}$.

Let $S(z)=az+b,\ |a|=1$ be an arbitrary motion. The mapping

$$S_1(z)=\tfrac{1}{a}z-\tfrac{b}{a},\ \left|\tfrac{1}{a}\right|=\tfrac{1}{|a|}=1$$

is motion and moreover the following holds true

$$S\left(S_1(z)\right)=S_1\left(S(z)\right)=z,\text{ for each } z\in \mathbf{C},\text{ i.e. } S^{-1}=S_1\in \boldsymbol{D}.\ ■$$

5.4. Definition. The motion $S(z)=z+b$ is said to be *translation* for the vector b, and it is denoted by S_b.

5.5. Theorem. The translation which is not identity mapping has no fix points.

Proof. The proof is directly implicated by 4.10. ■

5.6. Theorem. The set of translations $\boldsymbol{T}$ under the composition of mappings is a commutative subgroup of the group of motions $\boldsymbol{D}$.

Proof. If $S_1,S_2 \in \boldsymbol{T}$, then $S_1(z)=z+b,\ S_2(z)=z+d$. Thus,

$$S_1\left(S_2(z)\right)=S_1(z+d)=(z+d)+b=z+(d+b),$$

therefore, $S_1 \circ S_2 \in \boldsymbol{T}$. The latter means that the set $\boldsymbol{T}$ is closed under the composition of mappings.

If $S_1,S_2,S_3 \in \boldsymbol{T}$, then $S_1,S_2,S_3 \in \boldsymbol{D}$, Hence,

$$S_1 \circ (S_2 \circ S_3) = (S_1 \circ S_2) \circ S_3 ,$$

i.e. the associative law holds true.

Let $S_1,S_2 \in \boldsymbol{T}$, then $S_1(z) = z + b,\ S_2(z) = z + d$. Hence,

$$S_1\left(S_2(z)\right) = S_1(z+d) = (z+d)+b = (z+b)+d$$
$$= S_2(z+b) = S_2\left(S_1(z)\right),$$

for each $z \in \mathbf{C}$, i.e. the commutative law holds true.

If $b = 0$ then $1 \cdot z + 0 = E(z) \in \boldsymbol{T}$.

Let $S(z) = z + b$ be translation. The mapping $S_1(z) = z - b$ is also translation and furthermore the following holds true

$$S\left(S_1(z)\right) = S_1\left(S(z)\right) = z \text{, for each } z \in \mathbf{C} \text{, i.e. } S^{-1} = S_1 \in \boldsymbol{D} \text{. } \blacksquare$$

5.7. Definition. The direct similarity with ratio 1 and angle π is said to be a *point reflection.*

The shape **F** is said to be a point reflection iff there exists such a reflection *S*, that $S(\mathbf{F}) = \mathbf{F}$.

Thus, $S(z) = az + b,\ a,b \in \mathbf{C},\ a \neq 0$ is a point reflection if $a = -1$. Due to the last, $S(z) = b - z$ denotes a point reflection.

Moreover 4.10, implies that the point reflection $S(z) = b - z$ has center *C* with affix $c = \frac{b}{2}$ and is denoted by $S = S_C$. In our further considerations the set of point reflections will be denoted by ***CS***.

Let *A*(*a*) be any arbitrary point in a plane. The image of that point under the point reflection $S(z) = b - z$ is the point $A'(b-a)$. Since the centre *C* of the point reflection has an affix $c = \frac{b}{2}$, we get that

$$\overrightarrow{AC} = \tfrac{b}{2} - a = (b-a) - \tfrac{b}{2} = \overrightarrow{CA'} .$$

So, the centre *C* of the point reflection is the midpoint of the line segment AA' .

5.8. Theorem. a) The composition of two point reflections is translation.

b) The composition of a point reflection and a translation is point reflection.

Proof. a) Let $S_1(z) = b - z$ and $S_2(z) = d - z$, $b,d \in \mathbf{C}$ be arbitrary point reflections. Then,

$$S_1\left(S_2(z)\right) = S_1(d-z) = b - (d-z) = z + (b-d) .$$

So, the composition $S_1 \circ S_2$ is translation for the vector $b - d$.

б) Let $S_1(z) = b - z$ and $S_2(z) = z + d$, $b,d \in \mathbf{C}$ be arbitrary point reflection and translation, respectively. Therefore,

$$S_1\left(S_2(z)\right) = S_1(z+d) = b - (z+d) = b - d - z$$

and

$$S_2\left(S_1(z)\right) = S_2(b-z) = d + (b-z) = b + d - z ,$$

i.e. $S_1 \circ S_2$ and $S_2 \circ S_1$ are point reflections with centers $\frac{b-d}{2}$ and $\frac{b+d}{2}$, respectively. ■

5.9. Definition. The mapping $S:\mathbf{C}\to\mathbf{C}$ is said to be *involuntary* iff the mapping is invertible, i.e. there exists such S^{-1} that $S^{-1}=S$.

5.10. Theorem. The direct similarity which is not identity is involuntary if and only if it is a point reflection.

Proof. Theorem 4.2 implies that the direct similarity is involuntary if and only if

$$az+b=\tfrac{z-b}{a}\text{, for each } z\in\mathbf{C},$$

i.e. if and only if $a=\frac{1}{a}$ and $b=-\frac{b}{a}$. The last two equalities are satisfied if and only if $a=-1$. Therefore, S is involuntary if and only if S is a point reflection. ■

5.11. Corollary. The set $\boldsymbol{T}\cup\boldsymbol{CS}$ under the operation composition of mappings is a non-commutative subgroup of the group of motions $\boldsymbol{D}$.

Proof. The proof is a directly implicated by Theorem 5.6 and Theorem 5.8. ■

5.12. Definition. The motion which is not translation is said to be *rotation*.

Thereby each rotation $S(z)=az+b,\ |a|=1$ satisfies that $a\neq 1$, we deduce that each rotation has center C with affix $c=\frac{b}{1-a}$. If C and α are the center of the rotation and the angle of the rotation, respectively,we use to say that we have a rotation about C with angle α and we use to write $S=S_{C,\alpha}$. In our further discussion the set of the rotations will be denoted by $\boldsymbol{R}$. Apparently, the point reflections are rotations with angle π, and therefore $\boldsymbol{CS}\subset\boldsymbol{R}$.

Let $S(z)=az+b,\ |a|=1,\ a\neq 1$ be rotation about C with angle α. Then, the inverse mapping S^{-1} defined by $S^{-1}(z)=\bar{a}z-\bar{a}b$ is rotation about C with angle $-\alpha$.

5.13. Theorem. a) The composition of two rotations is either rotation or translation.

b) The composition of rotation and translation is rotation.

Proof. a) Let

$$S_1(z)=az+b,\ |a|=1,\ a\neq 1 \text{ and } S_2(z)=cz+d,\ |c|=1,\ c\neq 1$$

be two rotations. So,

$$S_1\left(S_2(z)\right)=S_1(cz+d)=a(cz+d)+b=(ac)z+(ad+b)\,. \qquad (1)$$

Apparently, if $ac=1$, then $S_1\circ S_2$ is translation; if $ac\neq 1$, then $S_1\circ S_2$ is rotation about C, with affix $\frac{ad+b}{1-ac}$, and with angle $\alpha_1+\alpha_2$, where α_1 and α_2 are the angles of the rotations S_1 and S_2, respectively.

b) Let

$$S_1(z)=z+b \text{ and } S_2(z)=cz+d,\ |c|=1,\ c\neq 1$$

be an arbitrary translation and a rotation, respectively.

Since,

$$S_1\left(S_2(z)\right) = S_1(cz+d) = cz + (d+b)$$

it follows that $S_1 \circ S_2$ is rotation about C with affix $\frac{d+b}{1-c}$ and with angle α_2. Further,

$$S_2\left(S_1(z)\right) = S_2(z+b) = cz + (d+bc)$$

implies that $S_2 \circ S_1$ is rotation about C' with affix $\frac{d+bc}{1-c}$ and with angle α_2. ■

5.14. Consider the rotations

$$S_1(z) = az+b,\ |a|=1,\ a \neq 1 \text{ and } S_2(z) = cz+d,\ |c|=1,\ c \neq 1.$$

While proving Theorem 5.13 we established that the composition $S_1 \circ S_2$ is either rotation or translation, depending on whether $ac \neq 1$ or $ac = 1$, respectively. Apparently,

$$S_2\left(S_1(z)\right) = S_2(az+b) = c(az+b)+d = (ac)z + (bc+d), \tag{2}$$

implies that $S_2 \circ S_1$ is either rotation or translation, too, depending on whether $ac \neq 1$ or $ac = 1$, respectively. Logically the following question arises: whether and under which conditions S_1 and S_2 commute under to the composition of mappings, i.e. when does the following relation hold true

$$S_2\left(S_1(z)\right) = S_1\left(S_2(z)\right), \text{ for each } z \in \mathbf{C}. \tag{3}$$

If we substitute (1) and (2) into (3), after reducing, we obtain that S_1 and S_2 commute if and only if $ad+b = bc+d$. The latter actually means that S_1 and S_2 commute if and only if $\frac{b}{1-a} = \frac{d}{1-c}$. So, we proved the following theorem.

Theorem. Two rotations commute if and only if their centers of rotations coincide. ■

5.15. Consider the rotations about a common centre. Let

$$S_1(z) = az+b,\ |a|=1,\ a \neq 1 \text{ and } S_2(z) = cz+d,\ |c|=1,\ c \neq 1,$$

be so that $\frac{b}{1-a} = \frac{d}{1-c}$ holds. Then,

$$S_2\left(S_1(z)\right) = (ac)z + (bc+d),\ a \neq 1,\ c \neq 1,\ |a|=1 \text{ and } |c|=1.$$

Obviously, $|ac| = 1$. If $ac = 1$, then $|c| = 1$ implies $c\bar{c} = 1$, and therefore $a = \bar{c} = \frac{1}{c}$. By substituting in $\frac{b}{1-a} = \frac{d}{1-c}$, we obtain that $bc + d = 0$, that is $S_2\left(S_1(z)\right) = z = E(z)$. If $ac \neq 1$, then the condition $\frac{b}{1-a} = \frac{d}{1-c}$ is equivalent to $\frac{bc+d}{1-ac} = \frac{d}{1-c}$, i.e. the composition $S_2 \circ S_1$ is rotation with center which coincides with the centers of the rotations S_1 and S_2. Finally, if have on mind the fact that the identity mapping can be understood as a rotation about an arbitrary centre, then the previous considerations and also Theorem 5.14 imply the validity of the following Theorem.

Theorem. The set of rotations about a common centre under the operation composition of mappings is a commutative sub-group of the group of motions ***D***. ■

5.16. Example. Let *ABCDEF* be a regular hexagon, *K* be the midpoint of the diagonal *BD*, and *M* be the midpoint of the side *EF*. Prove that $\triangle AMK$ is an equilateral triangle.

Solution. Let the hexagon *ABCDEF* be inscribed into the unit circle (figure 2). Then $1, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}, e^{i\pi}, e^{i\frac{4\pi}{3}}, e^{i\frac{5\pi}{3}}$ are the affixes of the vertices *C*, *D*, *E*, *F*, *A*, *B*, respectively. Thus the affixes of the points *K* and *M* are

$$k=\frac{e^{i\frac{\pi}{3}}+e^{i\frac{5\pi}{3}}}{2}=\frac{1}{2} \text{ and } m=\frac{e^{i\frac{2\pi}{3}}+e^{i\pi}}{2}=-\frac{3}{4}+i\frac{\sqrt{3}}{4},$$

respectively. Further,

$$(k-a)e^{i\frac{\pi}{3}}+a=\left(\frac{1}{2}-e^{i\frac{4\pi}{3}}\right)e^{i\frac{\pi}{3}}+e^{i\frac{4\pi}{3}}=$$

$$=\frac{1}{2}e^{i\frac{\pi}{3}}-e^{i\frac{5\pi}{3}}+e^{i\frac{4\pi}{3}}=-\frac{3}{4}+i\frac{\sqrt{3}}{4}=m,$$

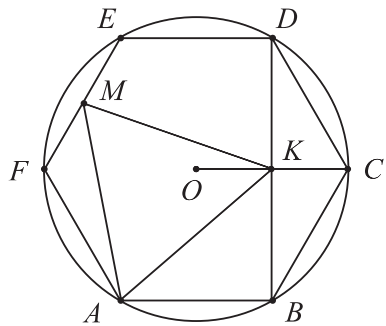

Figure 2

i.e. $(k-a)e^{i\frac{\pi}{3}}=m-a$ implies, that the side *MA* of $\triangle AMK$ is obtained when the side *AK* is rotated about the vertex *A* with angle $\frac{\pi}{3}$. So, $\triangle AMK$ is an equilateral triangle. ■

5.17. Example. Let M_k, $k=1,2,3,4$ with affixes z_k, $k=1,2,3,4$, respectively, be given distinct points in the complex plane. Prove the following statement:

The identity

$$z_2-z_1=\pm i(z_4-z_3) \tag{1}$$

holds true if and only if

$$\overline{M_1M_2}=\overline{M_3M_4} \text{ and } M_1M_2\perp M_3M_4 \tag{2}$$

Solution. Since the condition (1) we get $|z_2-z_1|=|z_4-z_3|$, that is $\overline{M_1M_2}=\overline{M_3M_4}$. Likewise,

$$z_2-z_1=\pm i(z_4-z_3)=e^{\pm i\frac{\pi}{2}}(z_4-z_3),$$

that is the number z_2-z_1 is obtained by rotation of the number z_4-z_3 about the origin *O* with angle $\pm\frac{\pi}{2}$. Thus, $M_1M_2\perp M_3M_4$. Therefore, the condition (2) is implied by the condition (2).

Conversely, since

$$\overline{M_1M_2}=|z_2-z_1|,\ \overline{M_3M_4}=|z_4-z_3| \text{ and } \overline{M_1M_2}=\overline{M_3M_4}$$

the following is satisfied $z_2-z_1=re^{it}$, $z_4-z_3=re^{is}$. Therefore

$$z_2-z_1=e^{i(t-s)}(z_4-z_3). \tag{3}$$

The second condition in (2) implies

$$t - s = \pm\frac{\pi}{2} + 2k\pi,\ k \in \mathbf{Z}.$$

By substituting in (3) we get $z_2 - z_1 = \pm i(z_4 - z_3)$. Thus, the condition (2) implies the condition (1). ■

6. HOMOTHETY

6.1. Definition. The direct similarity with angle 0 or π, which is not translation is called to be *homothety*.

Due to this, the direct similarity $S(z) = az + b,\ a, b \in \mathbf{C},\ a \neq 0$ is homothety if and only if $a \in \mathbf{R} \setminus \{0,1\}$. *Ratio of homothety* is a real (not a complex number as it was for generally directly similarities) number *a,* such that $a,\ (a \neq 0,1)$. Apparently, the point reflections are homotheties with ratio $\frac{1}{a}$. Theorem 4.2, implies that the inverse mapping of a homothety with ratio *a* is also a homothety, but the ratio is $\frac{1}{a}$. In our further considerations the set of the homotheties will be denoted by ***H***.

6.2. Theorem. a) The composition of two homotheties is either a homothety or a translation.

b) The composition of homothety and translation is homothety.

Proof. a) Let

$$S_1(z) = a_1 z + b_1,\ a_1 \in \mathbf{R} \setminus \{0,1\}$$

and

$$S_2(z) = a_2 z + b_2,\ a_2 \in \mathbf{R} \setminus \{0,1\}$$

be two homotheties. Then,

$$S_2\left(S_1(z)\right) = a_1 a_2 z + a_2 b_1 + b_2.$$

Obviously, if $a_1 a_2 = 1$, then $S_2 \circ S_1$ is translation for the vector $a_2 b_1 + b_2$, and if $a_1 a_2 \neq 1$, then $S_2 \circ S_1$ is homothety with center *C* and a ratio of homothety $a_1 a_2$. The affix of such the center is $\frac{a_2 b_1 + b_2}{1 - a_1 a_2}$.

b) Let the homothety and the translation be given by

$S_1(z) = a_1 z + b_1,\ a_1 \in \mathbf{R} \setminus \{0,1\}$ and $S_2(z) = z + b_2$, respectively.

Thereby,

$$S_2\left(S_1(z)\right) = a_1 z + b_1 + b_2,\ a_1 \in \mathbf{R} \setminus \{0,1\}$$

holds true, we get that $S_2 \circ S_1$ is homothety with center *C*, the affix of the center is $\frac{b_1 + b_2}{1 - a_1}$, and the ratio of the homothety is a_1. Since,

$$S_1\left(S_2(z)\right) = a_1 z + b_1 + a_1 b_2,\ a_1 \in \mathbf{R} \setminus \{0,1\}$$

holds true,we get that $S_1 \circ S_2$ is a homothety with center C, the affix of the center is $\frac{a_1 b_2 + b_1}{1-a_1}$ and the ratio of the homothety is a_1. ■

6.3. Corollary. The set $\boldsymbol{T} \cup \boldsymbol{H}$ is a non-commutative subgroup of the direct similarity group $\boldsymbol{DS}$ under the composition of mappings.

Proof. The proof is directly implied by Definition 6.1 and Theorems 4.2, 5.6 and 6.2. ■

6.4. Theorem. Any two homotheties and their composition, if it is not translation, have collinear centers.

Proof. a) Let

$$S_1(z) = a_1 z + b_1,\ a_1 \in \mathbf{R} \setminus \{0,1\} \text{ and } S_2(z) = a_2 z + b_2,\ a_2 \in \mathbf{R} \setminus \{0,1\}$$

be two homotheties with centers C_1 and C_2, whose affixes are $c_1 = \frac{b_1}{1-a_1}$ and $c_2 = \frac{b_2}{1-a_2}$, respectively, and also let the composition

$$S_2\left(S_1(z)\right) = a_1 a_2 z + a_2 b_1 + b_2$$

be homothety with center C, whose affix is $\frac{a_2 b_1 + b_2}{1-a_1 a_2}$. Then,

$$\frac{c_2 - c}{c_1 - c} = \frac{\frac{b_2}{1-a_2} - \frac{a_2 b_1 + b_2}{1-a_1 a_2}}{\frac{b_1}{1-a_1} - \frac{a_2 b_1 + b_2}{1-a_1 a_2}} = \frac{a_1 a_2 - a_2}{1-a_2}$$

is real number, and thereby Corollary 1.4, the points C_1, C_2 and C are collinear. ■

6.5. Theorem. The line (p) under a direct similarity $S(z) = az + b$ is mapped to a parallel line (p') if and only if the direct similarity is either homothety or translation.

Proof. If the line (p) has complex gradient η, then its image (p') under the direct similarity $S(z) = az + b$ has complex gradient $\eta\frac{a}{\bar{a}}$.The straight lines (p) and (p') are parallel if and only if $\eta\frac{a}{\bar{a}} = \eta$, i.e. if and only if $a = \bar{a}$, or in other words if and only if $a \in \mathbf{R}$. Hence, the direct similarity $S(z) = az + b$ maps the line (p) to a parallel line (p') if and only if the direct similarity is either a homothety or a translation. ■

6.6. Let $|z - c_1| = R_1$ and $|z - c_2| = R_2$ be the equations of circles (K_1) and (K_2), respectively.

If $R_1 \neq R_2$, then the mapping $S : \mathbf{C} \to \mathbf{C}$ defined by

$$w = S(z) = \frac{R_2}{R_1} z + \frac{R_1 c_2 - R_2 c_1}{R_1} \tag{1}$$

is a homothety with ratio $\frac{R_2}{R_1}$ and center $\frac{R_1 c_2 - R_2 c_1}{R_1 - R_2}$. Since (1), we obtain the following expression for z

$$z=\frac{R_1w-R_1c_2+R_2c_1}{R_2}$$

and if we substitute it into the (K_1) circle equation, we get the following equation

$$\left|\frac{R_1w-R_1c_2+R_2c_1}{R_2}-c_1\right|=R_1,$$

which in fact is equivalent to the (K_2) circle equation.

If $R_1=R_2$, the mapping (1) is translation for the vector c_2-c_1 and furthermore it maps (K_1) onto (K_2).

Analogously, the mapping $S:\mathbf{C}\to\mathbf{C}$ defined by

$$w=S(z)=-\frac{R_2}{R_1}z+\frac{R_1c_2+R_2c_1}{R_1} \tag{2}$$

is homothety with ratio $-\frac{R_2}{R_1}$ and center $\frac{R_1c_2+R_2c_1}{R_1+R_2}$, and furthermore it maps (K_1) onto (K_2).

The above statement implies the validity of the following Theorem.

Theorem. Any two circles are homotethic, i.e. there exists a homotethy which maps one of the circles to the other one. ■

6.7. Example. Let *B* and *C* be arbitrary distinct points on a given circle, such that they are not diametrically opposite and let the tangents to the given circle at these points intersect at point *A*. Let *P* be an arbitrary point on the circle. Let A_1, B_1, C_1 be the feet of the perpendiculars from *P* to the lines *BC*, *CA*, *AB*, respectively.

Hence, $\overline{PA}_1^2=\overline{PB}_1\cdot\overline{PC}_1$. Prove it!

Solution. Without loss of generality we may assume that the given circle is the unit circle, and 1 is the affix of *P* (why?).

Let *b* and *c* be the affixes of the points *B* and *C*, respectively. According to Remark 3.12 e) and d) it follows, that the affixes of the points *A* and A_1 are the following

$$a=\frac{2bc}{b+c} \text{ and } a_1=\frac{b+c+1-bc}{2},$$

respectively. In order to determine the affix b_1 of the point B_1 we will use the fact that the point B_1 is on the line *AC* and furthermore that PB_1 is perpendicular to *AC*. So,

$$\frac{b_1-c}{a-c}=\frac{\overline{b}_1-\overline{c}}{\overline{a}-\overline{c}} \text{ and } \frac{b_1-1}{a-c}=\frac{\overline{b}_1-1}{\overline{a}-\overline{c}}.$$

By substituting the expression for *a* in the last two equations, and after reducing, we get the following system:

$$\begin{cases} b_1+\overline{b}_1c^2=2c \\ b_1-\overline{b}_1c^2=1-c^2 \end{cases}$$

therefore $b_1=\frac{1+2c-c^2}{2}$. Analogously, $c_1=\frac{1+2b-b^2}{2}$. Due to this,

$$\overline{PA}_1^2 = |1-a_1|^2 = \tfrac{1}{4}|bc-b-c+1|^2 = \tfrac{1}{2}|b-1|^2 \cdot \tfrac{1}{2}|c-1|^2$$
$$= |1-b_1|\cdot|1-c_1| = \overline{PB}_1 \cdot \overline{PC}_1. \blacksquare$$

6.8. According to Theorem 5.5 and Definition 4.11, the direct similarity which is neither identity nor translation, has exactly one fixed point, and such fixed point is the center of the direct similarity. The line (p): $z = \eta\bar{z} + c$ is called to be a *fixed line* for the direct similarity S if $S(p) = p$, i.e. if the direct similarity maps (p) to itself. The circle (K): $|z-c| = R$ is called to be a *fixed circle* for the direct similarity S if $S(K) = K$.

6.9. Theorem 4.5 implies that the image of the circle (K): $|z-c| = R$ under the direct similarity $w = S(z) = az+b$ is the circle (K') : $|z-(ac+b)| = R\cdot|a|$. So, the circle (K) is fixed under the direct similarity if and only if $|a| = 1$ and $c = \frac{b}{1-a}$, i.e. if and only if S is a motion which is not a translation and the center of the such motion coincides with the center of the circle.

So, we proved the following theorem.

Theorem. a) The direct similarity has a fixed circle if and only if the direct similarity is a motion which is not translation.

b) The The only fixed circles under a motion which is not translation, are the circles centered at the center of the motion. ■

6.10. Theorem 4.4 implies that the image of the line (p): $z = \eta\bar{z} + c$ under the direct similarity $w = S(z) = az+b$ is the line (p'):
$$w = \left(\tfrac{a}{\bar{a}}\eta\right)\bar{w} + b + ac - \tfrac{a}{\bar{a}}\bar{b}\eta .$$
Therefore, the line (p) is fixed under the direct similarity S if and only if
$$\tfrac{a}{\bar{a}}\eta = \eta \text{ and } b + ac - \tfrac{a}{\bar{a}}\bar{b}\eta = c,$$
i.e. if and only if $a \in \mathbf{R}$ and
$$b + ac - \bar{b}\eta = c .$$

The already stated assertion implies that the line (p): $z = \eta\bar{z} + c$ is a fixed line under the direct similarity if and only if $a = 1$ and $b = \bar{b}\eta$ or $a \neq 1$ and $\frac{b}{2} = \eta\frac{\bar{b}}{2} + c$, i.e. if and only if either S is translation and the line (p) is parallel to the translation vector or S is a homothety and the line (p) passes through its center.

Thus, we proved the following theorem.

Theorem. A fixed line under a direct similarity exists if and only if the direct similarity is:

a) translation – the fixed line is each line parallel to the translation vector,

б) homothety – the fixed line is each line which passes through its center. ■

6.11. Remark. The proof of Theorem 6.6 implies that two concentric circles have one and only one center of similarity which coincides with the center of the circles (figure 3), and non-concentric circles have either one or two centers of similarity, depending on either their radii are congruent or not, respectively. In case when the radii of non-concentric circles are not congruent, the center of similarity of homothety (1) is said to be an *outer center of similarity*, and the center of homothety (2) is said to be an *inner center of similarity* (figure 4).

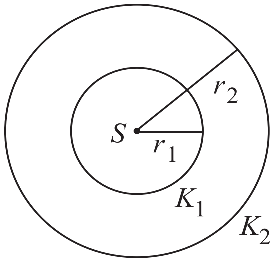

Figure 3

6.12. Remark. By Theorem 6.5, each homothety an arbitrary line maps to a parallel line. This statement enables the center of similarity of a homoethety to be constructed in case where we have non-concentric circles (K_1) and (K_2). Through the center S_1 we draw a diameter AB of the circle (K_1) and through the center S_2 we draw a radius S_2C parallel to the diameter AB (figure 4). If $R_1 \neq R_2$, then the lines AC and BC meet the line S_1S_2 at points O_1 and O_2, which are an outer and an inner center of the considered homotheties, respectively. If $R_1 = R_2$, then AC is parallel to S_1S_2, and the lines BC and S_1S_2 intersect at the inner center of similarity.

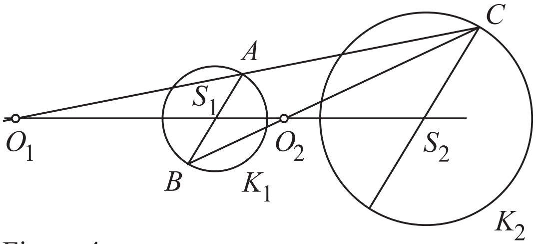

Figure 4

For the common tangent (t) to the circles (K_1) and (K_2), we get:

1. If $R_1 = R_2$, then (t) is parallel to S_1S_2, and
2. If $R_1 \neq R_2$, then (t) passes through one of the centers of similarity (why?). So, if $R_1 \neq R_2$, then in order to construct the common tangents to (K_1) and (K_2), we should firstly determine their centers of similarity and then draw the tangents to one of the circles.

6.13. Consider the circles (K_i), $i = 1,2,3$ given by the following equations

$$|z - c_i| = R_i, \ i = 1,2,3,$$

respectively, $R_i \neq R_j$, for $i \neq j$, and their centers are not collinear (figure 5). The proof of Theorem 6.6 implies, that

$$\frac{R_1c_2 - R_2c_1}{R_1 - R_2}, \ \frac{R_2c_3 - R_3c_2}{R_2 - R_3} \ \text{and} \ \frac{R_1c_3 - R_3c_1}{R_1 - R_3}$$

are the affixes of the homothety centers S_{12}, S_{23} and S_{13}, where (K_1) is mapped to (K_2), (K_2) to (K_3), and (K_1) to (K_3), respectively. Further,

$$\frac{\frac{R_1c_3-R_3c_1}{R_1-R_3}-\frac{R_1c_2-R_2c_1}{R_1-R_2}}{\frac{R_2c_3-R_3c_2}{R_2-R_3}-\frac{R_1c_2-R_2c_1}{R_1-R_2}}=\frac{R_1(R_2-R_3)}{R_2(R_1-R_3)}\in\mathbf{R}.$$

So, by corollary 1.4, the points S_{12}, S_{23} and S_{13} are collinear. Analogously, the following can be proven:

- the points S_{12}, S'_{23}, S'_{13} are collinear
- the points S'_{12}, S'_{23}, S_{13} are collinear
- the points S'_{12}, S_{23}, S'_{13} are collinear.

Hence, we proved the following theorem.

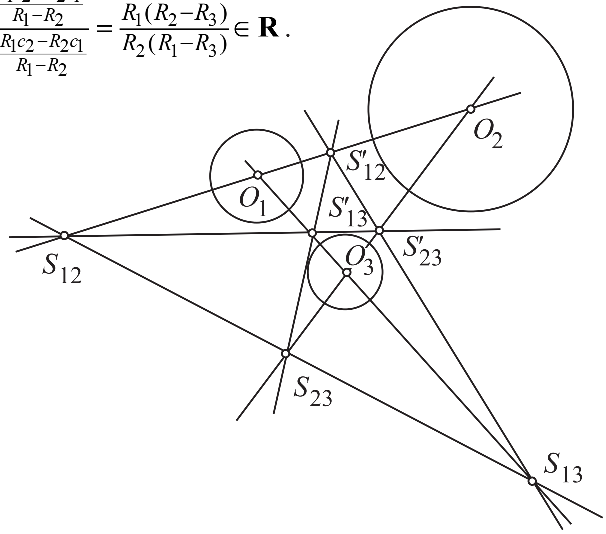

Figure 5

Theorem. If the centers of the circles (K_i), $i=1,2,3$, whose radii are not congruent, are non-collinear, then the centers of homothety $S_{12}, S_{23}, S_{13}, S'_{12}, S'_{23}, S'_{13}$ are on four lines, so that each line consists of exactly three of themy. ■

6.14. Example. Construct a circle which passes through a given point and touches two different given lines.

Solution. Let (*a*) and (*b*) be given lines and *A* be a given point. We will consider only the case where the lines (*a*) and (*b*) intersect, and the point *A* is on neither one of the lines (*a*) and (*b*). *A* is also not on the bisector of the angle formed by the lines (*a*) and (*b*) (as shown in figure 6). The other cases are left as exercises.

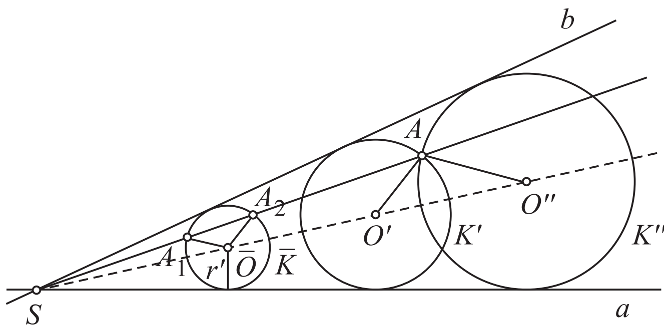

Figure 6

Let $S=(a)\cap(b)$ and let *K*(*O*,*r*) be the required circle. Since (*a*) and (*b*) are outer tangents to the circle (*K*), it is true that the center *O* of the circle is on the bisector (*s*) of the angle formed by the lines (*a*) and (*b*), (we consider the angle for which *A* is an inner point). If H is a homothety with center *S* and arbitrary ratio of homothety, then $\mathrm{H}(a)=a$, $\mathrm{H}(b)=b$ and $\mathrm{H}(K)=\overline{K}$ is a circle which touches the lines (*a*) and (*b*). So, if we want to construct the circle (*K*), we must firstly construct an arbitrary circle $(\overline{K})$ which touches the lines (a) and (b). Let A_1 and A_2 be the points where the circle $(\overline{K})$ meets the line SA. If H_1 and H_2 are homotheties with center S and ratio of homotheties $\overline{OA}:\overline{OA}_1$ and $\overline{OA}:\overline{OA}_2$, respectively, then $\mathrm{H}_1(A_1)=A$ and $\mathrm{H}_2(A_2)=A$. Therefore, the circles $\mathrm{H}_1(\overline{K})=K'$ and $\mathrm{H}_2(\overline{K})=K''$ pass through the point A and touch the lines (a) and (b). Their centers are $\mathrm{H}_1(\overline{O})=O'$ and $\mathrm{H}_2(\overline{O})=O''$, respectively. Due to this, it is necessary to draw parallel lines to the lines OA_1 and OA_2. The points where these parallel lines intersect the line (s), are actually the points O' and O''.

According to the previous considerations we conclude that the given problem has two solutions.■

7. INDIRECT SIMILARITY

7.1. Definition. The mapping $S:\mathbf{C}\to\mathbf{C}$ defined by

$$S(z)=a\overline{z}+b,\ \ a,b\in\mathbf{C},\ \ a\neq 0 \tag{1}$$

is said to be *indirect similarity*. In our further discussion the set of indirect similarities will be denoted by ***IS***.

7.2. Theorem. The indirect similarity $S:\mathbf{C}\to\mathbf{C}$ defined as (1) is bijection. Its inverse mapping $S^{-1}:\mathbf{C}\to\mathbf{C}$ is defined by

$$S^{-1}(z)=\frac{1}{\overline{a}}\overline{z}-\frac{\overline{b}}{\overline{a}},\ \ a,b\in\mathbf{C},\ \ a\neq 0\,, \tag{2}$$

and furthermore $S^{-1}\in \boldsymbol{IS}$.

Proof. If $S(z_1)=S(z_2)$, then

$$a\overline{z_1}+b=a\overline{z_2}+b\,,$$

therefore $z_1=z_2$, i.e. S is an injection. If $w\in\mathbf{C}$, then $z=\frac{\overline{w}-\overline{b}}{\overline{a}}$ satisfies

$$S(z)=S\left(\frac{\overline{w}-\overline{b}}{\overline{a}}\right)=a\frac{w-b}{a}+b=w$$

i.e. S is a surjection. So, S is a bijection.

The mapping $S_1(z)=\frac{1}{\overline{a}}\overline{z}-\frac{\overline{b}}{\overline{a}}$ is an indirect similarity and furthermore the following holds true $S\left(S_1(z)\right)=S_1\left(S(z)\right)=z$, i.e. $S^{-1}=S_1\in \boldsymbol{IS}$. ■

7.3. Theorem. The composition of two indirect similarities is also direct similarity, and the composition of a direct and an indirect similarity is indirect similarity.

Proof. If $S_1,S_2\in \boldsymbol{IS}$, then the following holds true

$$S_1(z)=a\overline{z}+b,\ \ a,b\in\mathbf{C},\ \ a\neq 0\ \text{ and }\ S_2(z)=c\overline{z}+d,\ \ c,d\in\mathbf{C},\ \ c\neq 0\,.$$

Thus,

$$S_1(S_2(z)) = S_1(c\bar{z}+d) = a\overline{(c\bar{z}+d)}+b = (a\bar{c})z+(a\bar{d}+b),$$

where $a\bar{c}, a\bar{d}+b \in \mathbf{C}$, $a\bar{c} \neq 0$, i.e. $S_1 \circ S_2 \in \boldsymbol{DS}$.

So, the composition of two indirect similarities is a direct similarity.

If $S_1 \in \boldsymbol{DS}$ and $S_2 \in \boldsymbol{IS}$, then

$$S_1(z) = az+b,\ a,b \in \mathbf{C},\ a \neq 0 \text{ and } S_2(z) = c\bar{z}+d,\ c,d \in \mathbf{C},\ c \neq 0.$$

Hence,

$$S_1(S_2(z)) = S_1(c\bar{z}+d) = a(c\bar{z}+d)+b = (ac)\bar{z}+(ad+b),$$

where $ac, ad+b \in \mathbf{C}$, $ac \neq 0$, i.e. $S_1 \circ S_2 \in \boldsymbol{IS}$. Analogously, $S_2 \circ S_1 \in \boldsymbol{IS}$.

So, the composition of a direct and an indirect similarity is indirect similarity. ■

7.4. Both the direct and indirect similarities are commonly said to be similarities. In our further discussion the set of the similarities will be denoted by $\boldsymbol{S}$. With direct checking, the associative law for similarities with respect to composition of mappings can be proved. The above stated assertion and Theorems 4.2, 7.2 and 7.3 imply the validity of the following Theorem.

Theorem. The set of the similarities $\boldsymbol{S}$ is non-commutative group under the composition of mappings. ■

7.5. Theorem. Each indirect similarity is exactly determined by two pairs of corresponding points.

Proof. Let S be an arbitrary indirect similarity such that $S(z_1) = w_1$ and $S(z_2) = w_2$. So, $S(z) = a\bar{z}+b$, where $a,b \in \mathbf{C}$, $a \neq 0$ are coefficients that should be determined. According to Theorem 7.2 each indirect similarity is a bijection, and therefore $z_1 \neq z_2$ implies $w_1 \neq w_2$. By substituting in $S(z) = a\bar{z}+b$, we get the following system of equations

$$\begin{cases} w_1 = a\bar{z}_1 + b \\ w_2 = a\bar{z}_2 + b \end{cases} \tag{3}$$

By solving the system (3) with respect to a and b, we get

$$a = \frac{w_1 - w_2}{\bar{z}_1 - \bar{z}_2},\ b = \frac{\bar{z}_1 w_2 - \bar{z}_2 w_1}{\bar{z}_1 - \bar{z}_2} \text{ and } a \neq 0,$$

i.e. the ratios of the indirect similarity are completely determined by two pairs of corresponding points $(z_1, S(z_1))$ and $(z_2, S(z_2))$. ■

7.6. Theorem. The image of a line (p) under indirect similarity is the line (p').

Proof. Let $S_1(z) = a\bar{z}+b,\ a,b \in \mathbf{C},\ a \neq 0$ be an indirect similarity and (p): $z = \eta\bar{z}+c$ be a given line. Hence, $z = \frac{\overline{w-b}}{\bar{a}}$ and by substituting into the line equation of (p), we get that line (p) is mapped to the line (p') with the following line equation $w = \left(\frac{a}{\overline{a}}\overline{\eta}\right)\overline{w} + \left(b - ac\overline{\eta} - \frac{a\overline{b}}{\overline{a}}\overline{\eta}\right)$. ■

7.7. Theorem. The image of a circle (K) under an indirect similarity is the circle (K').

Proof. Let $S_1(z) = a\overline{z} + b,\ a, b \in \mathbf{C},\ a \neq 0$ be an indirect similarity and (K): $|z - c| = R$ be a given circle. Hence, $z = \frac{\overline{w} - \overline{b}}{\overline{a}}$ and by substituting into the circle equation of (K), we get that the circle (K) is mapped to the circle (K') with the following circle equation

$$|w - (b + a\overline{c})| = R \cdot |a|. \ ■$$

7.8. Theorem. If A, B are arbitrary distinct points and A', B' are their images, respectively, under the indirect similarity (1) and if $a = re^{i\varphi}$, then $\overline{A'B'} = r\overline{AB}$. Furthermore, if α and α_1 are the directed angles between the real axis and the lines AB and $A'B'$, respectively, then $\alpha + \alpha_1 = \varphi$.

Proof. Let z_1, z_2, w_1, w_2 be affixes of the points A, B, A', B', respectively. The following equalities are satisfied,

$$z_2 - z_1 = \overline{AB} \cdot e^{i\alpha} \text{ and } w_2 - w_1 = \overline{A'B'} \cdot e^{i\alpha_1},$$

where α and α_1 are the angles formed by the real axis and the vectors $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$, respectively. Thereby $w_1 = a\overline{z_1} + b$ and $w_2 = a\overline{z_2} + b$, we get the following equation

$$w_2 - w_1 = a\overline{(z_2 - z_1)},$$

i.e.

$$\overline{A'B'} \cdot e^{i\alpha_1} = r\overline{AB} \cdot e^{i(\varphi - \alpha)},$$

which implies that $\overline{A'B'} = r\overline{AB}$ and $\alpha + \alpha_1 = \varphi$. ■

7.9. Definition. Two forms are indirect similar if there exists an *indirect similarity* so that under that similarity one of the forms is mapped to the other one.

The real number r given in the previous theorem is called to be the *ratio of the indirect similarity* (1).

7.10. Corollary. If ABC and $A'B'C'$ are indirect similar triangles, then

$$\overline{A'B'} : \overline{A'C'} = \overline{AB} : \overline{AC} \text{ and } \angle A'B'C' = -\angle ABC.$$

Proof. The proof is directly implicated by Theorem 7.8. ■

7.11. Theorem. Let $z_1, z_2, z_3,\ w_1, w_2, w_3$ be affixes of the points A, B, C, A', B', C', respectively. The triangles ABC and $A'B'C'$ are indirectly similar if and only if the following holds true

$$z_1(\overline{w_2} - \overline{w_3}) + z_2(\overline{w_3} - \overline{w_1}) + z_3(\overline{w_1} - \overline{w_2}) = 0. \qquad (4)$$

Proof. The triangles ABC and $A'B'C'$ are indirectly similar if and only if there exists an indirect similarity like (1) such that the following holds true

$$w_i = a\overline{z_i} + b, \text{ for } i = 1,2,3 .$$

By the last equations we obtain the following

$$w_1 - w_2 = a\overline{(z_1 - z_2)} \text{ and } w_1 - w_3 = a\overline{(z_1 - z_3)} .$$

After dividing and reducing the last two equations we get:

$$\frac{w_1 - w_2}{w_1 - w_3} = \overline{\frac{z_1 - z_2}{z_1 - z_3}},$$

which is equivalent to (4). ■

7.12. Example. Let $ABCD$ be a given rectangle and M and N be the midpoints of the sides AD and BC, respectively. Let P be a point on the extension of DC through D and Q is the intersection point of the lines PM and AC. Prove that $\angle QMN = \angle MNP$.

Solution. Let the origin coincides with the point N and $B(-x)$, $C(x)$, $D(x+iy)$, $A(-x+iy)$, $P(x+iz)$ (figure 7). Hence, $M(iy)$ and the equation of the line PM is $z(x-ip+iy) - \overline{z}(x+ip-iy) - 2ixy = 0$, i.e.

$$z - \overline{z}\frac{x+ip-iy}{x-ip+iy} - \frac{2ixy}{x-ip+iy} = 0 . \qquad (*)$$

Let $S\left(\frac{iy}{2}\right)$ be the intersection of the lines MN and AC. Then AC coincides to AS, thus the equation of AC is $z\left(x+\frac{iy}{2}\right) - \overline{z}\left(x-\frac{iy}{2}\right) - ixy = 0$, i.e.

$$z - \overline{z}\frac{2x-iy}{2x+iy} - \frac{2ixy}{2x+iy} = 0 . \qquad (**)$$

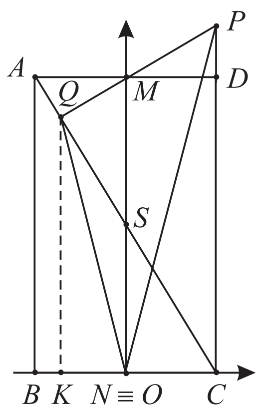

Figure 7

Since $Q = AC \cap PM$, by utilizing the identities (*) and (**) we find the affix of Q to be the following

$$\overline{q} = \frac{xy}{y+2p} + i\frac{yp}{y+2p}, \text{ i.e. } q = \frac{xy}{y+2p} - i\frac{yp}{y+2p} .$$

Further, if K denotes the projection of the point Q on the x-axis, then its affix is $k = \frac{xy}{y+2p}$. In order to prove the statement, we should only prove that the triangle CPN is indirectly similar to the triangle KQN. Therefore by Theorem 7.11 it is sufficient to check the validity of the following equality

$$\frac{x+ip-x}{0-x} = \frac{\frac{xy}{y+2p} + i\frac{yp}{y+2p} - \frac{xy}{y+2p}}{0 - \frac{xy}{y+2p}},$$

which is obviously satisfied. ■

7.13. Definition. An indirect similarity with ratio $|a| = 1$ is called to be an *indirect isometry*. The motions and the indirect isometries are called to be *isometries*.

In our further discussion the set of the direct isometries will be denoted by ***II***, and the set of the isometries by ***I***. The already stated assertion implies the validity of the following Theorem.

Theorem. The set ***I*** of the isomeries with the composition of mappings is a non-commutative group. ■

7.14. Theorem. The indirect similarity is involutory if and only if it is a *reflection*.

Proof. Let the indirect similarity $S:\mathbf{C}\to\mathbf{C}$ be involutory. Hence, $S(z)=S^{-1}(z)$, for each $z\in\mathbf{C}$, and therefore

$$a\overline{z}+b=\tfrac{1}{\overline{a}}\overline{z}-\tfrac{\overline{b}}{\overline{a}}\text{ , for each } z\in\mathbf{C}.$$

The latter implies $|a|=1$ and $\frac{b}{\overline{b}}=-a$. Thus, the equation $z=a\overline{z}+b$ is a line equation. Let us consider the point z and its image $w=S(z)$. We have $w=a\overline{z}+b$, and according to Example 1.9 the points are symmetric with respect to the line $z=a\overline{z}+b$, i.e. $S:\mathbf{C}\to\mathbf{C}$ is a reflection.

Conversely, it is enough to apply Example 1.9 directly. ■

7.15. The point z is *fixed point* of the indirect similarity (1) if and only if $z=a\overline{z}+b$. Hence, $\overline{z}=\overline{a}z+\overline{b}$ and applying the previous equation we get

$$z(1-a\overline{a})=a\overline{b}+b\ . \qquad (5)$$

There are three possibilities:

1) If $a\overline{a}\neq 1$, then (1) is not an isometry and it has only one fixed point z, $z=\frac{a\overline{b}+b}{1-a\overline{a}}$.
2) If $a\overline{a}=1$ and $a\overline{b}+b\neq 0$, then (1) is an indirect isometry, but it is not a reflection and there are no fixed points.
3) If $a\overline{a}=1$ and $a\overline{b}+b=0$, then $|a|=1$ and $\frac{b}{\overline{b}}=-a$. By the proof of Theorem 7.13 we deduce that (1) is a reflection. Furthermore, $z=a\overline{z}+b$ implies that the point z is on the line of reflection.

The above statement implies the validity of the following Theorem.

Theorem. The indirect similarity which is not an isometry has a unique fixed point $z=\frac{a\overline{b}+b}{1-a\overline{a}}$. If the indirect isometry is not a reflection, then there are no fixed points. If the indirect isometry is a reflection, then the only fixed points are points on the line of reflection. ■

7.16. Definition. If the indirect similarity (1) is not an isometry, then the fixed point $\frac{a\bar{b}+b}{1-a\bar{a}}$ is called to be a *center of the indirect similarity*.

7.17. Definition. The line (p): $z = \eta\bar{z} + c$ is called to be a *fixed line* under the indirect similarity (1) if $S(p) = p$, i.e. if the indirect similarity maps the line (p) to itself. The circle (K): $|z-c| = R$ is called to be a *fixed circle* under the indirect similarity (1) if $S(K) = K$.

7.18. Theorem. a) If the indirect similarity (1) is a reflection, then the line of reflection and its perpendicular lines are the only fixed lines under (1).

b) If the indirect similarity (1) is a reflection, then only the circles with center on the line of reflection are fixed circles under (1).

Proof. a) Let (1) be a reflection, (p): $z = a\bar{z} + b$ be the line of reflection, $|a| = 1$, $\frac{b}{\bar{b}} = -a$ and (q): $z = \eta\bar{z} + c$ be an arbitrary line. Hence $w = a\bar{z} + b$. We obtain $z = \frac{\bar{w}-\bar{b}}{\bar{a}}$. Now, substituting in the equation of (q) we get

$$\frac{\bar{w}-\bar{b}}{\bar{a}} = \eta\frac{w-b}{a} + c$$

i.e.

$$w = \frac{a}{\eta\bar{a}}\bar{w} + \frac{b}{\eta\bar{a}} + b - \frac{c}{\eta\bar{a}}.$$

The line (q) is fixed line under the reflection (1) if and only if

$$\frac{a}{\eta\bar{a}} = \eta \text{ and } \frac{b}{\eta\bar{a}} + b - \frac{c}{\eta\bar{a}} = c.$$

Since $\frac{a}{\eta\bar{a}} = \eta$ and $|a| = 1$, we have $\eta = \pm a$. Furthermore, if $\eta = a$, then

$$\frac{b}{\eta\bar{a}} + b - \frac{c}{\eta\bar{a}} = c$$

implies that $b = c$. If $\eta = -a$, then the equation

$$\frac{b}{\eta\bar{a}} + b - \frac{c}{\eta\bar{a}} = c$$

is satisfied for each $c \in \mathbf{C}$ and $(q) \perp (p)$.

b) Let (1) be a reflection with a line of reflection (p): $z = a\bar{z} + b$, $|a| = 1$, $\frac{b}{\bar{b}} = -a$ and let (K): $|z-c| = R$ be an arbitrary circle. Since $w = a\bar{z} + b$, we obtain $z = \frac{\bar{w}-\bar{b}}{\bar{a}}$. By substituting in the equation of the circle (K), we get $|w-(b+a\bar{c})| = R$. The circle (K) is fixed under the reflection (1) if and only if $c = a\bar{c} + b$, or in other words if and only if its center is on the line of reflection. ■

7.19. Example. Construct a circle which passes through two distinct given points and touches a given line.

Solution. Let the points A, B and the line (c) be given. Obviously, if A and B are in different semi-planes with respect to the line (c) or both are on the line (c), then the given problem has no solution. If $A\in(c)$ and $B\notin(c)$, then the problem has a unique solution. The center O of the required circle will be the intersection of the bisector of the segment AB and the line (p), which passes through A and is perpendicular to (c). If the points A and B are in the same semi-plane with respect to the line (c) and AB is parallel to (c), then the given problem has two solutions. The one is the circle which passes through the points A, B and M, where M is the intersection of the bisector of the segment AB and the line (c). The other one is the line AB (figure 8).

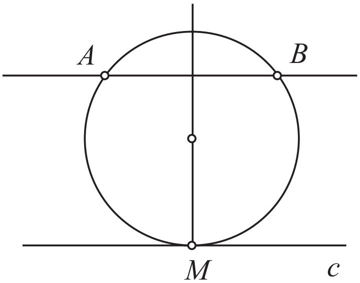

Figure 8

Let the points A and B be in the same semi-plane with respect to the line (c) and the line AB be not parallel to the line (c) (figure 9). Since, the required circle $K(O,R)$ passes through the points A and B, its center will be on the bisector (l) of the segment AB. Let σ_l be a reflection with a line of reflection (l). By the previous theorem, $\sigma_l(K)=K$. Since the line (c) is tangent to the circle $K(O,R)$, Corollary 7.10 implies that $\sigma_l(c)=c'$ is tangent to $K(O,R)$. If the lines (c) and (c') are not parallel, then the problem can be transformed to Example 6.14.

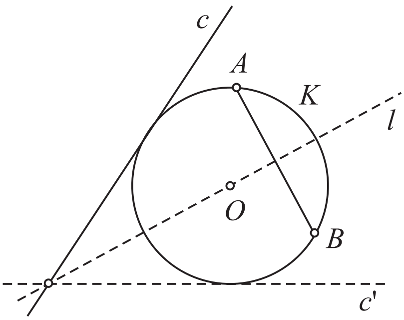

Figure 9

If the lines (c) and (c') are parallel, then the point A is between them. Let d be the distance between the lines (c) and (c'). The circle $\overline{K}\left(A,\frac{d}{2}\right)$ meets the bisector (s) in points O_1 and O_2. So, the required circles are $K_1\left(O_1,\frac{d}{2}\right)$ and $K_2\left(O_2,\frac{d}{2}\right)$, (figure 10). ■

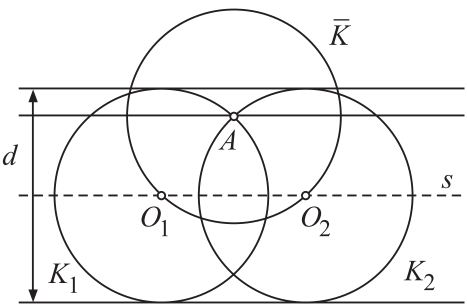

Figure 10

7.20. It is naturally to wonder, if there are any fixed lines under an indirect similarity, which is not a reflection, i.e. $|a|\neq 1$.

According to Theorem 7.6, the image of the line (p): $z=\eta\overline{z}+c$ under the indirect similarity (1) is the line (p') with equation

$$w=\left(\frac{a}{\overline{a}}\overline{\eta}\right)\overline{w}+\left(b-ac\overline{\eta}-\frac{a\overline{b}}{\overline{a}}\overline{\eta}\right).$$

The lines (p) and (p') coincide if and only if

$$\frac{a}{\overline{a}}\overline{\eta}=\eta \text{ and } b-ac\overline{\eta}-\frac{a\overline{b}}{\overline{a}}\overline{\eta}=c\,.$$

Reducing the last two equalities, we get

$$\eta^2 = \frac{a}{\bar{a}} = \frac{a^2}{|a|^2}\text{, i.e. } \eta_1 = \frac{a}{|a|},\ \eta_2 = -\frac{a}{|a|}$$

and

$$c_1 = \frac{b\bar{a} - \bar{b}|a|}{\bar{a}(1+|a|)},\ c_2 = \frac{b\bar{a}+\bar{b}|a|}{\bar{a}(1-|a|)},$$

respectively. Thus, the indirect similarity which is not an isometry has two fixed lines (p_1) and (p_2) such that

$$(p_1):\ z = \frac{a}{|a|}\bar{z} + \frac{b\bar{a}-\bar{b}|a|}{\bar{a}(1+|a|)} \text{ and } (p_2):\ z = -\frac{a}{|a|}\bar{z} + \frac{b\bar{a}+\bar{b}|a|}{\bar{a}(1-|a|)}.$$

According to Corollary 1.8. A b) the lines and are perpendicular.

Since

$$\frac{a}{|a|}\overline{\left(\frac{a\bar{b}+b}{1-a\bar{a}}\right)} + \frac{b\bar{a}-\bar{b}|a|}{\bar{a}(1+|a|)} = \frac{a}{|a|}\frac{b\bar{a}+\bar{b}}{1-|a|^2} + \frac{b\bar{a}-\bar{b}|a|}{\bar{a}(1+|a|)}$$

$$= \frac{|a|(b\bar{a}+\bar{b}) + (b\bar{a}-\bar{b}|a|)(1-|a|)}{\bar{a}(1-|a|^2)}$$

$$= \frac{b\bar{a}+\bar{b}a\bar{a}}{\bar{a}(1-|a|^2)} = \frac{b+\bar{b}a}{1-|a|^2}$$

we get that the center $\frac{b+\bar{b}a}{1-|a|^2}$ of the indirect similarity (1), which is not an isometry, is on the line (p_1). Analogously,

$$-\frac{a}{|a|}\overline{\left(\frac{a\bar{b}+b}{1-a\bar{a}}\right)} + \frac{b\bar{a}+\bar{b}|a|}{\bar{a}(1-|a|)} = -\frac{a}{|a|}\frac{b\bar{a}+\bar{b}}{1-|a|^2} + \frac{b\bar{a}+\bar{b}|a|}{\bar{a}(1-|a|)}$$

$$= \frac{-|a|(b\bar{a}+\bar{b}) + (b\bar{a}+\bar{b}|a|)(1+|a|)}{\bar{a}(1-|a|^2)}$$

$$= \frac{b\bar{a}+\bar{b}a\bar{a}}{\bar{a}(1-|a|^2)} = \frac{b+\bar{b}a}{1-|a|^2}$$

i.e. the center $\frac{b+\bar{b}a}{1-|a|^2}$ of the indirect similarity (1), which is not an isometry, is on the line (p_2).

Thus, we proved the following theorem.

Theorem. The indirect similarity (1), which is not isometry, has such two perpendicular fixed lines that pass through the center of similarity. ■

7.21. Definition Let (1) be an indirect similarity, which is not isometry. The lines (p_1) and (p_2) with equations

$$z = \frac{a}{|a|}\bar{z} + \frac{b\bar{a}-\bar{b}|a|}{\bar{a}(1+|a|)} \text{ and } z = -\frac{a}{|a|}\bar{z} + \frac{b\bar{a}+\bar{b}|a|}{\bar{a}(1-|a|)},$$

respectively, are called to be *lines of the indirect similarity* (1).

Clearly, by Theorem 7.19 the lines of the indirect similarity are the only fixed lines under indirect similarity which is not an isometry.

8. INVERSION

8.1. Let $m \in \mathbf{R}$, $m > 0$ and $a \in \mathbf{C}$ and let $I : \mathbf{C} \setminus \{a\} \to \mathbf{C} \setminus \{a\}$ be a mapping defined by

$$I(z) = a + \frac{m}{\overline{z-a}}\,. \tag{1}$$

i) If $I(z_1) = I(z_2)$, then

$$a + \frac{m}{\overline{z_1 - a}} = a + \frac{m}{\overline{z_2 - a}}$$

implies $z_1 = z_2$, i.e. *I* is an injection.

ii) For $w \in \mathbf{C} \setminus \{a\}$, there exists such $z = a + \frac{m}{\overline{w-a}}$ that $I(z) = w$, i.e. *I* is a surjection.

Now, *i*) and *ii*) imply that *I* is a bijection.

Definition. The mapping

$$I : \mathbf{C} \setminus \{a\} \to \mathbf{C} \setminus \{a\}$$

defined by (1) is called to be an *inversion with center a and radius* $\sqrt{m}$.

8.2. The point *z* is fixed point under the inversion (1) if and only if the following equality is satisfied

$$z = a + \frac{m}{\overline{z-a}}\,,$$

or in other words, if and only if

$$|\,z - a\,| = \sqrt{m}\,.$$

Thus we proved the following theorem.

Theorem. The point *z* is fixed point under the inversion (1) if and only if *z* is on the circle $|\,z - a\,| = \sqrt{m}$. ■

8.3. Definition. The circle (K_0) : $|\,z - a\,| = \sqrt{m}$, is called to be the *inversion circle* of (1).

8.4. Theorem. The inversion is involutory mapping.

Proof. Let an inversion be defined by (1). Thus, for each $z \in \mathbf{C}$ the following holds true

$$I\big(I(z)\big) = I\left(a + \frac{m}{\overline{z-a}}\right) = a + \frac{m}{\overline{a + \frac{m}{\overline{z-a}} - a}} = z = E(z)$$

i.e. $I^2 = E$ and since *I* is a bijection, then $I = I^{-1}$, i.e. the inversion is involutory. ■

8.5. Let *O* be the center of inversion (1), *A* be an arbitrary point in the plane, such that *A* differs from *O*, and $\mathrm{I}(A) = A'$. Let *a*, *z* and *z** be the affixes of the points *O*, *A* and A', respectively. Thus, it holds true

$$z^* - a = a + \frac{m}{\overline{z-a}} - a = \frac{m}{\overline{z-a}} = \frac{m}{|z-a|^2}(z - a)\,.$$

The latter implies

$$\arg(z^*-a)=\arg(z-a) \text{ and } |z^*-a|\cdot|z-a|=m .$$

Thus, we have proved the following theorem.

Theorem. a) Under inversion each point A, distinct from O (the center of the inversion (1)), is mapped to (inverts to) a point A' which is on the ray $\overrightarrow{OA}$ and

$$\overline{OA}\cdot\overline{OA'}=m . \qquad (2)$$

b) The points A and A', with affixes z and z^*, respectively, are inverse with respect to the circle $|z-a|=\sqrt{m}$ if and only if

$$(z^*-a)(\overline{z}-\overline{a})=m . \ \blacksquare$$

8.6. Theorem. Each inner point of a referent circle inverts to an outer point of the referent circle, and vice versa.

Proof. If A is an inner point of the referent circle $K_0\left(O,\sqrt{m}\right)$ and $\mathrm{I}(A)=A'$, then $\overline{OA}<\sqrt{m}$ and according to (2) it follows, that $\overline{OA'}>\sqrt{m}$. So, A' is an outer point for the circle K_0.

The converse statement can be proved analogously. $\blacksquare$

8.7. Let us explain the construction of A' as an inverse of A under the inversion (1). Let A, with affix z_0, be an inner point of a referent circle $K_0\left(O,\sqrt{m}\right)$. Therefore, $|z_0-a|^2<m$. The equation of the line OA is

$$z-a=\frac{z_0-a}{\overline{z}_0-\overline{a}}(\overline{z}-\overline{a}) .$$

Through the point A we draw a line (q) perpendicular to OA. The equation of that line is the following

$$z-z_0=-\frac{z_0-a}{\overline{z}_0-\overline{a}}(\overline{z}-\overline{z}_0) .$$

The points of intersection of the line (q) and the circle $K_0\left(O,\sqrt{m}\right)$ are obtained as solutions of the following system

$$\begin{cases} z-z_0=-\dfrac{z_0-a}{\overline{z}_0-\overline{a}}(\overline{z}-\overline{z}_0) \\ |z-a|=\sqrt{m} \end{cases}$$

One of them is the point T with affix

$$z_1=z_0+i\frac{\sqrt{m-|z_0-a|^2}}{|z_0-a|}(z_0-a) .$$

Through the point T we draw a tangent (t) to the circle $K_0\left(O,\sqrt{m}\right)$. The equation of that tangent is

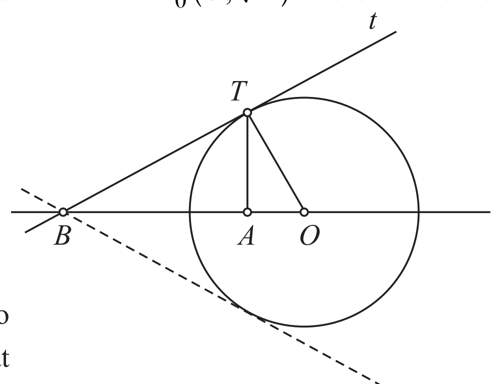

Figure 11

$$z - z_1 = -\frac{z_1 - a}{\overline{z}_1 - \overline{a}}(\overline{z} - \overline{z}_1)\,.$$

Further, we determine the point *B*, as intersection point of the tangent (*t*) and the line *OA*. Its affix is

$$z = a + \frac{m}{\overline{z}_0 - \overline{a}}\,.$$

Thus, $\mathrm{I}(A) = B = A'$.

Let *A* be an outer point for the circle $K_0\left(O, \sqrt{m}\right)$. According to Theorem 8.4 the following holds true $\mathrm{I}^2 = E$. The latter implies the construction $\mathrm{I}(A) = A'$. We draw the tangent (*t*) to the circle K_0 through the *A*. The orthogonal projection of *T*, the point where the tangent (*t*) touches K_0, on the line *OA* is the point $\mathrm{I}(A) = A'$.

8.8. Let *c* and *b*, be the affixes of the points *A* and *B* respectively. The affixes of their images under the inversion (1) are the following:

$$c' = a + \frac{m}{\overline{c} - \overline{a}} \text{ and } b' = a + \frac{m}{\overline{b} - \overline{a}}\,,$$

respectively. The complex gradients of the lines *OA*, *OB*, *AB*, OA', OB' and $A'B'$ are

$$\eta_1 = \frac{c-a}{\overline{c}-\overline{a}},\ \eta_2 = \frac{b-a}{\overline{b}-\overline{a}},\ \eta_3 = \frac{b-c}{\overline{b}-\overline{c}},$$

$$\eta_4 = \eta_1,\ \eta_5 = \eta_2,$$

$$\eta_6 = \frac{(b-a)(c-a)(\overline{b}-\overline{c})}{(\overline{b}-\overline{a})(\overline{c}-\overline{a})(b-c)}$$

respectively. Since $\frac{\eta_2}{\eta_3} = \frac{\eta_6}{\eta_4}$ and $\frac{\eta_1}{\eta_3} = \frac{\eta_6}{\eta_5}$ and also theorem 1.7, we deduce, that (figure 12):

$$\angle OBA = \angle B'A'O \text{ and}$$
$$\angle OAB = \angle A'B'O\,.$$

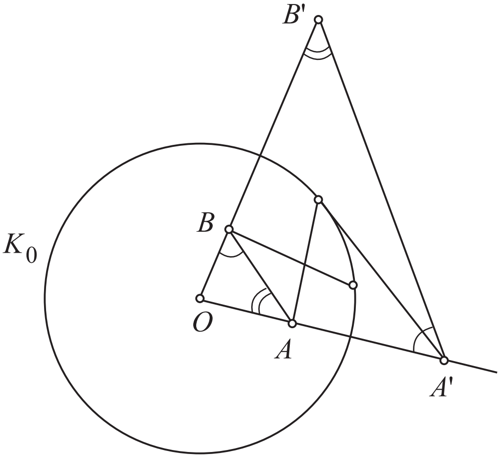

Figure 12

Thus, we have proved the following theorem.

Theorem. Let *O* be the center of the inversion (1), *A* and *B* be arbitrary points and A' and B' are their images under the inversion (1). Then,

$$\angle OBA = \angle B'A'O \text{ and } \angle OAB = \angle A'B'O\,. \blacksquare$$

8.9. Under the inversion (1) the line (p) with a self-conjugate equation

$$Az + B\overline{z} + C = 0,\ C \in \mathbf{R},\ B = \overline{A}$$

is mapped to the curve with equation

$$Aa + B\overline{a} + C + \frac{Am}{\overline{w} - \overline{a}} + \frac{Bm}{w - a} = 0\,. \qquad (3)$$

Two following cases are possible:

i) If $Aa+B\overline{a}+C=0$, i.e. the line passes through the inversion center, then (3) implies that the image of (p) is the line

$$Aw+B\overline{w}+C=0,\ \ C\in\mathbf{R},\ \ B=\overline{A}.$$

The latter is actually the equation of the line (p).

ii) If $Aa+B\overline{a}+C\neq 0$, i.e. the line does not pass through the inversion center, then (3) implies that the image of (p) is the circle

$$w\overline{w}+\overline{A_1}w+A_1\overline{w}+B_1=0,$$

where

$$A_1=\frac{Bm}{Aa+B\overline{a}+C}-a,\ \ B_1=a\overline{a}-\frac{Aam+Bm\overline{a}}{Aa+B\overline{a}+C}.$$

By direct checking we get $a\overline{a}+\overline{A_1}a+A_1\overline{a}+B_1=0$. The latter means that the image of a line (p) which does not pass through the center of the inversion (1) is circle which passes through the inversion center.

The above stated assertion implies the validity of the following Theorem.

Theorem. Each line through the inversion center O is mapped to itself, and each line, which does not pass through O is mapped to a circle through O. ■

8.10. If the line (p) does not pass through the inversion center O, then the proof of Theorem 8.9 implies that the center O_1 of the circle, in which (p) maps to, has affix

$$z_1=-A_1=a-\frac{Bm}{Aa+B\overline{a}+C}.$$

Let P be the orthogonal projection of the O on the line (p). According to 2.3 the affix of the point P is

$$z_0=\frac{Aa-B\overline{a}-C}{2A},$$

Therefore the affix of its image $P'=\mathrm{I}(P)$ is

$$z^*=a-\frac{2Bm}{Aa+B\overline{a}+C}.$$

Since,

$$\frac{z^*+a}{2}=a-\frac{Bm}{Aa+B\overline{a}+C}=z_1$$

we deduce that O_1 is the midpoint of OP'.

The above stated assertion defines the construction of the circle I(p) if the line (p) does not pass through the center O of an inversion I. Firstly, we determine the orthogonal projection P of the inversion center O on the line (p). Thus we get the point $P'=\mathrm{I}(P)$, (figures 13 and 14). Further, we construct a circle with diameter OP'. So, we get the circle I(p).

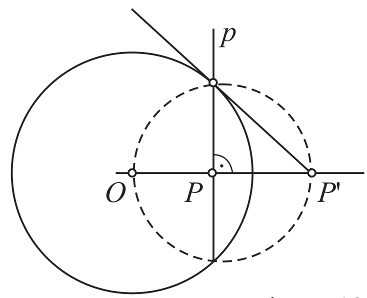

Figure 13

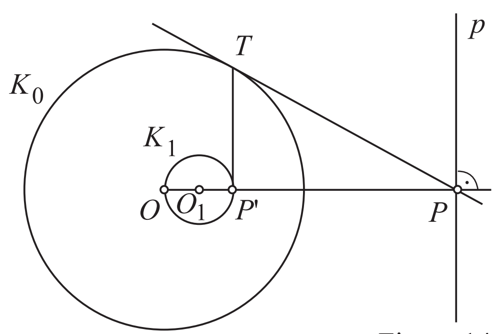

Figure 14

8.11. Example. Let (p) and (q) be given lines. Is there an inversion I such that $\mathrm{I}(p) = q$?

Solution. According to Theorem 8.9 such an inversion exists if and only if the lines (p) and (q) coincide. Further, for each inversion with center on (p) and arbitrary radius the following holds true $\mathrm{I}(p) = q$. ■

8.12. Theorem. If a circle K_1 passes through the center O of the inversion I, then $\mathrm{I}(K_1)$ is a line which does not pass through O.

Proof. The proof is directly implicated by Theorems 8.4 and 8.9. ■

8.13. Example. Given are a line (p) and a circle $K_1(O_1, R)$. Is there an inversion I such that $\mathrm{I}(p) = K_1$?

Solution. According to Theorem 8.9 if the required inversion I exists, then its center O is on the circle $K_1(O_1, R)$ and the line OO_1 is perpendicular to the line (p).

The following three cases are possible:

i) The line (p) and the circle K_1 meet at points M and N. Starting from the center O_1 of the circle K_1 we draw a line (q) perpendicular to (p) and find the points O and O' as intersections of (q) and K_1. The discussion in 8.10, implies that the inversions I and I_1 determined by the circles $K(O, \overline{OM})$ and $K'(O', \overline{O'M})$ satisfy the given conditions.

ii) The line (p) touches the circle K_1 at the point M. Starting from the center O_1 of the circle K_1 we draw a line (q) perpendicular to (p). The point O is found as intersection of (q) and K_1. The discussion in 8.10 implies that the inversion I determined by the circles $K(O, \overline{OM})$ satisfies the given conditions.

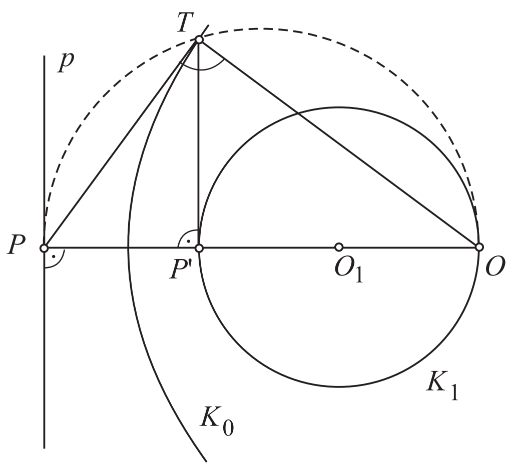

Figure 15

iii) The line (p) and the circle K_1 have no common points. Starting from the center O_1 of the circle K_1 we draw a line (q) perpendicular to (p). The point P is determined as intersection of (q) and (p). The points O and P' are the intersection of (q) and K_1 such that O_1, P', P are positioned in that order (figure 15). We construct a semicircle with diameter OP, through P' we draw a line perpendicular to OP and determine the point T as their intersection. The discussion in 8.10 implies that the inversion I determined by the circle $K(O, \overline{OT})$ satisfies the given conditions. ■

8.14. Now, we have to consider the image of the circle $K_1(O_1, R)$ under inversion I when K_1 does not pass through the inversion center *O*.

Let $K_1 : |z-b| = R$ be a given circle, which does not pass through the center of the inversion (1), i.e. $|a-b| \neq R$. By substituting

$$z = a + \frac{m}{\overline{w-a}}$$

in the equation of the circle K_1, and after equivalent transformations we get that the circle K_1 inverts to a circle $\mathrm{I}(K_1)$ with equation

$$w\overline{w} + \overline{A}_1 w + A_1 \overline{w} + B_1 = 0,$$

where

$$A_1 = \frac{m(b-a)}{R^2-|b-a|^2} - a, \quad B_1 = |a|^2 - \frac{m^2+ma(\overline{b}-\overline{a})+m\overline{a}(b-a)}{R^2-|b-a|^2}.$$

With direct checking we get

$$a\overline{a} + \overline{A}_1 a + A_1 \overline{a} + B_1 \neq 0.$$

The above implies the validity of the following Theorem.

Theorem. If a circle K_1 does not pass through the center *O* of the inversion I, then $\mathrm{I}(K_1)$ is circle such that it does not pass through *O*. ■

8.15. If the circle K_1 does not pass through the inversion center *O*, then the proof of Theorem 8.14 implies that the center O'_1 of the circle $\mathrm{I}(K_1)$ has affix

$$z_1 = -A_1 = a - \frac{m(b-a)}{R^2-|b-a|^2}.$$

The affixes z_1, *a* and *b* of the points O'_1, *O* and O_1 satisfy

$$\frac{z_1-a}{b-a} = \frac{m}{|b-a|^2-R^2} \in \mathbf{R}.$$

According to Corollary 1.4, the above means that they are collinear. But, the line OO_1 is fixed under the inversion I. Therefore according to the discussion in 8.5 we conclude that the diameter of K_1 which is on this line inverts to a diameter of $\mathrm{I}(K_1)$ which is on OO_1.

The above stated assertion implies the following construction of the circle $\mathrm{I}(K_1)$, when K_1 does not pass through the center *O* of the inversion I. We draw a straight line OO_1, we find the points *M*, *N* (points where K_1 meets OO_1) and further we obtain $M' = \mathrm{I}(M)$ and $N' = \mathrm{I}(N)$. After that, we construct a circle with diameter $M'N'$. Hence, we get the circle $\mathrm{I}(K_1)$ (figure 16).

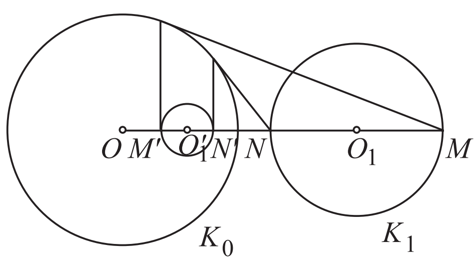

Figure 16

8.16. Example. Given are the circles $K_1(O_1, R)$ and $K_2(O_2, R)$. Is there an inversion I so that $\mathrm{I}(K_1) = K_2$?

Solution. Let the equations of the circles K_1 and K_2 be $|z - c_1| = R_1$ and $|z - c_2| = R_2$, respectively. We will consider five cases:

i) If $c_1 = c_2$, i.e. the circles are concentric, then the mapping

$$\mathrm{I}(z) = c_1 + \frac{R_1 R_2}{\bar{z} - \bar{c}_1}$$

is inversion, so that $\mathrm{I}(K_1) = K_2$. Clearly, c_1 is the center of such the inversion and $R_1 R_2$ is the radius. The referent circle can be constructed if we draw an arbitrary half-line and use the fact that the common points of the circles K_1 and K_2 are inverse.

ii) If $c_1 \neq c_2$, $R_1 \neq R_2$ and $|c_1 - c_2| \neq |R_1 - R_2|$, then the mapping

$$\mathrm{I}(z) = \frac{c_1 R_2 - c_2 R_1}{R_2 - R_1} + \frac{\frac{R_1 R_2 \left|(R_2 - R_1)^2 - |c_1 - c_2|^2\right|}{(R_2 - R_1)^2}}{\bar{z} - \frac{\bar{c}_1 R_2 - \bar{c}_2 R_1}{R_2 - R_1}}$$

is such an inversion, that $\mathrm{I}(K_1) = K_2$. We notice that the inversion center coincides with the outer center of homothety, given by 6.6, which can be constructed as explained in Remark 6.7. When constructing the inversion circle, it is necessary to follow the procedure given in Example 8.13.

iii) If $c_1 \neq c_2$, $R_1 \neq R_2$ and $|c_1 - c_2| = |R_1 - R_2|$, then the mapping

$$\mathrm{I}(z) = \frac{c_1 R_2 + c_2 R_1}{R_2 + R_1} + \frac{\frac{R_1 R_2 \left|(R_2 + R_1)^2 - |c_1 - c_2|^2\right|}{(R_2 + R_1)^2}}{\bar{z} - \frac{\bar{c}_1 R_2 + \bar{c}_2 R_1}{R_2 + R_1}}$$

is such an inversion, that $\mathrm{I}(K_1) = K_2$. We notice that the inversion center coincides with the inner center of homothety, given by 6.6, which can be constructed as explained in Remark 6.7. When constructing the inversion circle, it is necessary to follow the procedure given in Example 8.13.

iv) If $c_1 \neq c_2$, $R_1 = R_2 = R$ and $|c_1 - c_2| < 2R$, then the mapping

$$\mathrm{I}(z) = \frac{c_1 + c_2}{2} + \frac{R^2 - \frac{|c_1 - c_2|^2|}{4}}{\bar{z} - \frac{\bar{c}_1 + \bar{c}_2}{2}}$$

is such an inversion, that $\mathrm{I}(K_1) = K_2$. We notice that the inversion center is the midpoint of segment $O_1 O_2$, and the common points of the circles K_1 and K_2 are fixed. So, we can construct the referent circle.

v) If $c_1 \neq c_2$, $R_1 = R_2 = R$ and $|c_1 - c_2| > 2R$, then Theorem 8.4 implies that the required inversion does not exist.■

8.17. Consider the following theorem.

Theorem. If two lines, a line and a circle, or two circles have no common points, they are tangent, or they have two common points, then their images under the inversion

I have no common points, they are tangent, or they have two common points, respectively. ■

8.18. Definition. Let the line (p) and the circle K meet in the points M and N. We draw a tangent (t) to the circle K through M. Let α be the smaller angle between (p) and (t). The angle α is called to be *angle of intersection* of the circle K and the line (p).

Let the circles K and K^* meet in the points M and N. We draw tangents (t_1) and (t_2) to the circles K and K^* through M. Let α be the smaller angle between the lines (t_1) and (t_2). The angle α is called to be *angle of intersection* of the circles K and K^*.

We say that the circle K meets the circle K^* *orthogonally* if the measure of the angle of intersection of K and K^* is $\frac{\pi}{2}$. K and K^* are said to be *orthogonal*.

Theorem 8.2 proves that the fixed points under the inversion I with circle K_0 are the points on K_0 only. This means that the circle is fixed under I. Theorem 8.8 implies that there does not exist any line which is fixed under I, but each line through the inversion center is fixed under I.

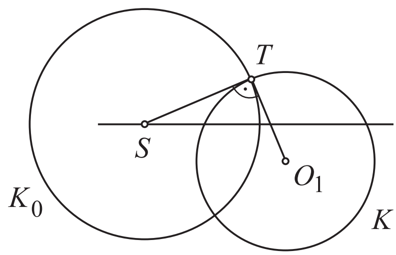

Figure 17

The question now is whether there exist any other circle (different from K_0), which is fixed under the inversion I. By Theorem 8.6, it follows that if there exists such a circle $K_1 : |z-b| = R$, then it is mandatory to meet the reference circle K_0. Therefore, it must have two fixed points. One of them is denoted by T with affix z_1 (figure 17). Now, the proof of Theorem 8.14 implies that $K_1 : |z-b| = R$ is fixed under inversion (1) if and only if

$$b = a - \frac{m(b-a)}{R^2-|b-a|^2},$$

i.e. if and only if $m + R^2 = |b-a|^2$. The last equality is equivalent to $\frac{z_1-a}{\overline{z_1}-\overline{a}} = -\frac{z_1-b}{\overline{z_1}-\overline{b}}$. Due to this, K_1 is fixed under inversion (1) if and only if the tangents to K_1 and K_0 through T are perpendicular to each other.

So, we proved the following Theorem.

Theorem. A circle K_1, different from K_0 is fixed under inversion I if and only if K_1 intersects K_0 orthogonally. ■

8.19. Theorem. An angle between two lines, a line and a circle or between two circles is preserved under inversions.

Proof. Let an inversion be given by (1) and let (p) and (q) be two lines with the following self-conjugate equations:

$$Az + B\overline{z} + C = 0,\ B = \overline{A},\ C \in \mathbf{R} \text{ and } A_1 z + B_1\overline{z} + C_1 = 0,\ B_1 = \overline{A_1},\ C_1 \in \mathbf{R}$$

respectively. The following two cases are possible:

i) Let the lines (*p*) and (*q*) pass through the inversion center. It follows by the proof of Theorem 8.9, that (*p*) and (*q*) are fixed, and therefore the angle between them is preserved.

ii) Let one of the lines, for example (*p*), passe through the inversion center and let the other one (*q*) does not pass through the inversion center. The proof of Theorem 8.9 implies that the line (*p*) is fixed, and the line (*q*) inverts to the circle

$$\mathrm{I}(q): \left| z - a + \frac{B_1 m}{A_1 a + B_1 \overline{a} + C_1} \right| = \frac{m|A_1|}{|A_1 a + B_1 \overline{a} + C_1|}.$$

Let the line (*p*) and the circle I(*q*) meet at the inversion center with affix *a*. The equation of the tangent (q_1) to the circle I(*q*) through *a* is the following

$$A_1 z + B_1 \overline{z} + C_1 - aA_1 - \overline{a}B_1 = 0.$$

We check directly that the complex gradients η_1, η_2, η_3 of the lines (*p*), (*q*), (q_1) satisfy $\frac{\eta_1}{\eta_2} = \frac{\eta_1}{\eta_3}$. Further, the statement given in this Theorem is implied by Theorem 1.7.

iii) The lines (*p*) and (*q*) do not pass through the center of inversion. The proof of Theorem 8.9 implies that the inverses of the lines (*p*) and (*q*) are circles with the following equations

$$\left| z - a + \frac{Bm}{Aa + B\overline{a} + C} \right| = \frac{m|A|}{|Aa + B\overline{a} + C|} \text{ and } \left| z - a + \frac{B_1 m}{A_1 a + B_1 \overline{a} + C_1} \right| = \frac{m|A_1|}{|A_1 a + B_1 \overline{a} + C_1|},$$

respectively. The circles I(*p*) and I(*q*) meet in the inversion center with affix e *a*. The equations of the tangents (p_1) and (q_1) to the circles I(*p*) and I(*q*) through *a* are the following

$$Az + B\overline{z} + C - aA - \overline{a}B = 0 \text{ and } A_1 z + B_1 \overline{z} + C_1 - aA_1 - \overline{a}B_1 = 0,$$

respectively. The complex gradients $\eta_1, \eta_2, \eta_3, \eta_4$ of the lines (*p*), (*q*), (p_1), (q_1) satisfy the following equality $\frac{\eta_1}{\eta_2} = \frac{\eta_3}{\eta_4}$. Due to this, Theorem 1.7 implies the validity of the given Theorem.

The remaining part of Theorem is proved analogously, using Theorems 8.9 8.12 and 8.14. ■

8.20. Example. Let the line (*p*) and the circle *K*(*O*,*R*) have no common points. Prove that there exists an inversion I such that I(*p*) and I(*K*) are two concentric circles.

Solution. Through the center *O* of the circle *K* we draw a line (*q*) perpendicular to (*p*) and let $P = (p) \cap (q)$, (figure18). Let *T* be an arbitrary point on the circle *K*, so that *T* is not on (*q*), and *S* be one of the common points of the circle $K_1(P, \overline{PT})$ and the line (*q*).

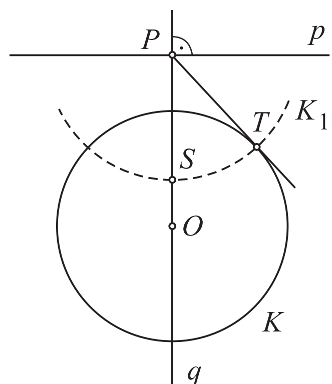

Figure 18

We define an inversion I with center S and arbitrary radius. According to Theorems 8.9 and 8.14, the inverses (images) I(p) and I(K) are circles. We have to determine their centers.

The line (p) meets orthogonally the line (q) and the circle K_1. Due to this, Theorem 8.19 implies that I(p) meets orthogonally I(q) and $\mathrm{I}(K_1)$. Theorem 8.9 implies that $\mathrm{I}(q)=q$, and Theorem 8.12 implies that $\mathrm{I}(K_1)$ is a line. Since, the lines (q) and $\mathrm{I}(K_1)$ meet the circle I(p) orthogonally, we get that its center $O"$ is a common point of these two lines. Hence, $O"=(q)\cap \mathrm{I}(K_1)$.

The circle K meets orthogonally the line (q) and the circle K_1. Due to this, Theorem 8.19 implies that I(K) meets I(q) and $\mathrm{I}(K_1)$ orthogonally. This means that I(K) meets (q) and $\mathrm{I}(K_1)$ orthogonally. Hence, the center O' of the circle I(K) is obtained as $O'=(q)\cap \mathrm{I}(K_1)$.

Finally, $O'\equiv O"$, i.e. I(p) and I(K) are two concentric circles.■

8.21. We solve the problem below by applying homothety (Example 7.18). In this section we will give the solution by applying inversion.

Example. Construct a circle which passes through two given points and touches a given line.

Solution. Let points A and B and a line c be given. We suppose that the points A and B lie in the same semi-plane with respect to the line c and that the line AB is not parallel to the line c.

Let I be an inversion with center A and radius $\overline{AB}^2$. Hence, $\mathrm{I}(B)=B$ and I(c) is the circle K_1 which passes through A. The required circle K passes through the point A, therefore I(K) is a line which passes through B and touches the circle K_1, i.e. I(K) is the tangent to the circle K_1 through the point B. The already stated assertion implies the following construction (figure 19):

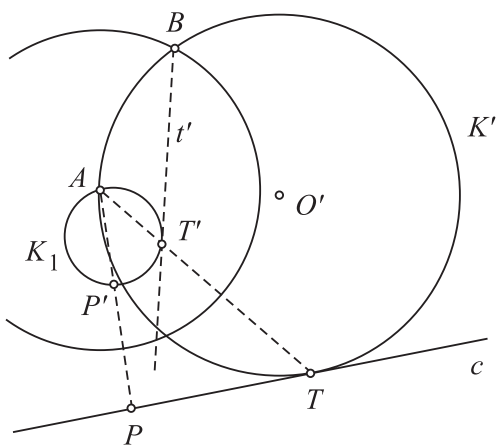

Figure 19

- We define an inversion I with center A and radius $\overline{AB}^2$.
- We construct the circle $K_1=\mathrm{I}(c)$.
- Through B we draw the tangents (t') and $(t")$ to the circle .
- The required circles are $K'=\mathrm{I}(t')$ and $K"=\mathrm{I}(t")$. ■

9. MÖBIUS TRANSFORMATION

9.1. Definition. The mapping

$$S(z)=\tfrac{az+b}{cz+d},\ \ ad-bc\neq 0\,, \tag{1}$$

where a, b, c, d are complex numbers, is called to be a *Möbius transformation*.

The Möbius transformation is determined for each $z\neq -\frac{d}{c},\infty$. If $c=0$, then the Möbius transformation is determined for each finite z. If $c\neq 0$, then we extend the definition by

$$S\left(-\tfrac{d}{c}\right)=\infty \text{ and } S(\infty)=\tfrac{a}{c}\,. \tag{2}$$

If $c=0$, then it is sufficient to let $S(\infty)=\infty$. Thus, $S:\mathbf{C}_\infty\to\mathbf{C}_\infty$ is a well defined mapping.

Note that, the condition $ad-bc\neq 0$ is equivalent to $S(z)$ is an injection (show it!).

9.2. Example. Determine the image of the unit circle $|z|=1$ under the transformation

$$w=u\tfrac{z-v}{\bar{v}z-1},\ \ u,v\in\mathbf{C} \text{ and } |v|\neq 1\,.$$

Solution. Since

$$w\bar{w}=u\bar{u}\,\tfrac{z\bar{z}-v\bar{z}-z\bar{v}+v\bar{v}}{v\bar{v}z\bar{z}+\bar{z}v-z\bar{v}+1}$$

for $z\bar{z}=1$ we get $w\bar{w}=u\bar{u}$. That means, that the unit circle is mapped to the circle $|w|=|u|$. ■

9.3. Theorem. The Möbius transformation defined by (1) and (2) is a bijection from $\mathbf{C}_\infty$ to $\mathbf{C}_\infty$.

Proof. Let S be a Möbius transformation defined by (1) and (2).

If $w\in\mathbf{C}$ and $w\neq\frac{a}{c}$, then since $w=\frac{az+b}{cz+d}$ we get $z=\frac{dw-b}{-cw+a}$. So, when $w\neq\frac{a}{c},\infty$ there exists $z=\frac{dw-b}{-cw+a}$ such that $S(z)=w$.

If $w=\infty$, then $S\left(-\frac{d}{c}\right)=\infty$, and if $w=\frac{a}{c}$, then $S(\infty)=\frac{a}{c}$. Due to that, S is surjection. But, we already noted that S is injection, and therefore S is bijection.

The above stated implies that the mapping $S^{-1}:\mathbf{C}_\infty\to\mathbf{C}_\infty$ defined by

$$S^{-1}(z)=\tfrac{dz-b}{-cz+a},\ \text{if } z\neq\tfrac{a}{c} \text{ and by } S^{-1}(\infty)=-\tfrac{d}{c}\,,\ S^{-1}\left(\tfrac{a}{c}\right)=\infty$$

is well defined, and furthermore

$$S\left(S^{-1}(z)\right)=S^{-1}\left(S(z)\right)=z \text{ holds.}$$

Finally, S^{-1} is the inverse mapping to S and since

$$ad-(-b)(-c)=ad-bc\neq 0$$

we get that it is Möbius transformation. ■

9.4. Theorem. The family ***M*** of all Möbius transformations under the composition of mapping is a group.

Proof. Let

$$S_1(z)=\frac{az+b}{cz+d},\ ad-bc\neq 0 \text{ and } S_2(z)=\frac{ez+f}{gz+h},\ eh-fg\neq 0$$

be two Möbius transformations. Then,

$$S(z)=(S_1\circ S_2)(z)=S_1\left(S_2(z)\right)=S_1\left(\frac{ez+f}{gz+h}\right)=\frac{a\frac{ez+f}{gz+h}+b}{c\frac{ez+f}{gz+h}+d}=\frac{(ae+gb)z+(af+bh)}{(ce+gd)z+(cf+dh)}$$

and

$$(ae+gb)(cf+dh)-(af+bh)(ce+gd)=(ad-vc)(eh-fg)\neq 0\,.$$

So, the mapping $S=S_1\circ S_2$ is Möbius transformation, i.e. $(\boldsymbol{M},\circ)$ is a groupoid.

a) *Associativity*. For all $S_1,S_2,S_3\in\boldsymbol{M}$ we have

$$S_1\circ(S_2\circ S_3)=(S_1\circ S_2)\circ S_3\,. \tag{3}$$

Namely, both sides of (3) are equal to the Möbius transformation $S_1\left(S_2\left(S_3(z)\right)\right)$. Therefore, $(\boldsymbol{M},\circ)$ is a semi-group.

b) *Existence of identity.* Obviously, the identity mapping $E(z)=z$ is Möbius transformation, that $E(z)=z$ is an identity in the semi-group $(\boldsymbol{M},\circ)$.

c) Thereby Theorem 9.3, it is true that each element of $(\boldsymbol{M},\circ)$ has its inverse.

So, statements a), b) and c) imply that $(\boldsymbol{M},\circ)$ is group. ■

9.5. Remark. The group $(\boldsymbol{M},\circ)$ is a non-abelian.

Indeed, for

$$S_1(z)=z+a,\ a\neq 0 \text{ and } S_2(z)=\frac{1}{z}$$

the following holds true

$$S_1\left(S_2(z)\right)=\frac{1}{z}+a \text{ and } S_2\left(S_1(z)\right)=\frac{1}{z+a}\text{, i.e. } S_1\circ S_2\neq S_2\circ S_1\,.$$

9.6. Theorem. If *S* is Möbius transformation, then *S* is a composition of elementary transformations in a complex plane.

Proof. Let

$$S(z)=\frac{az+b}{cz+d},\ ad-bc\neq 0$$

be an arbitrary Möbius transformation.

If $c=0$, then

$$S(z)=\frac{a}{d}z+\frac{b}{d}\,.$$

Therefore, if

$$S_1(z)=\frac{a}{d}z \text{ and } S_2(z)=z+\frac{b}{d}$$

we get $S=S_2\circ S_1$, i.e. *S* is a composition of elementary transformations.

If $c\neq 0$, then for

$$S_1(z)=z+\frac{d}{c},\ S_2(z)=\frac{1}{z},\ S_3(z)=\frac{bc-ad}{c}z,\ S_4(z)=z+\frac{a}{c}$$

we get $S=S_4\circ S_3\circ S_2\circ S_1$. Thus, also in this case *S* is composition of elementary transformations. ■

9.7. Example. Let $w=\frac{i-z}{i+z}$ be a given transformation. Determine the image of:

a) the real axis, b) the circle $|z|=1$

under the given transformation.

Solution. The given transformation can be rewritten as $w=-1+\frac{2i}{i+z}$.

a) The equation of the real axis is $z-\bar{z}=0$, and under the transformation $w_1=z+i$, it is mapped to the line $w_1-\overline{w_1}=2i$. Further, the line $w_1-\overline{w_1}=2i$ is mapped to the circle $\left|w_2+\frac{i}{2}\right|=\frac{1}{2}$ under $w_2=\frac{1}{w_1}$, and under the transformation $w_3=2iw_2$ the circle $\left|w_2+\frac{i}{2}\right|=\frac{1}{2}$ is mapped to the circle $|w_3-1|=1$. Finally, under the transformation $w=-1+w_3$ the circle $|w_3-1|=1$ is mapped to the circle $|w|=1$. The above means that the given transformation maps the real axis to the circle $|w|=1$.

b) The translation $w_1=z+i$ maps the circle $|z|=1$ to the circle $|w_1-i|=1$.

Further, the transformation $w_2=\frac{1}{w_1}$ maps the circle $|w_1-i|=1$ to the line $w_2-\overline{w_2}=-i$, and the transformation $w_3=2iw_2$ maps the line $w_2-\overline{w_2}=-i$ to the line $w_3+\overline{w_3}=2$. Finally, the transformation $w=-1+w_3$ maps the line $w_3+\overline{w_3}=2$ to the line $w+\bar{w}=0$. The above means that the given Möbius transformation maps the circle $|z|=1$ to the line $w+\bar{w}=0$. ■

9.8. Consider the Möbius transformation (1). The mapping $S_1(z)=-\frac{d}{c}+\frac{1}{\overline{z+\frac{d}{c}}}$ is an inversion with respect to the circle $\left|z+\frac{d}{c}\right|=1$, and the mapping $S_2(z)=p\bar{z}+q$, where

$$p=\frac{bc-ad}{c^2},\quad q=\frac{bc-ad}{c^2}\frac{\bar{d}}{\bar{c}}+\frac{a}{c}$$

is an indirect similarity. Further, since

$$\begin{aligned}S_2\left(S_1(z)\right)&=S_2\Big(-\frac{d}{c}+\frac{1}{\overline{z+\frac{d}{c}}}\Big)=\frac{bc-ad}{c^2}\cdot\overline{\left(-\frac{d}{c}+\frac{1}{\overline{z+\frac{d}{c}}}\right)}+\frac{bc-ad}{c^2}\frac{\bar{d}}{\bar{c}}+\frac{a}{c}\\&=\frac{bc-ad}{c^2}\cdot\left(-\frac{\bar{d}}{\bar{c}}+\frac{1}{z+\frac{d}{c}}\right)+\frac{bc-ad}{c^2}\frac{\bar{d}}{\bar{c}}+\frac{a}{c}\\&=-\frac{bc-ad}{c^2}\frac{\bar{d}}{\bar{c}}+\frac{bc-ad}{c^2}\frac{\bar{d}}{\bar{c}}+\frac{bc-ad}{c}\frac{1}{cz+d}+\frac{a}{c}\\&=\frac{bc-ad+a(cz+d)}{c(cz+d)}=\frac{az+b}{cz+d},\end{aligned}$$

it follows the validity of the following Theorem.

Theorem. The Möbius transformation (1) can be expressed as composition $S=S_2\circ S_1$ of the inversion $S_1(z)=-\frac{d}{c}+\frac{1}{\overline{z+\frac{d}{c}}}$ with respect to the circle $\left|z+\frac{d}{c}\right|=1$ and the indirect similarity $S_2(z)=p\bar{z}+q$, where

$$p=\frac{bc-ad}{c^2},\quad q=\frac{bc-ad}{c^2}\frac{\bar{d}}{\bar{c}}+\frac{a}{c}.\ ■$$

10. GEOMETRIC PROPERTIES OF A MÖBIUS TRANSFORMATION

10.1. Theorem. Arbitrary Möbius transformation maps each circle of $\mathbf{C}_\infty$ to a circle of $\mathbf{C}_\infty$.

Proof. According to Theorem 9.6, each Möbius transformation is a composition of elementary transformations in the complex plane and moreover the elementary transformation $S_2(z) = \frac{1}{z}$ given in the proof of Theorem 9.6 is composition of the inversion $I(z) = \frac{1}{\bar{z}}$ and the reflection $S(z) = \bar{z}$. Then, the validity of the Theorem is implied directly by the previously proved properties of the elementary transformations in the complex plane. ■

10.2. Applying the properties of elementary transformations in a complex plane, analogously to the proof of Theorem 8.19, can be proved the following very important Theorem.

Theorem. Arbitrary Möbius transformation preserves the angle between circles in the extended complex plane $\mathbf{C}_\infty$. ■

10.3. Definition. Consider the circle *K*(*O*,*R*). The points *M* and *M** are said to be *symmetric with respect to the circle* *K*, if $\mathrm{I}(M) = M^*$, where I is an inversion determined by the circle *K*.

10.4. Before discussing the properties of the symmetric points with respect to a circle, and related to the Möbius transformation we will give the following Lemma, which characterizes the symmetric points *M* and *M** with respect to the circle *K*(*O*,*R*).

Lemma. The points *M* and *M** are symmetric with respect to the circle *K*(*O*,*R*) if and only if each circle γ through these points meets orthogonally the circle *K*(*O*,*R*).

Proof. Let the points *M* and *M** be symmetric with respect to the circle *K*(*O*,*R*) (figure 20).

Consider the circle γ through the points *M* and *M** and $K \cap \gamma = \{P\}$. According to Theorem 8.14, the circle γ under inversion I, determined by the circle *K*, is mapped to a circle γ_1, which passes through the points *M*, *M** and *P*. Therefore, the circles γ and γ_1 coincide, i.e. the circle γ is fixed under the inversion I. Now, the validity of the given Theorem is directly implicated by Theorem 8.18.

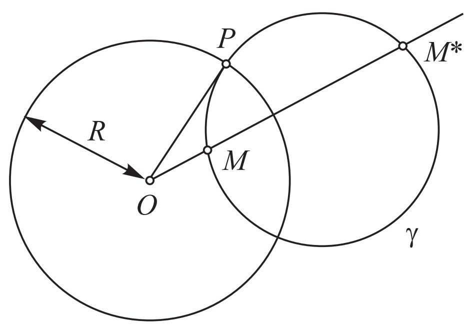

Figure 20

Conversely, let each circle γ, which passes through the points *M* and *M**, meets orthogonally the circle K (figure 20). The Theorem 8.18 implies that the circle γ is fixed under the inversion I, determined by the circle K. Then the line (in the extended complex plane viewed as a circle), which passes through the points M and M^*, also meets the circle K orthogonally i.e. the line passes through the center O of the circle K. But, γ is fixed under the inversion I, and therefore $\mathrm{I}(M) = M^*$, i.e. the points M and M^* are symmetric with respect to the circle K. ■

10.5. Theorem. A Möbius transformation maps a pair of symmetric points with respect to a circle, to a pair of symmetric points with respect to the image of that circle.

Proof. Let the points z and z^* be symmetric with respect to the circle K and let $w = S(z)$ be an arbitrary Möbius transformation. According to Theorem 10.1 the image $K^{\sim}$ of the circle K is a circle. We have to prove that the points w and w^* are symmetric with respect to $K^{\sim}$. According to Lemma 10.4 it is sufficient to prove that each circle $\gamma^{\sim}$, which passes through the points w and w^*, crosses $K^{\sim}$ at right angle.

The inverse image of the circle $\gamma^{\sim}$ under the Möbius transformation $w = S(z)$ is a circle which passes through the points z and z^*. This circle crosses the circle K at right angle. Therefore, $\gamma^{\sim}$ crosses $K^{\sim}$ at right angle, because Theorem 10.2 states that the Möbius transformation preserves the angle between intersecting circles at any point in the extended complex plane. ■

10.6. Further, we will prove one important Theorem about Möbius transformation.

Theorem. A Möbius transformation $S: \mathbf{C}_\infty \to \mathbf{C}_\infty$ maps the unit circle $|z| \le 1$ to the unit circle $|w| \le 1$ if and only if

$$w = S(z) = e^{i\varphi} \frac{z-v}{1-\bar{v}z}, \quad v \in \mathbf{C}, \quad 0 \le \varphi < 2\pi, \quad |v| < 1. \tag{1}$$

Proof. Let $z \in \mathbf{C}$, $|z| \le 1$. Likewise, the following holds true:

$$|w|^2 = w\bar{w} = e^{i\varphi} \frac{z-v}{1-\bar{v}z} e^{-i\varphi} \frac{\bar{z}-\bar{v}}{1-\bar{z}v} = \frac{|z-v|^2}{|1-\bar{v}z|^2} \le 1,$$

i.e. the Möbius transformation (1) maps the unit circle $|z| \le 1$ to the unit circle $|w| \le 1$.

Conversely, let $S: \mathbf{C}_\infty \to \mathbf{C}_\infty$ be Möbius transformation such that it maps the unit circle $|z| \le 1$ to the unit circle $|w| \le 1$ and let us suppose that there exists such a point v, $v \ne 0$, $|v| < 1$ that is mapped to the point $w = 0$. The point symmetric to null with respect to the circle $|w| = 1$ is the infinity distanced point. According to Theorem 10.5 it follows that $w = \infty$ when $z = \frac{1}{\bar{v}}$, and therefore the required Möbius transformation is the following

$$w = k\frac{z-v}{z-\frac{1}{\bar{v}}},$$

where k is a constant. The latter can be transformed and rewritten as it follows

$$w = -k\bar{v}\frac{z-v}{1-\bar{v}z} = k'\frac{z-v}{1-\bar{v}z}. \qquad (2)$$

Since $\bar{z} = \frac{1}{z}$ when $|z| = 1$ we get

$$|1-\bar{v}z| = \left|\frac{1}{z} - \bar{v}\right| = |\bar{z} - \bar{v}| = |z - v|.$$

But, the circle $|z| = 1$ is mapped to the circle $|w| = 1$, and therefore $|k'| = 1$, i.e. $k' = e^{i\varphi}$, for some $0 \le \varphi < 2\pi$, i.e. the formula (2) transforms to (1). Clearly, likewise the formula (2) holds true when $v = 0$. ■

10.7. Definition. A point z is said to be a *fixed point* under the Möbius transformation

$$S(z) = \frac{az+b}{cz+d} \text{ if } z = S(z), \text{ i.e. } z = \frac{az+b}{cz+d}.$$

Clearly, z is a fixed point under the Möbius transformation if $S(z) = \frac{az+b}{cz+d}$

$$cz^2 + (d-a)z - b = 0. \qquad (3)$$

If $c \ne 0$, then the fixed points are the following:

$$z_{1/2} = \frac{a-d \pm \sqrt{(a-d)^2 + 4bc}}{2c}. \qquad (4)$$

If $c = 0$, then the fixed points are $z_1 = \frac{b}{d-a}$ and $z_2 = \infty$. Further, if $b = c = 0$ and $a = d$, then the Möbius transformation is the identity mapping $S(z) = z$ and therefore each point of $\mathbf{C}_\infty$ is fixed point.

By (4), if $(a-d)^2 + 4bc = 0$, then $z_1 = z_2$. The last means that we have a repeated or a double fixed point i.e. the two fixed points coincide. When $c = 0$, the condition for repeated points implies that $a = d$, and in this case we get that $z = \infty$ is a double fixed point for the translation $S(z) = z + \frac{b}{d}$.

10.8. Comment. When defining the Möbius transformation

$$S(z) = \frac{az+b}{cz+d}, \ ad - bc \ne 0,$$

four complex numbers a, b, c and $d$are used. But, one of c or d differs from 0, and therefore if we divide both, the numerator and the denominator, by this number, we get that the Möbius transformation can be expressed using three coefficients. Therefore, it is naturally to expect that the images of three given points determine a unique Möbius transformation. The following Theorem confirms our assumtion..

10.9. Theorem. There is a unique Möbius transformation *S*, such that the points
$$z_1,\, z_2,\, z_3 \quad (z_i \neq z_j, i \neq j)$$
under such transformation, are mapped to the points $w_1,\, w_2,\, w_3 \ (w_i \neq w_j, i \neq j)$, respectively.

Proof. *Existence***.** The mappings S_1 and S_2 defined by
$$S_1(z) = \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-z_1)} \text{ and } S_1(w) = \frac{(w-w_1)(w_3-w_2)}{(w-w_2)(w_3-w_1)}, \tag{5}$$
map the points $z_1,\, z_2,\, z_3$ in the plane *z* and the points $w_1,\, w_2,\, w_3$ in the plane *w*, to the points 0, ∞, 1 in the plane ς , respectively. Finally, the mapping
$$S = S_2^{-1} \circ S_1 \tag{6}$$
which is determinedfrom the plane *z* to the plane *w*, $S(z) = w$, as
$$\frac{z-z_1}{z-z_2} \cdot \frac{z_3-z_2}{z_3-z_1} = \frac{w-w_1}{w-w_2} \cdot \frac{w_3-w_2}{w_3-w_1} \tag{7}$$
is Möbius transformation, which maps the points $z_1,\, z_2,\, z_3$ to the points $w_1,\, w_2,\, w_3$, respectively.

*Uniqueness***.** Let λ, $\lambda(z_i) = w_i$, $i = 1,2,3$ be an arbitrary Möbius transformation. Consider the mapping $\mu = S_2 \circ \lambda \circ S_1^{-1}$, where S_1 and S_2 are mappings defined by (5). Clearly, μ is Möbius transformation and the points 0, ∞, 1 are fixed points under this transformation. Since the condition $\mu(\infty) = \infty$ it follows that $\mu(\varsigma) = \alpha\varsigma + \beta$. The condition $\mu(0) = 0$ implies that $\beta = 0$, and the condition $\mu(1) = 1$ implies that $\alpha = 1$. Therefore, $\mu(\varsigma) = \varsigma$, i.e.
$$S_2 \circ \lambda \circ S_1^{-1} = E\,.$$
Since $(\boldsymbol{M}, \circ)$ is a group, we get $\lambda = S_2^{-1} \circ S_1$, i.e. $\lambda = S$. ■

10.10. Remark. In the equality (7), each of the points z_i and w_i appears exactly twice, once in the numerator and once in the denominator. It is easy to prove that the equality holds true when one of the points z_i or w_i (either one z_i and one w_i) is the infinity point. Then, it is necessary the numerator and denominator, where this point appears, to be replaced by 1. For example, if $z_2 = w_1 = \infty$, then the formula (7) can be transformed and rewritten as
$$\frac{z-z_1}{1} \cdot \frac{1}{z_3-z_1} = \frac{1}{w-w_2} \cdot \frac{w_3-w_2}{1}\,.$$
Therefore, Theorem 10.9 holds true for any point in the extended complex plane $\mathbf{C}_\infty$.

10.11. Corollary. It exists a unique Möbius transformation *S*, such that it maps the points $z_1,\, z_2,\, z_3,\, z_4 \ (z_i \neq z_j, i \neq j)$ to the points $w_1,\, w_2,\, w_3,\, w_4 \ (w_i \neq w_j, i \neq j)$ if and only if

$$\frac{z_4-z_1}{z_4-z_2}\cdot\frac{z_3-z_2}{z_3-z_1}=\frac{w_4-w_1}{w_4-w_2}\cdot\frac{w_3-w_2}{w_3-w_1}. \tag{7'}$$

Proof. Let the equality (7') be satisfied and let S be such the Möbius transformation that maps the points z_1, z_2, z_3 $(z_i \neq z_j, i \neq j)$ to the points w_1, w_2, w_3 $(w_i \neq w_j, i \neq j)$, respectively. Then the Möbius transformation is defined by (7), and therefore

$$\frac{z_4-z_1}{z_4-z_2}\cdot\frac{z_3-z_2}{z_3-z_1}=\frac{S(z_4)-w_1}{S(z_4)-w_2}\cdot\frac{w_3-w_2}{w_3-w_1}.$$

Thus,

$$\frac{S(z_4)-w_1}{S(z_4)-w_2}\cdot\frac{w_3-w_2}{w_3-w_1}=\frac{w_4-w_1}{w_4-w_2}\cdot\frac{w_3-w_2}{w_3-w_1},$$

i.e. $S(z_4)=w_4$.

If there exists a Möbius transformation with the stated properties, then that transformation maps the points z_1, z_2, z_3 $(z_i \neq z_j, i \neq j)$ to the points w_1, w_2, w_3 $(w_i \neq w_j, i \neq j)$, and therefore the Möbius transformation is as (7), and $S(z_4)=w_4$. So, (7') holds true. ■

10.12. Remark. Theorems 10.9 and 10.1 imply that each circle K in $\mathbf{C}_\infty$ can be mapped to a circle K^* in $\mathbf{C}_\infty$. It is sufficient to map any three distinct points of K to any three distinct points of K^*.

10.13. Example. Determine the Möbius transformation such that it maps the points $-1,\ i,\ 1+i$ to the points

a) $0,\ 2i,\ 1-i$; б) $i,\ \infty,\ 1$,

respectively.

Solution. According to Theorem 10.9 and Remark 10.10 the required Möbius transformations are

a) $w=\frac{-2i(z+1)}{-4z-1-5i}$, б) $w=\frac{(1+2i)z+6-3i}{5(z-i)}$. ■

10.14. Remark. The proof of Theorem 10.9 implies that each Möbius transformation S have up to two fixed points z_1, z_2, i.e. such points z_1, z_2, so that $S(z_1)=z_1$, $S(z_2)=z_2$, when $S \neq E$. The Möbius transformation with two fixed points z_1, z_2 is determined by

$$\frac{w-z_1}{w-z_2}=A\frac{z-z_1}{z-z_2},\quad z_1,z_2\neq\infty \tag{8}$$

or

$$w-z_1=A_1(z-z_1),\quad z_2=\infty. \tag{9}$$

The coefficients A and A_1 are given by

$$A = \frac{z_3 - z_2}{z_3 - z_1} \cdot \frac{w_3 - z_1}{w_3 - z_2}, \quad A = \frac{w_3 - z_1}{w_3 - z_2}, \tag{10}$$

and they do not depend on the choice of z_3 when $w_3 = \frac{az_3 + b}{cz_3 + d}$. It is easy to express the coefficients A and A_1 in terms of a, b, c and d, if (10) holds true and we set $w_3 = \frac{b}{d}$ when $z_3 = 0$.

10.15. At the end of this section we will provide a proof of the following Theorem.

Theorem. The set of Möbius transformations $\boldsymbol{F}$ such that the unit circle $|z| \le 1$ they mapped to the unit circle $|w| \le 1$ is a subgroup of the group of Möbius transformations.

Proof. Let $S_1, S_2 \in \boldsymbol{F}$. Then, Theorem 10.6 implies that

$$S_1(z) = e^{i\alpha} \frac{z - a}{1 - \bar{a}z}, \ |a| < 1 \text{ and } S_2(z) = e^{i\beta} \frac{z - b}{1 - \bar{b}z}, \ |b| < 1,$$

therefore,

$$S_1(S_2(z)) = e^{i(\alpha - \beta)} \frac{\overline{e^{i\beta} + a\bar{b}}}{e^{i\beta} + a\bar{b}} \frac{z - \frac{e^{i\beta}b + a}{e^{i\beta} + a\bar{b}}}{1 - \left(\overline{\frac{e^{i\beta}b + a}{e^{i\beta} + a\bar{b}}}\right) z},$$

where

$$\left| \frac{e^{i\beta}b + a}{e^{i\beta} + a\bar{b}} \right| = \left| \frac{b + ae^{-i\beta}}{1 + \bar{a}e^{i\beta}\bar{b}} \right| < 1 \text{ and } \left| e^{i(\alpha - \beta)} \frac{\overline{e^{i\beta} + a\bar{b}}}{e^{i\beta} + a\bar{b}} \right| = 1.$$

Applying Theorem 10.6 once again, we get that $S_1 \circ S_2 \in \boldsymbol{F}$. So, the set $\boldsymbol{F}$ is closed under the composition of mappings.

Let $S_1, S_2, S_3 \in \boldsymbol{F}$. Then $S_1, S_2, S_3 \in \boldsymbol{M}$, therefore

$$S_1 \circ (S_2 \circ S_3) = (S_1 \circ S_2) \circ S_3$$

i.e. the associative law holds true.

Letting $v = 0$ and $\varphi = 0$ in Theorem 10.6 we get

$$\boldsymbol{F} \ni E(z) = z = e^{i \cdot 0} \frac{z - 0}{1 - z \cdot \bar{0}}.$$

Let $S \in \boldsymbol{F}$. Then, $S(z) = e^{i\varphi} \frac{z - v}{1 - \bar{v}z}$, $|v| < 1$. Consider the transformation

$$S_1(z) = e^{i(-\varphi)} \frac{z - (-ve^{i\varphi})}{1 - (\overline{-ve^{i\varphi}})z}, \ |-ve^{i\varphi}| < 1.$$

Clearly, $S_1 \in \boldsymbol{F}$ and the following holds true

$$S(S_1(z)) = S_1(S(z)), \text{ i.e. } S^{-1} = S_1 \in \boldsymbol{F}. \ \blacksquare$$

CHAPTER III
GEOMETRY OF CIRCLE AND TRIANGLE

1. CENTRAL AND INSCRIBED ANGLES

1.1. Definition. An angle whose vertex coincides with the center of a given circle K is called a *central angle*.

1.2. Theorem. If two central angles in a same circle are congruent, then their corresponding arcs are congruent, too.

Proof. Without loss of generality, we consider K as a unit circle. Let $\angle AOB = \angle COD$ (figure 1) and the affixes of points A, B, C and D be a, b, c and d, respectively. Therefore

$$\frac{b}{a} = \frac{d}{c}, \text{ i.e. } \frac{c}{a} = \frac{d}{b}.$$

Since

$$S(a) = \frac{c}{a}a = c \text{ and } S(b) = \frac{c}{a}b = \frac{d}{b}b = d$$

we get that under the mapping $S(z) = \frac{c}{a}z$, which in fact is a rotation around $\angle AOC$, the point A maps to a point C, and B to D. Therefore, the arc AB maps to the arc CD. ■

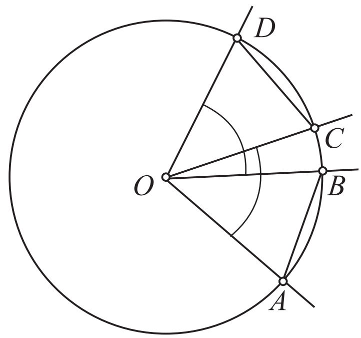

Figure 1

1.3. Similarly the reverse theorem of theorem 1.2 can be proved. The proof is left as an exercise.

Theorem. If two arcs in a same circle are congruent, then their corresponding central angles are congruent, too. ■

1.4. Definition. An angle whose vertex is on a given circle K, and its rays meet the circle is called an *inscribed angle*.

1.5. Theorem. The size of an inscribed angle is half of the size of its corresponding central angle.

Proof. Without loss of generality, we consider K as a unit circle. Let consider the arc $\overset{\frown}{AB}$ where points A and B has affixes 1 and $e^{i\varphi}$, $\varphi \in (0,\pi)$, respectively. Then, $\angle AOB = \varphi$. Let M be a point of the complementary arc of the circle (figure 2), i.e.it has affix $e^{i\theta}$, $\theta \in (\varphi, 2\pi)$.

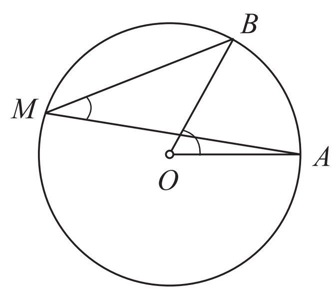

Figure 2

Then,

$$\angle AMB = \arg\frac{e^{i\varphi}-e^{i\theta}}{1-e^{i\theta}} = \arg\frac{e^{i\frac{\varphi}{2}}(e^{i\frac{\varphi}{2}}-e^{i(\theta-\frac{\varphi}{2})})}{-2ie^{i\frac{\theta}{2}}\sin\frac{\theta}{2}}$$

$$= \arg\frac{e^{i\frac{\varphi}{2}}(e^{i(\frac{\theta}{2}-\frac{\varphi}{2})}-e^{-i(\frac{\theta}{2}-\frac{\varphi}{2})})}{2i\sin\frac{\theta}{2}}$$

$$= \arg e^{i\frac{\varphi}{2}}\frac{\sin(\frac{\theta}{2}-\frac{\varphi}{2})}{\sin\frac{\theta}{2}} = \arg e^{i\frac{\varphi}{2}} = \frac{\varphi}{2}$$

i.e. $\angle AMB = \frac{\angle AOB}{2}$. ■

1.6. Corollary. All inscribed angles with congruent arcs in a given circle K, are congruent.

Proof. The proof is directly implied by Theorem 1.5. ■

1.7. Corollary (Thales' Theorem). Each inscribed angle on a diameter of a circle is a right angle.

Proof. The proof is directly implied by Theorem 1.5. ■

1.8. Remark. In paragraph II 8.7 during the construction of the image of an arbitrary point under inversion, we have drawn a tangent to a circle at point which is out of that circle. The effective construction of the tangent is implied by the Thales' Theorem.

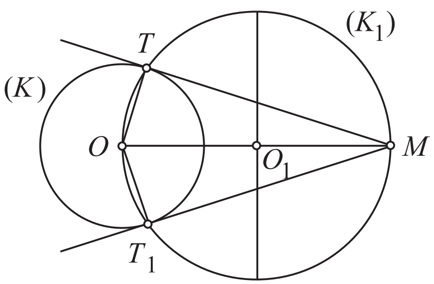

Figure 3

Let a circle $K(O,R)$ and a point M which is outer for the circle K be given. The tangent at M to K is constructed with the following procedure (figure 3).

a) we construct the midpoint O_1 of the line segment OM,

b) we construct a circle $K_1(O_1, \overline{MO_1})$,

c) we find the intersecting points $K \cap K_1 = \{T, T_1\}$ and

d) we draw lines MT and MT_1.

1.9. The above discussed line segments MT and MT_1, are called tangent segments to the circle $K(O,R)$ at point M. The tangent segments satisfy the following Theorem.

Theorem. Let M be a point out of the circle $K(O,R)$. If MT and MT_1 are tangent segments drawn at point M to the circle K, then $\overline{MT} = \overline{MT_1}$.

Proof. Without loss of generality we consider K as a unit circle (why?). Let the affixes of points T and T_1 be t and t_1, respectively. The lines MT and MT_1 are tangents, and since Remark II 3.13, their equations are

$$z + t^2\overline{z} = 2t \text{ and } z + t_1^2\overline{z} = 2t_1$$

and affix of M is $m = \frac{2tt_1}{t+t_1}$. Finally,

$$\overline{MT} = |m-t| = \left|\frac{2tt_1}{t+t_1} - t\right| = \left|\frac{t(t_1-t)}{t+t_1}\right| = \left|\frac{t_1(t-t_1)}{t+t_1}\right| = \left|\frac{2tt_1}{t+t_1} - t_1\right| = |m-t_1| = \overline{MT}1. \blacksquare$$

1.10. Theorem. The angle between a chord AB and the tangent to the circle (t) drawn at one of the points A or B is congruent to the inscribed angle on the chord AB.

Proof. Without loss of generality we can consider the unit circle $K(O,R)$. Let affixes of points A and B be a and b, respectively. The equation of the tangent (t) drawn at the point B is as following

$$z = -b^2\overline{z} + 2b,$$

And the equation of a line AB is

$$z = -ab\overline{z} + a + b.$$

Therefore, the angle β between line AB and the tangent (t) satisfies the following $e^{2i\beta} = \frac{b}{a}$. Since the proof of Theorem II 1.4, the inscribed angle α satisfies that

$$e^{2i\alpha} = \frac{b}{a} = e^{2i\beta},$$

So, $\alpha = \beta$. $\blacksquare$

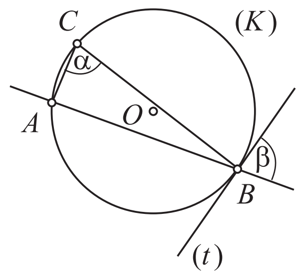

Figure 4

2. POWER OF A POINT WITH RESPECT TO A CIRCLE

2.1. Let $K(O,R)$ be a given circle and M be an arbitrary point of a plane. Through M we draw an arbitrary line (p) which meets the circle K at points A and B. We will prove that the product $\overline{MA}\cdot\overline{MB}$ does not depend on the choice of the line (p).

If the point M is on a circle K, then M is one of the points A or B. Therefore one of $\overline{MA}$ or $\overline{MB}$ is null. So, $\overline{MA}\cdot\overline{MB} = 0$.

Let point M be out of the circle K (figure 5) and (t) be one of the tangents to K drawn at M, and T be the tangent point. The circle $K_1(M, \overline{MT})$ orthogonally crosses the circle K, therefore K is fixed under the inversion defined by the circle K_1. Therefore, from the definition of inversion we have that

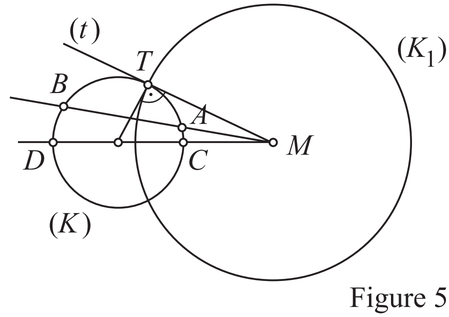

Figure 5

$$\overline{MA}\cdot\overline{MB}=\overline{MD}\cdot\overline{MC}=(\overline{MO}-R)(\overline{MO}+R)=\overline{MO}^2-R^2 .$$
In this case, since the properties of inversion (Theorem 8.5), it is true that
$$\overline{MA}\cdot\overline{MB}=\overline{MT}^2 .$$

Let the point M be an inner point for the circle K and let (p) be an arbitrary line which passes through M. Without loss of generality we assume that the center of the circle coincides with the origin. Then, if m is an affix of M, the equation of line (p) is the following $z-m=\eta(\overline{z}-\overline{m})$, and the equation of the circle is the following $z\overline{z}=R^2$. From the equation of (p) we express $\overline{z}$, and when we substitute in the equation of the circle we obtain a quadratic equation as following
$$\overline{\eta}z^2+(\overline{m}-\overline{\eta}m)z-R^2=0$$
whose solutions
$$z_{1/2}=\frac{-\overline{m}+\overline{\eta}m\pm\sqrt{(\overline{m}-\overline{\eta}m)^2+4\overline{\eta}R^2}}{2\overline{\eta}}$$
are the affixes of the points A and B, points of intersection between the line and the circle. Due to this,
$$\overline{MA}\cdot\overline{MB}=|m-z_1|\cdot|m-z_2|=|m\overline{m}-R^2|=R^2-\overline{MO}^2 .$$
The arbitrariness of (p) implies that the product $\overline{MA}\cdot\overline{MB}$ does not depend on the choice of the line (p) through the point M.

The already stated implies that the product $\overline{MA}\cdot\overline{MB}$ does not depend on the choice of the line (p) through M. It depends only on the length of the radius R of a circle K and on a distance $d=\overline{MO}$ between the point M and the center O of the circle K. The value (the real number) d^2-R^2 is called as *power of the point M* with respect to the circle K. Clearly, if M is a point on K, then the power is 0, furthermore if M is point outside the circle K, then the power is a positive real number, and if M is within the circle K, then the power is a negative real number.

2.2. Definition. Let $K_1(O_1,R_1)$ and $K_2(O_2,R_2)$ be given circles and let H be homothety with center S so that $\mathrm{H}(K_1)=K_2$. If (p) is line such that it passes through S and meets the circles K_1 and K_2 at points P_1, P_2 and Q_1, Q_2 respectively and if $\mathrm{H}(P_1)=Q_1$, $\mathrm{H}(P_2)=Q_2$, then the points P_1 and Q_2 (P_2 and Q_1) are said to be *antihomothetic* (figure 6).

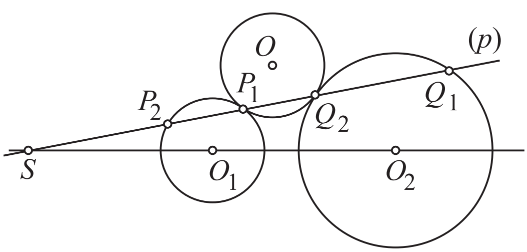

Figure 6

If the circle K touches the circles K_1 and K_2, both internally or externally, then we shall say *K touches K_1 and K_2 in a same way*, and if it touches one internally and the other one externally, then we shall say that *K touches K_1 and K_2 in a different way*.

2.3. Lemma. The product of the distances between the center of homothety of two circles and two antihomothetic points is a constant value.

Proof. Let S be external center of a homothety H of the circles $K_1(O_1,R_1)$ and $K_2(O_2,R_2)$, ($R_1 \neq R_2$) and let points P_1 and Q_2 (P_2 and Q_1) be antihomothetic (figure 6). Then, the homothety ratio is

$$a = \frac{\overline{SQ_1}}{\overline{SP_1}} = \frac{\overline{SQ_2}}{\overline{SP_2}}$$

So,

$$\overline{SP_1} \cdot \overline{SQ_2} = \overline{SP_1} \cdot \overline{SP_2} \cdot \frac{\overline{SQ_2}}{\overline{SP_2}} = \overline{SP_1} \cdot \overline{SP_2} \cdot a = \text{const}$$

Thus, $\overline{SP_1} \cdot \overline{SP_2}$ is a power of the point S with respect to the circle K_1. ■

2.4. Lemma. Let $K(O,R)$ be such a circle that touches the circles $K_1(O_1,R_1)$ and $K_2(O_2,R_2)$. Then,

a) if K touches K_1 and K_2 in a same way, then the points where K touches K_1 and K_2 are antihomothetic with respect to the external center of similarity S for circles K_1 and K_2,

b) if K touches K_1 and K_2 in a different way, then the points where K touches K_1 and K_2 are antihomothetic with respect to the internal center of similarity S for circles K_1 and K_2,

Proof. b) Let a circle K touches the circles K_1 and K_2 at points P_1 and Q_2 in a different way, respectively (figure 7) and let the affixes of O, O_1, O_2, P_1 and Q_2 be c, c_1, c_2, p and q, respectively. Then the affix of internal center of homothety is

$$s = \frac{R_1c_2 + R_2c_1}{R_1 + R_2}.$$

The points O, O_1, P_1 are collinear and it holds true that $\overline{O_1P_1} = R_1$, therefore the following equalities are satisfied

$$p - c_1 = \frac{c_1 - c}{\overline{c_1} - \overline{c}}(\overline{p} - \overline{c_1})$$

and

$$(p - c_1)(\overline{p} - \overline{c_1}) = R_1^2$$

By reducing we get

$$(p - c_1)^2 = \frac{c_1 - c}{\overline{c_1} - \overline{c}} R_1^2,$$

i.e.

$$(p - c_1)^2 = \frac{(c_1 - c)^2}{|c_1 - c|^2} R_1^2.$$

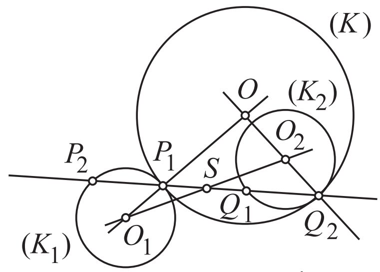

Figure 7

Thus, the affix p of the point P_1 is as following

$$p = c_1 - \frac{c_1 - c}{|c_1 - c|} R_1 = c_1 - \frac{c_1 - c}{R + R_1} R_1.$$

Analogously, the affix q of point Q_2 is as following

$$q = c_2 + \frac{c_2-c}{|c_2-c|}R_2 = c_2 + \frac{c_2-c}{R-R_2}R_2 .$$

Finally,

$$\frac{s-p}{s-q} = -\frac{R_1(R-R_2)}{R_2(R+R_1)} \in \mathbf{R} ,$$

i.e. the points S, P_1 and Q_2 are collinear. Due to this, the points P_1 and Q_2 are antihomothetic.

The statement a) can be proved analogously. The details are left to be proven as an exercise. ■

2.5. Lemma. Each center of homothety of circles $K_1(O_1,R_1)$ and $K_2(O_2,R_2)$ has equal power with respect to each circle which touches the circles K_1 and K_2 .

Proof. Since Lemma 2.4, if points where *K* touches the circles K_1 and K_2 are P_1 and Q_2 respectively, then P_1 and Q_2 are antihomothetic points (figure 23 and figure 24). If *S* is homothety center, then due to Lemma 2.3 the product $\overline{SP_1} \cdot \overline{SQ_2}$ is a constant value, and that is actually the power of a point *S* with respect to the circle *K*. ■

2.6. Example. Construct a circle such that it runs through the points *A* and *B* and touches the line (*c*).

Solution. We will consider the most general case when the points *A* and *B* are on a same semi-plane with respect to the line (*c*) and furthermore, the lines (*c*) and *AB* are not parallel to each other (figure 8).

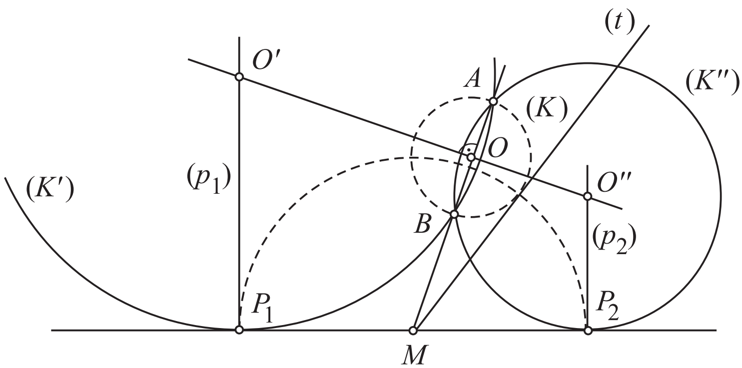

Figure 8

The power of the point $M = AB \cap (c)$ with respect to the arbitrary circle which crosses through the points *A* and *B* is $\overline{MA} \cdot \overline{MB}$. If $K\left(O, \frac{\overline{AB}}{2}\right)$ and if (*t*) is a tangent to (*K*) drawn at the point *M*, then $\overline{MT}^2$ is a power of *M* with respect to (*K*), and therefore it is actually the power of *M* with respect to the required circle. That means, the points P_1 and P_2 where the line (*c*) meets the circle $(M, \overline{MT})$ in fact are points where line (*c*) meets the circles which passes through the points *A* and *B*. If (p_1) and (p_2) are perpendicular to (c) and P_1 and P_2 respectively, are foots of perpendicular and if (s) is a bisector of the line segment AB, then $(p_1)\cap(s)=O'$ and $(p_2)\cap(s)=O''$ are centers of required circles. Clearly, the given problem has two solutions. ■

3. RADICAL AXIS AND RADICAL CENTER

3.1. Let two circles $K'(O',R')$ and $K''(O'',R'')$ be given. Let's determine the locus Γ such that it has an equal power with respect to the circles (K') and (K'').

Let affixes of the centers O' and O'' be c_1 and c_2, respectively. Clearly, $M\in\Gamma$ if and only if

$$\overline{MO'}^2-R'^2=\overline{MO''}^2-R''^2,$$

i.e. if and only if

$$|z-c_1|^2-R'^2=|z-c_2|^2-R''^2.$$

The last equality is equivalent to

$$z=-\frac{c_2-c_1}{\overline{c_2}-\overline{c_1}}\overline{z}+\frac{|c_2|^2-|c_1|^2+R'^2-R''^2}{\overline{c_2}-\overline{c_1}}.$$

Finally, the required locus is a line perpendicular to $O'O''$ (figure 9). This line shall be called the *radical axis* of K_1 and K_2.

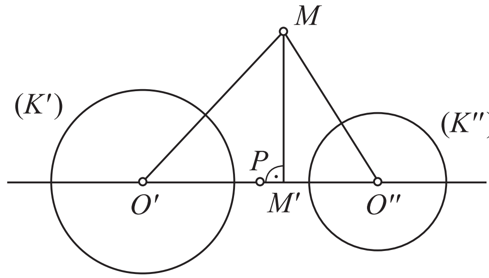

Figure 9

3.2. Lemma. A radical axis of two circles with no common points and also a part of a radical axis of two crossing circles (outer for the intersecting circles) is a locus of the centres of circles which orthogonally crosses the given circles.

Proof. If $K(O,R)$ orthogonally crosses the circles $K'(O',R')$ and $K''(O'',R'')$, then the tangents at O to (K') and (K'') are congruent, i.e. O has an equal power with respect to (K') and (K''). Hence, O is on the radical axis of (K') and (K''), figure 10.

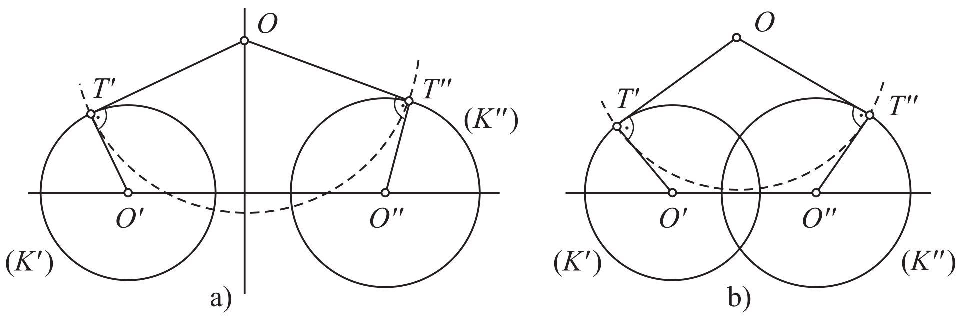

Figure 10

Let $K(O,R)$ be a circle centred at a point on the radical axis of the circles (K') and (K'') and

$$R = \overline{T'O} = \overline{T''O}.$$

Therefore, OT' and OT'' are tangents to (K') and (K''), respectively, the circles (K) and (K'), and also (K) and (K'') cross at right angle, i.e. (K) orthogonally crosses (K') and (K''). ■

3.3. Definition. The circle $K(O,R)$ *halves* the circle $K'(O',R')$, if (K) intersects (K') at two diametric opposite points.

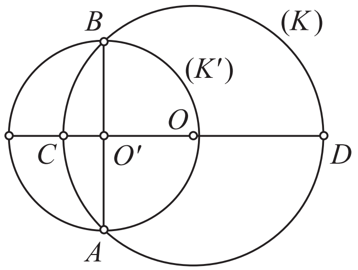

Figure 11

Let us state that if the circle $K(O,R)$ crosses the circle $K'(O',R')$ at two diametric opposite points A and B (figure 11), then the power of the point O' with respect to the circle $K(O,R)$ implies that

$$\overline{O'C} \cdot \overline{O'D} = \overline{O_1 A}^2$$

Therefore, when writing $d = \overline{O'O}$ we get that

$$(R-d)(R+d) = R'^2, \text{ i.e. } R^2 = R'^2 + d^2.$$

3.4. Lemma. The inner part of a radical axis of intersecting circles $K'(O',R')$ and $K''(O'',R'')$ is a locus of the centers of circles (K), such that both (K') and (K'') halve (K).

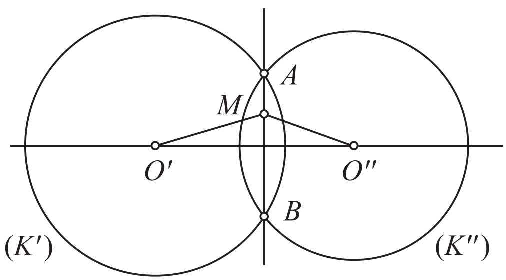

Figure 12

Proof. Clearly, if M is a center of circle which is halved by two given circles (K') and (K''), then M must be an inner point for circles (K') and (K'') (figure 12), therefore, such a circle exists only if (K') and (K'') intersect each other. The comment after definition 3.3 implies that

$$R^2 = R'^2 - d'^2 \text{ and } R^2 = R''^2 - d''^2,$$

thus

$$R'^2 - d'^2 = R''^2 - d''^2,$$

i.e. M has the equal power with respect to the both circles (K') and (K''), i.e. M is placed on the inner part of the radical axis of (K') and (K''). ■

3.5. Let be given three circles $K_i(O_i,R_i)$, $i=1,2,3$. We will determine the locus of points in a plane which has an equal power with respect to the three given circles. Let p_{12}, p_{23}, p_{13} be the radical axis of (K_1) and (K_2), (K_2) and (K_3), (K_3) and (K_1), respectively. Hence, if there is any point P with an equal power with respect to the circles (K_1), (K_2) and (K_3), then that point must be on a radical axis (p_{12}) and (p_{23}). Two cases are possible.

a) If centers O_i, $i=1,2,3$ of the circles are not collinear (figure 13), then the radical axis (p_{12}) and (p_{23}) intersect each other. The point $P = (p_{12}) \cap (p_{23})$ has an equal power with respect to the circles (K_i), $i=1,2,3$. Therefore the radical axis (p_{13})

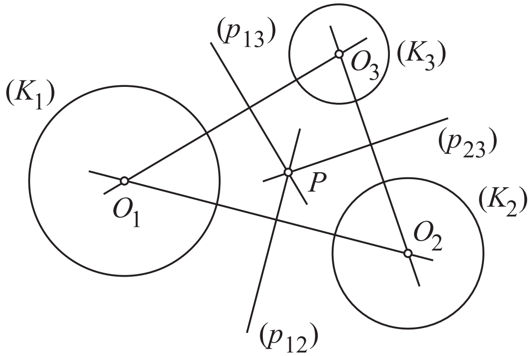

Figure 13

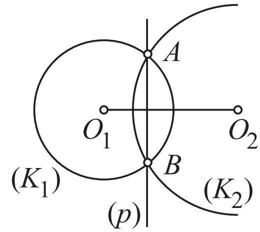

Figure 14

which passes through the point *P*, a point with equal power with respect to the circles (K_i), $i=1,2,3$. This point is to be called a *radical center* of (K_i), $i=1,2,3$.

b) If the centers O_i, $i=1,2,3$, of the circles are collinear, then the racial axes are parallel, and furthermore each of them is either different or coincide. When the first case is satisfied, there is no any point with required property, when the second case is satisfied the required locus is a line.

3.6. The already stated implies the effective construction of radical axis of two circles. Namely, if circles (K_1) and (K_2) intersect at *A* and *B*, then the radical axis is a straight line *AB* (figure 14), and if circles touch at *T*, then the radical axis is a common tangent to (K_1) and (K_2) at *T* (figure 15).

If circles (K_1) and (K_2) do not intersect, then we construct an arbitrary circle (K_3) such that it intersects both (K_1) and (K_2). The intersecting point of radical axes (p_{13}) and (p_{23}) is a radical center *P* of the circles (K_i), $i=1,2,3$, and therefore the radical axis is a line which passes through *P* and is perpendicular to O_1O_2 (figure 16).

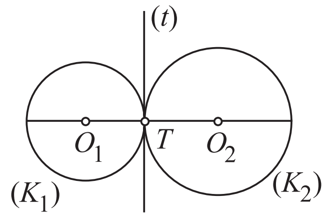

Figure 15

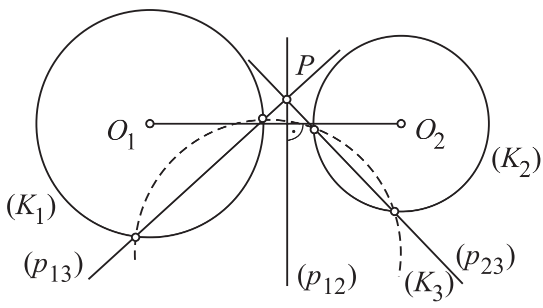

Figure 16

3.7. Remark. Let us notice that we can discuss about a radical axis of a point and a circle, and likewise about a radical axis of two points. Clearly, the radical axis of two points *A* and *B* is a bisector of the line segment *AB*. When talking about the radical axis of point *A* and circle (K_1), if $A \in (K_1)$, then it is a tangent at the point *A*, and if *A* is outside the circle, then the radical axis can be constructed by using the radical center of *A*, (K_1)

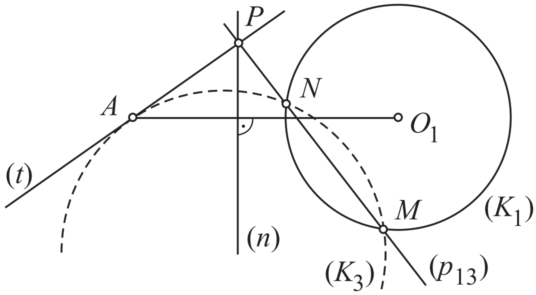

Figure 17

and a circle (K_3) such that it passes through A and intersects (K_1) (figure 17). Namely, the radical axis of A and (K_3) in fact is a tangent (t) to (K_3) at A, and the radical axis (p_{13}) passes through the points of intersection M and N. So, the radical center P is an intersection of (t) and (p_{13}), and therefore the required radical axis of A and (K_1) is a straight line (n) at P, perpendicular to AO_1.

Analogously, we can discuss about the radical center of a point and two circles, of two points and a circle and also about radical center of three points. Clearly, the radical center of three non-collinear points A, B and C is a center of circle which passes through A, B and C.

3.8. Lemma. If $K_i(O_i, R_i)$, $i=1,2,3$ are three not concurrent circles with centers which are not collinear, then it exists a unique circle (K) such that either (K) intersects all three circles or all three circles halve the circle (K).

Proof. Let P be a radical center of $K_i(O_i, R_i)$, $i=1,2,3$. Then, P is either inside or outside of each three circles. When the first case is satisfied the power of the circle is $-m^2$ and the circle $K(P,m)$ halves the circles $K_i(O_i, R_i)$, $i=1,2,3$, and when the second case is satisfied the power of the circle is m^2 and the circle $K(P,m)$ orthogonally intersects the circles $K_i(O_i, R_i)$, $i=1,2,3$. ■

3.9. Example. Construct a circle which passes through the points A and B and touches (K_1).

Solution. Through the points A and B, we draw an arbitrary circle (K) such that (K) meets the circle (K_1) at C and D (figure 18). Then, the radical axis of (K) and (K_1) is a straight line CD, and the radical axis of (K) and the required circle (K*) is a line AB. Thus, the intersection P of lines AB and CD, if such point exists, is radical center of (K), (K*) and (K_1), which implies that the radical axis of (K_1) and (K*) is a tangent to (K_1) at P.

Hence, drawing the tangents (t_1) and (t_2) at P to (K_1), we find the points of touching T_1 and T_2 between the required circles and (K_1). So, the given problem has at most two solutions. ■

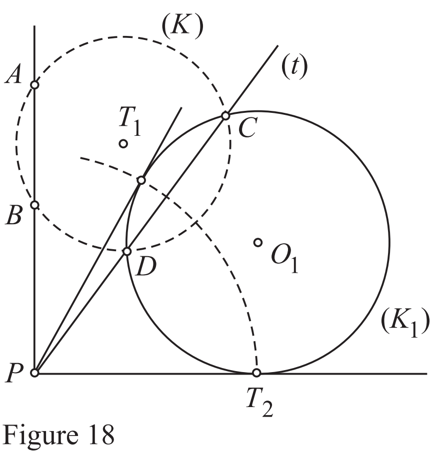

Figure 18

4. A PENCIL AND A BUNDLE OF CIRCLES

4.1. Let circles $K_1(O_1,R_1)$ and $K_2(O_2,R_2)$ be given and let (p) be their radical axis. The set of circles such that (p) is radical axis for each two of them ((p) is considered as a circle) is said to be a *pencil of circles*. The straight line (p) is said to be a *radical axis* of the pencil.

Since 3.1 the centres of each circle of the pencil are on the line O_1O_2 which is perpendicular to (p) and is said to be a *central line of a pencil.*

4.2. Clearly, each pencil is defined by two circles, but our further discussion will imply that pencil of circles is likewise fully defined by a radical axis and a circle. Let M be an arbitrary point of a plane. If M is on a radical axis, then it exists a circle which passes through M. Let M not be on a radical axis. Likewise in 3.5 we will consider three cases:

a) If circles (K_1) and (K_2) intersect at A and B, then the unique circle (K_0) passes through the points M, A and B. It is easy to conclude that the circle (K_0) belongs to a pencil of circles determined by the circles (K_1) and (K_2).

b) If circles (K_1) and (K_2) touch at A, then the circle $K_0(O,\overline{OA})$ (O is intersection of bisector of line segment AM and the central line of the pencil), belongs to a pencil of circles determined by (K_1) and (K_2).

c) Let (K_1) and (K_2) have no intersection, m be an affix of M and T with affix t be an arbitrary point on a radical axis. Then, it exists a unique point M_1 with affix

$$m_1 = t + \frac{|t-o_1|^2 - R_1^2}{|m-t|^2}(m-t)\,.$$

Since,

$$\overline{TM}\cdot\overline{TM}_1 = |m-t|\cdot\left|t + \frac{|t-o_1|^2-R_1^2}{|m-t|^2}(m-t) - t\right|$$

$$= |t-o_1|^2 - R_1^2 = \overline{TO}_1^2 - R_1^2,$$

the circle $K_0(O,\overline{OM})$ (O is intersection of the bisector of line segment MM_1 and the central line of a pencil), belongs to a pencil of circles determined by (K_1) and (K_2).

Thus, we proved the following theorem.

Theorem. Through each point of a plane passes exactly one and only one circle of a given pencil of circles. ■

4.3. Let the pencil Π be defined by a radical axis (p) and a circle (K_0). Since the proof of Theorem 4.2, depending on the relationship between (p) and (K_0), there are three types of pencils. Namely,

a) If (p) and (K_0) intersect at A and B (figure 19), then each circle K of Π intersects the radical axis at A and B (the *Poncelet points* of the pencil) and

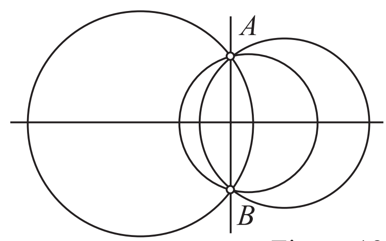

Figure 19

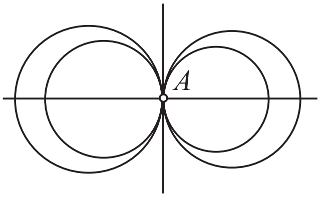

Figure 20

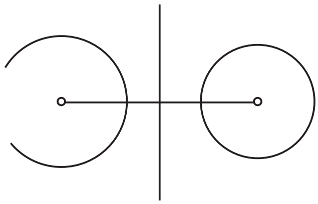

Figure 21

vice verse, each circle which passes through A and B belongs to a pencil Π. Thus, the pencil consists of each circle which passes through A and B. So, we will say that the pencil Π has two basic points or in other words is said to be a *hyperbolic pencil* .

b) If (p) and (K_0) touch at A (figure 20), then each circle (K) of Π touches the radical axis at A and vice verse, each circle such that it touches (p) at A belongs to Π. Thus, the pencil of circles consists of each circle which touches (p) at A. So, we will say that the pencil Π has one basic point or in other words is said to be a *parabolic pencil* .

c) If (p) and (K_0) have not any sharing points (figure 21), then each other circle of Π has no sharing points with (p) and furthermore, each two circles of Π do not have any sharing points. So, we will say that the pencil Π has no any basic point or in other words is said to be an *elliptic pencil*.

4.4. Lemma. The set of circles Π_1 (each of them is orthogonal to the circles of a pencil Π) is a pencil of circles, too.

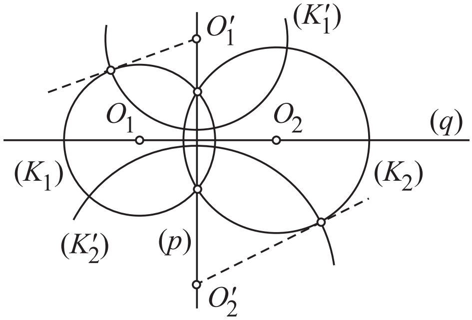

Figure 22

Proof. Let $K_i(O_i, R_i)$, $i = 1,2$ be two arbitrary circles of the pencil Π with radical axis (p) and central line (q), and let $K'_i(O'_i, R'_i)$, $i = 1,2$ be two arbitrary circles of the set Π_1 (figure 22). Since the circles $K'_i(O'_i, R'_i)$, $i = 1,2$ are orthogonal to $K_i(O_i, R_i)$, $i = 1,2$ their centres O'_i, $i = 1,2$ are on a radical axis (p) of $K'_i(O'_i, R'_i)$, $i = 1,2$. On the other hand, the points O_i, $i = 1,2$ are with equal power with respect to $K'_i(O'_i, R'_i)$, $i = 1,2$, i.e. (q) is a radical axis of $K'_i(O'_i, R'_i)$, $i = 1,2$. Now, the statement is implied by the arbitrariness of the circles $K'_i(O'_i, R'_i)$, $i = 1,2$. ■

4.5. Definition. If each circle of the pencil of circles Π is orthogonal to the pencil Π_1, then the pencils Π and Π_1 shall be called *conjugate pencil of circles*.

4.6. Lemma. a) If one of two conjugate pencils of circles is elliptic, then the other one is hyperbolic, and vice versa.

b) If one of two conjugate pencils of circles is parabolic, then the other one is parabolic, too.

Proof. a) Let Π be an elliptic pencil with radical (p) and central line (q) (figure 23). The intersection P of (p) and (q) is outer for each circle of the pencil Π, and therefore it is a centre of (K'), which belongs to a conjugate pencil Π_1. Since (K') meets the radical axis (q) of the pencil Π_1 at A and B, we get that the pencil Π_1 is hyperbolic.

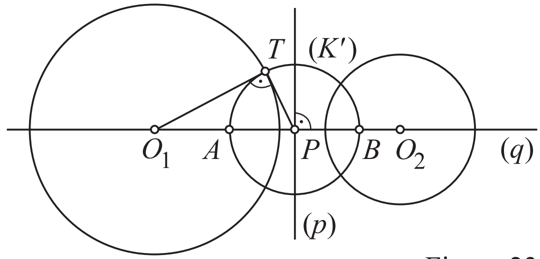

Figure 23

Conversely, let Π_1 be hyperbolic pencil with radical axis (q) and central line (p) (figure 23). The circle of the pencil Π_1 centred at $P=(p)\cap(q)$ meets the radical axis (q) at A and B. If $K_1(O_1,R_1)$ be an arbitrary circle of Π, and T a point of intersection between (K_1) and (K'), then $\overline{O_1T}<\overline{O_1P}$, and therefore (K_1) has no any intersection point with the radical axis (p) of Π. So, the pencil of circles Π is elliptic.

b) The statement is directly implied by the Definition of parabolic pencil of circles. The details are left as an exercise. ■

4.7. Example. Let be given a pencil of circles Π with a radical axis (p) and a circle $K_1(O_1,R_1)$. Construct a circle $K(O,R)$ such that it belongs to the pencil Π and touches the given circle $K_2(O_2,R_2)$.

Solution. Let (p_{12}) be a radical axis of the circles (K_1) and (K_2). Since the required circle (K) belongs to a pencil Π, the radical axis of (K_1) and (K) will be the line (p). So, the radical centre of (K), (K_1) and (K_2) is $P=(p)\cap(p_{12})$, and therefore the radical axis of (K) and (K_2) is a tangent to (K_2) such that it passes through P. After that, we construct the tangents to (K_2) through P, if such tangent exist, and further construct the required circle (K) centred at the perpendicular of the tangent drawn at the point of touching to (K_2). Then we find the central line of the pencil determined by (K_1) and (p). ■

4.8. Example. Given a pencil Π with a radical axis (p) and a circle $K_1(O_1,R_1)$. Construct a circle $K(O,R)$ such that it belongs to the pencil Π and touches the given line (a) which differs from (p).

Solution. If (a) is parallel to (p), then the exersize given problem may be considered as construction of a circle that belongs to the pencil Π and passes through the point $(a)\cap(q)$, where (q) is the central line of the tensile Π.

Therefore, let's assume that the line (a) intersects the line (p) and let M be the point of intersection. The tangent distance from M to (K_1) is equivalent to the tangent distance from M to the required circle (K), thus the circle (K) tangents the line (a) at the point P so that $\overline{MP}=\overline{MT}$, where T is the point of tangent of the tangent drawn from M to (K_1). Now the center of (K) is in the intersection of the perpendicular to (a) at the point P and the central line (q) of the pencil Π. ■

4.9. Definition. The set of all the circles, such that each three of them have a common radical center P, and all the lines such that pass through the point P, is called a *bundle of circles*. The point P is called the *radical center of the bundle* and the degree of the point P with respect to an arbitrary circle of that bundle is called the *degree of the bundle*.

4.10. Every bundle of circles Γ is determined by

- center and degree,
- a center and a circle
- a degree and two circles or
- three circles.

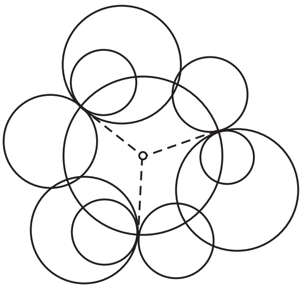

Figure 24

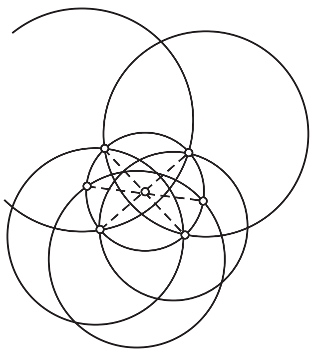

Figure 25

Depending on the mutual position of the center with respect to the circles of the bundle Γ, we can distinguish between three types of bundles such that:

a) If the degree is $m^2 > 0$, then the center P is an external point to every circle of the bundle Γ and according to the lemma 13.8 the circle $K(P,m)$ intersects orthogonally every circle of Γ. According to this, Γ consists of every circle and every line which orthogonally intersect the circle $K(P,m)$ (figure 24).

b) If the degree is $m^2 = 0$, then the bundle Γ consists of every circle and every line that pass through the centre P.

c) If the degree is $m^2 < 0$, then the centre P is an internal point for every circle of the bundle Γ and according to lemma 13.8 every circles of Γ half the circle $K(P,m)$. Therefore, Γ consists of every circle and every line which intersect the circle $K(P,m)$ in diametrically opposite points (figure 25).

4.11. Remark. Sometimes a bundle of circles is regarded as a set of all circles whose centers are on a given line (p) and every line perpendicular to (p), i.e. the set of every circle and every line which orthogonally intersect the line (p) and this bundle is a bundle of the first type, because as we said before the line in an extended complex plane might be regarded as a circle.

4.12. Lemma. The intersection of two bundles of circles is a pencil of circles or pencil of lines.

Proof. Let Γ_1 and Γ_2 be two bundles of circles with centers P and Q and degrees m and n, respectively. If $P \not\equiv Q$, then $\Gamma_1 \cap \Gamma_2$ is a pencil of circles with a radical axis PQ, and if $P \equiv Q \equiv O$, then it is a pencil of lines with a center at O. We will consider three different cases where $P \not\equiv Q$.

a) If $m=n=0$, then Γ_1 and Γ_2 are the sets of every circle which passes through *P* and *Q*, respectively, thus $\Gamma_1 \cap \Gamma_2$ is the set of every circle which passes through the points *P* and *Q*, i.e it is a hyperbolic pencil of circles

b) If $m>0$ and $n>0$, then $\Gamma_1 \cap \Gamma_2$ is the set of all circles which orthogonally intersect the circles $K\left(P,\sqrt{m}\right)$ and $K^*\left(Q,\sqrt{n}\right)$. According to the lemma 4.4 $\Gamma_1 \cap \Gamma_2$ is a pencil of circles.

c) If $m=0$ and $n>0$, then $\Gamma_1 \cap \Gamma_2$ is the set of all the circles which pass through the point *P* and orthogonally intersect the circle $K^*\left(Q,\sqrt{n}\right)$, and that is a pencil of circles with at least one base point, i.e. it is a hyperbolic or parabolic pencil of circles depending on whether $P \notin K^*\left(Q,\sqrt{n}\right)$ or $P \in K^*\left(Q,\sqrt{n}\right)$.

The other possible cases are left as an exercise for the reader. ■

5. ORTHOCENTAR AND CENTROID OF A TRIANGLE

5.1. Let consider the $\triangle ABC$, whose vertices *A*, *B* and *C* have affixes *a*, *b* and *c*, respectively. In the example II 3.3 we proved that

$$o=\frac{a\bar{a}(c-b)+b\bar{b}(a-c)+c\bar{c}(b-a)}{\begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix}}$$

is the affix of *O* circumcenter of $\triangle ABC$. Clearly, the radius of the circum circle of $\triangle ABC$ is $R=|a-o|$. The mapping $S:\mathbf{C}\to\mathbf{C}$ determined by $S(z)=\frac{1}{R}(z-o)$ is a direct similarity which maps $\triangle ABC$ into $\triangle A'B'C'$. According to consequence 4.8 we get that

$$\overline{A'B'}:\overline{A'C'}=\overline{AB}:\overline{AC} \text{ and } \angle A'B'C'=\angle ABC.$$

Let a', b', c' be the affixes of the vertices A', B', C', respectively, and let *t* be one of the three square roots of the complex number $\overline{a'b'c'}$. The mapping $S_1:\mathbf{C}\to\mathbf{C}$ determined by $S_1(z)=tz$ is a movement which maps $\triangle A'B'C'$ into $\triangle A''B''C''$. Moreover, if a'', b'', c'' are the affixes of A'', B'', C'', respectively, then

$$a''b''c''=t^3a'b'c'=1$$

and according to the consequence 4.8 we get that

$$\overline{A''B''}:\overline{A''C''}=\overline{A'B'}:\overline{A'C'} \text{ and } \angle A'B'C'=\angle A''B''C''.$$

The above stated implies that when consider the triangle with no loss of generality we take that its vertexes *A*, *B* and *C* with the affixes *a*, *b* and *c*, respectively, are on the unit center circle centered at the origin. Therefore $|a|=|b|=|c|=1$. Moreover, the coordinate system is chosen such that $abc=1$.

In our further consideration, unless it is not mentioned differently, we will consider that $\triangle ABC$ is inscribed in the unit circle centered at the origin O and that for the affixes a, b and c and the vertices A, B and C is true that $abc = 1$.

5.2. Let's consider the $\triangle ABC$ whose vertices A, B and C have the affixes a, b and c, respectively. According to the theorem 1.2 the equations of the lines AB, BC and CA are such that

$$z + ab\bar{z} = a + b\,,\ z + bc\bar{z} = b + c\,,\ z + ca\bar{z} = c + a\,, \qquad (1)$$

respectively, i.e. their complex angle coefficients are $-ab$, $-bc$ and $-ca$, respectively. According to consequence 1.8 the lines that go through C, A and B, and are perpendicular to AB, BC and CA are expressed as following

$$cz - \bar{z} = c^2 - ab\,,\ az - \bar{z} = a^2 - bc\,,\ bz - \bar{z} = b^2 - ca\,, \qquad (2)$$

respectively. The lines whose equations are given in (2) are to be called *altitudes* of $\triangle ABC$, drawn from the vertices C, A and B, respectively.

Similarly, the equations of the bisectors of the sides AB, BC and CA are the following

$$z - ab\bar{z} = 0\,,\ z - bc\bar{z} = 0\,,\ z - ca\bar{z} = 0\,, \qquad (3)$$

respectively.

Therefore $a \neq b$ we get that the system of equations

$$\begin{cases} az - \bar{z} = a^2 - bc \\ bz - \bar{z} = b^2 - ca \end{cases}$$

has a solution $h = a + b + c$. With a direct check we can prove that the complex number h satisfies the equation of the altitude drawn from the vertex C as well.

Thus, we proved the following theorem.

Theorem. The altitudes at $\triangle ABC$ concur at the point H with affix

$$h = a + b + c\,. \blacksquare$$

5.3. Definition. The point N discussed in the theorem5.2 is called the *orthocenter* of $\triangle ABC$.

5.4. Remark. a) For $\triangle ABC$ whose vertexes A, B and C have the affixes a, b and c, respectively, and which is not inscribed in the unit circle centered at the origin, it can be proven that for the affixes h and o of the orthocenter H and the circumcenter O, respectively, the following is true

$$h + 2o = a + b + c\,.$$

b) Let's consider the at $\triangle OXY$ such that one of its vertexes coincides with the origin and the other X and Y have the affixes x and y, respectively. Then, for the affix o of O_1 circum center of the $\triangle OXY$ we get that

$$o=\frac{0\bar{0}(y-x)+x\bar{x}(0-y)+y\bar{y}(x-0)}{\begin{vmatrix}0&\bar{0}&1\\x&\bar{x}&1\\y&\bar{y}&1\end{vmatrix}}=\frac{y\bar{y}x-x\bar{x}y}{x\bar{y}-\bar{x}y}=\frac{xy(\bar{y}-\bar{x})}{x\bar{y}-\bar{x}y}.$$

Moreover, for the affix h of the orthocenter H of $\triangle OXY$ we get that

$$h=0+x+y-2o=x+y-2\frac{xy(\bar{y}-\bar{x})}{x\bar{y}-\bar{x}y}=\frac{x^2\bar{y}-x\bar{x}y+y\bar{y}x-y^2\bar{x}-2y\bar{y}x+2x\bar{x}y}{x\bar{y}-\bar{x}y}$$

$$=\frac{x^2\bar{y}-y^2\bar{x}-y\bar{y}x+x\bar{x}y}{x\bar{y}-\bar{x}y}=\frac{x\bar{y}(x-y)+y\bar{x}(x-y)}{x\bar{y}-\bar{x}y}=\frac{(x-y)(x\bar{y}+y\bar{x})}{x\bar{y}-\bar{x}y}.$$

5.5. Example. Draw a line perpendicular to the diameter of the circle $K(O,R)$ from a point M which is not on that circle.

Solution. We draw lines from the point M to the ends of the diameter A and B. Then, the lines AM and BM meet the circle $K(O,R)$ at the points C and D, respectively. In accordance with consequence 1.7 the lines AD and BC are altitudes of the triangle whose sides are on the lines AC and BD. So the theorem 5.2 implies that the line MP is the required perpendicular drawn at the point M to the diameter AB (figure 26). ■

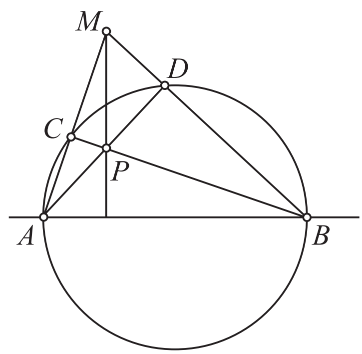

Figure 26

5.5. Example. If H and O are the orthocenter and the circum center of the $\triangle ABC$, respectively, then

$$\overline{OH}^2=9R^2-(\overline{AB}^2+\overline{AC}^2+\overline{BC}^2),$$

where R is the length of the circumradius. Prove this!

Solution. Without loss of generality, we can assume that $\triangle ABC$ is inscribed in a circle centered at the origin and a radius R. If the vertices A, B and C have the affixes a, b and c, then $|a|=|b|=|c|=R$ and the remark 15.4 implies that:

$$\begin{aligned}\overline{OH}^2&=|a+b+c|^2=(a+b+c)(\bar{a}+\bar{b}+\bar{c})\\&=a\bar{a}+b\bar{b}+c\bar{c}+a\bar{b}+\bar{a}b+b\bar{c}+\bar{b}c+c\bar{a}+\bar{c}a\\&=3(a\bar{a}+b\bar{b}+c\bar{c})-\left(|a-b|^2+|b-c|^2+|c-a|^2\right)\\&=9R^2-(\overline{AB}^2+\overline{AC}^2+\overline{BC}^2),\end{aligned}$$

Which was exactly supposed to be proven. ■

5.6. Theorem. If H is the orthocenter of $\triangle ABC$ and are the points symmetric to H with respect to the lines BC, CA, AB, respectively, then the points A_4, B_4, C_4 lie on the circle circumscribed around $\triangle ABC$.

Proof. The example II 1.9 and the theorem 5.2 it follows that the affixes of the points A_4, B_4, C_4 symmetric to the point H with respect to the lines BC, CA, AB are $-b^2c^2, -c^2a^2, -a^2b^2$, respectively. From

$$|a|=|b|=|c|=1$$

it follows that

$$|-b^2c^2|=|-c^2a^2|=|-a^2b^2|=1,$$

i.e. the points A_4, B_4, C_4 are on the circumcircle of the $\triangle ABC$. ■

5.7. Consequence. The projections A_2, B_2, C_2 of the vertexes A, B, C on the sides BC, CA, AB of $\triangle ABC$ have the affixes $\frac{h-b^2c^2}{2}, \frac{h-c^2a^2}{2}, \frac{h-a^2b^2}{2}$, respectively.

Proof. According to the theorem 5.6 the points A_2, B_2, C_2 are midpoints of the line segments HA_4, HB_4, HC_4, respectively. Now, the proof by the fact that the affix of the orthocenter H is h, and the affixes of the points A_4, B_4, C_4 are $-b^2c^2$, $-c^2a^2$, $-a^2b^2$ respectively. ■

5.8. Let $\triangle ABC$ be given, and let its vertices A, B, C have the affixes a, b, c, respectively. The affixes of the midpoints A_1, B_1, C_1 of the sides BC, CA, AB are $\frac{b+c}{2}, \frac{c+a}{2}, \frac{a+b}{2}$, respectively, which means that the equations of the lines AA_1, BB_1, CC_1 are

$$z-a=\frac{b+c-2a}{\overline{b}+\overline{c}-2\overline{a}}(\overline{z}-\overline{a}),\quad z-b=\frac{c+a-2b}{\overline{c}+\overline{a}-2\overline{b}}(\overline{z}-\overline{b}),\quad z-c=\frac{a+b-2c}{\overline{a}+\overline{b}-2\overline{c}}(\overline{z}-\overline{c}), \qquad (4)$$

The system of equations

$$\begin{cases} z-a=\frac{b+c-2a}{\overline{b}+\overline{c}-2\overline{a}}(\overline{z}-\overline{a}) \\ z-b=\frac{c+a-2b}{\overline{c}+\overline{a}-2\overline{b}}(\overline{z}-\overline{b}) \end{cases}$$

has the solution

$$t=\frac{a+b+c}{3}.$$

By a direct check we prove that the complex number t satisfies the equality of the line CC_1 as well. The point whose affix is the complex number t is denoted by T. Furthermore, we get that

$$\overline{AT}=\left|\frac{a+b+c}{3}-a\right|=\left|\frac{b+c-2a}{3}\right|=2\left|\frac{b+c-2a}{6}\right|=2\left|\frac{b+c}{2}-\frac{a+b+c}{3}\right|=2\overline{A_1T}.$$

Analogously we prove that

$$\overline{BT}=2\overline{B_1T} \text{ and } \overline{CT}=2\overline{C_1T}.$$

Thus we proved the following theorem.

Theorem. If A_1, B_1, C_1 are the midpoints of the sides BC, CA, AB of the $\triangle ABC$, then the lines AA_1, BB_1, CC_1 concur at a point T whose affix is $t=\frac{a+b+c}{3}$ and the point T divides the line segments AA_1, BB_1, CC_1 in a ratio of $2:1$. ■

5.9. Definition. The point T defined as in theorem 5.8 is called ***a*** *centroid* of $\triangle ABC$, and the lines AA_1, BB_1, CC_1 are its *medians*.

5.10. Example A. Let be given a quadrangle $ABCD$ and let T_a, T_b, T_c, T_d be the centroids of the triangles BCD, ACD, BAD, ABC, respectively. Prove that the line segments AT_a, BT_b, CT_c, DT_d intersect in one point, and each of them is divided in a ratio of $3:1$ starting at the vertices of the quadrangle.

Solution. Due to the theorem 15.8 we get that

$$t_a = \tfrac{b+c+d}{3},\ t_b = \tfrac{a+c+d}{3},\ t_c = \tfrac{a+b+d}{3} \text{ and } t_d = \tfrac{a+b+c}{3}.$$

Let A', B', C', D' be the points which divide the line segments AT_a, BT_b, CT_c, DT_d in a ratio of 3:1 starting from the vertices of the quadrangle, respectively. I 4.2. implies that $a' = b' = c' = d' = \tfrac{a+b+c+d}{4}$, which means that AT_a, BT_b, CT_c, DT_d concur at a point T with an affix $t = \tfrac{a+b+c+d}{4}$, and each of them is divided by in a ratio of 3:1 starting from the vertices of the quadrangle. ■

Comment. The point T discussed in the previous example is called *centroid* *ABCD*. The example A shows how we can define the centroid of a pentagon. Namely, we consider the line segments which connect a vertex of a pentagon with a centroid of a quadrlateral formed by the other four vertices of the pentagon and thus we get five line segments which intersect at T, which is to be called a *vertex of a pentagon*. It is easy to prove that if the affixes of the vertices of the pentagon $ABCDE$ are a', b', c', d', e', respectively, then the affix t of its centroid T is $t = \tfrac{a+b+c+d+e}{5}$. On a similar way we can define the centroid T of n-gon $A_1A_2...A_n$ and we can prove that its affix is

$$t = \frac{a_1+a_2+...+a_n}{n},$$

where a_i, $i = 1,2,...,n$ are the affixes of the vertices A_i, $i = 1,2,...,n$, respectively.

Example B. Let S be the center of a circumcenter, and H be the orthocenter of $\triangle ABC$. Furthermore, let the point Q be such that S is the midpoint of the line segment HQ and let T_1, T_2 and T_3 be the centroids of $\triangle BCQ$, $\triangle CAQ$ and $\triangle ABQ$, respectively. Prove that

$$\overline{AT_1} = \overline{BT_2} = \overline{CT_3} = \tfrac{4}{3}R,$$

where R is the circumradius of $\triangle ABC$.

Solution. Without loss of generality, we can say that the circumcircle of $\triangle ABC$ is the unit circle, i.e. that $o = 0$ and $|a| = |b| = |c| = 1$. We have $h = a + b + c$ and $o = \frac{h+q}{2}$, and therefore $q = -h = -a - b - c$. Furthermore, $t_1 = \frac{b+c+q}{3} = -\frac{a}{3}$ and similar to this we get that $t_2 = -\frac{b}{3}$, $t_3 = -\frac{c}{3}$. Now we have that

$$\overline{AT_1} = |a - t_1| = \left|a + \frac{a}{3}\right| = \left|\frac{4a}{3}\right| = \frac{4}{3} = \frac{4}{3}R\,,\ \overline{BT_2} = \overline{CT_3} = \frac{4}{3} = \frac{4}{3}R\,,$$

Which was actually supposed to be proven. ■

5.11. Theorem (Leibniz). If T is the centroid of $\triangle ABC$ and P is an arbitrary point of the plane of the triangle, then

$$\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 = 3\overline{PT}^2 + \overline{TA}^2 + \overline{TB}^2 + \overline{TC}^2\,.$$

Proof. Let a, b, c be the affixes of the vertexes A, B, C respectively and let p be the affix of the point P. From the theorem 5.8, we get

$$\begin{aligned}
3\overline{PT}^2 + \overline{TA}^2 + \overline{TB}^2 + \overline{TC}^2 &= 3\left|\frac{a+b+c-3p}{3}\right|^2 + \left|\frac{b+c-2a}{3}\right|^2 + \left|\frac{a+c-2b}{3}\right|^2 + \left|\frac{a+b-2c}{3}\right|^2 \\
&= 3p\overline{p} + a\overline{a} + b\overline{b} + c\overline{c} - p\overline{a} - \overline{p}a - p\overline{b} - \overline{p}b - p\overline{c} - \overline{p}c \\
&= |p-a|^2 + |p-b|^2 + |p-c|^2 \\
&= \overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2,
\end{aligned}$$

Which was supposed to be proven. ■

5.12. Example. If T is the centroid of $\triangle ABC$, then

$$\overline{TA}^2 + \overline{TB}^2 + \overline{TC}^2 = \frac{1}{3}(\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2)\,.$$

Prove this!

Solution. It is sufficient to get that $P \equiv A$, $P \equiv B$ and $P \equiv C$ in the equality of theorem 5.11 and after that to summaries all the obtained equalities. ■

5.13. Let it be given $\triangle ABC$ and let consider the homothety

.
$$w = -\frac{1}{2}z + \frac{h}{2}\,. \tag{5}$$

The equality $z = -\frac{1}{2}z + \frac{h}{2}$ implies that $z = \frac{h}{3}$, i.e. the center of the homothety (5) is the centroid T of $\triangle ABC$. The point A under the homothety (5) maps to a point with an affix

$$-\frac{a}{2} + \frac{h}{2} = \frac{b+c}{2}\,,$$

This means that at A_1 maps in the midpoint of the side BC. Analogously it can be proven that the points B and C map at the midpoints B_1 and C_1 of the sides AC and AB, respectively. Thus, we proved the following theorem.

Theorem. If T is the centroid of $\triangle ABC$ and A_1, B_1 and C_1 are the midpoints of the sides BC, AC and AB, respectively, then the homothety with a center in T and a ration $-\frac{1}{2}$ maps $\triangle ABC$ to $\triangle A_1B_1C_1$. ■

5.14. Consequence. If A_1, B_1 and C_1 are the midpoints of the sides BC, AC and AB of $\triangle ABC$, then the following holds true

$$A_1B_1 \parallel AB, \qquad B_1C_1 \parallel BC, \qquad C_1A_1 \parallel CA,$$
$$2\overline{A_1B_1} = \overline{AB}, \qquad 2\overline{B_1C_1} = \overline{BC}, \qquad 2\overline{C_1A_1} = \overline{CA}\,.$$

Proof. The theorems 5.13 and II 6.5 imply that
$$A_1B_1 \parallel AB,\ B_1C_1 \parallel BC,\ C_1A_1 \parallel CA,$$
and the theorems 5.12 and 4.6 imply that
$$2\overline{A_1B_1} = \overline{AB}, \quad 2\overline{B_1C_1} = \overline{BC}, \quad 2\overline{C_1A_1} = \overline{CA}\,. \ \blacksquare$$

5.15. Definition. Let A_1, B_1 and C_1 and be the midpoints of the sides *BC*, *AC* and *AB*, respectively, of $\triangle$ *ABC*. The line segments A_1B_1, B_1C_1, C_1A_1 are called the *medians* of the sides *AB*, *BC*, *CA*, respectively.

5.16. Remark. In consequence 5.14 we proved that the medians of the triangle are parallel to the suitable sides of the triangle and that the length of each one Is half of the length of the suitable side.

6. RIGHT ANGLED TRIANGLE

6.1. We call $\triangle$ *ABC right angled triangle* if its orthocenter *H* coincides with one of the vertices *A*, *B* or *C*. Due to this, $\triangle$ *ABC* is a right angled triangle if and only if $|h| = 1$, i.e if and only if
$$(a+b+c)(\overline{a}+\overline{b}+\overline{c}) = 1\,.$$
The last equality is equivalent to the equality
$$(a+b)(b+c)(c+a) = 0\,.$$
Which implies that the $\triangle$ *ABC* is a right angled triangle if and only if either
$$a+b=0 \text{ or } b+c=0 \text{ or } c+a=0\,.$$
Thus, we proved the following theorem.

Theorem. The triangle *ABC* is a right angled triangle if and only if either $a+b=0$ or $b+c=0$ or $c+a=0$. $\blacksquare$

6.2. Consequence. The triangle *ABC* is a right triangle if and only if one of the sides *AB*, *BC* or *CA* is the diameter of the circle circumscribed around it.

Proof. It is directly implied by the theorem 6.1. $\blacksquare$

The side of the right angled traingle *ABC* which is the diameter of its circumcircle is called a *hypotenuse*, and the remaining two sides are called *legs* of $\triangle$ *ABC*.

6.3. Theorem (Pythagoras). For every right angled triangle the square of the length of the hypotenuse is equal to the sum of the squares of the length of its legs.

Proof. Let *AB* be the hypotenuse of the right triangle *ABC*. According to theorem 16.1 we get that $a+b=0$ i.e. $b=-a$, thus

$$
\begin{aligned}
\overline{AC}^2+\overline{BC}^2 &= |c-a|^2+|c-b|^2 \\
&= |c-a|^2+|c+a|^2 \\
&= (c-a)(\overline{c}-\overline{a})+(c-a)(\overline{c}+\overline{a}) \\
&= 2|c|^2+2|a|^2=4|b|^2 \\
&= |2b|^2=|b-a|^2=\overline{AB}^2,
\end{aligned}
$$

which was supposed to be proven. ■

6.4. Example. If the hypotenuse of the right triangle is divided in three equal parts and the point of division are connected with the vertex of the right angle, then the sum of the squared of the length of the sides of so obtained triangle is equal to $\frac{2}{3}$ of the square of the hypotenuse. Prove it!

Solution. Without loss of generality, we take that *ABC* is a right angled triangle with a right angle in the vertex *C*, such that it is inscribed in the unit circle. Let the affixes on the vertices *A*, *B*, *C* are *a*, *b*, *c* respectively. If *D* and *E* are points of the hypotenuse *AB* such that

$$\overline{AD}=\overline{DE}=\overline{EB},$$

then their affixes are $\frac{2a+b}{3}$ and $\frac{a+2b}{3}$, respectively. According to theorem 6.1 we get that

$$
\begin{aligned}
\overline{CD}^2+\overline{DE}^2+\overline{EC}^2 &= \left|\tfrac{2a+b-3c}{3}\right|^2+\left|\tfrac{b-a}{3}\right|^2+\left|\tfrac{a+2b-3c}{3}\right|^2 \\
&= \left|\tfrac{a-3c}{3}\right|^2+\left|\tfrac{2a}{3}\right|^2+\left|\tfrac{a+3c}{3}\right|^2 \\
&= \tfrac{2a\overline{a}+6c\overline{c}}{3}=\tfrac{2}{3}\left(|a|^2+3|c|^2\right) \\
&= \tfrac{2}{3}\left(|a|^2+3|a|^2\right)=\tfrac{2}{3}|2a|^2 \\
&= \tfrac{2}{3}|b-a|^2=\tfrac{2}{3}\overline{AB}^2,
\end{aligned}
$$

which was supposed to be proven. ■

7. EULER LINE AND EULER CIRCLE

7.1. Theorem. The circum center O, the centroid T and the orthocenter H of $\triangle ABC$ are on a same line and furthermore $\overline{OH} = 3\overline{OT}$.

Proof. For the affixes h and t of the orthocenter H and the centroid T we get that

$$h = a + b + c = 3\tfrac{a+b+c}{3} = 3t\ ,$$

which means that O, T and H are collinear and thus $\overline{OH} = 3\overline{OT}$. ■

7.2. Definition. The line on which are the circum center O, the centroid T and the orthocenter H of $\triangle ABC$ is called *Euler line* for $\triangle ABC$.

7.3. Example. If T and O are the centroid and the circum center of $\triangle ABC$, respectively, then

$$\overline{OT}^2 = R^2 - \tfrac{1}{9}(\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2)\ ,$$

where R is the length of the circumradius of the $\triangle ABC$.

Solution. According to the theorem 7.1 we get $\overline{OH}^2 = 9\overline{OT}^2$. So, the statement is implied directly by example 5.5. ■

7.4. According to theorem II 2.2 the equation of the Euler line OH is $z = \frac{h}{\overline{h}}\overline{z}$, which means that its complex angle coefficient is $\frac{h}{\overline{h}}$. Let t_1, t_2, t_3 and t_4 be the complex angle coefficients of the lines BC, CA, AB and the Euler line of $\triangle ABC$. We get that, $t_1 = -bc,\ t_2 = -ca,\ t_3 = -ab$ and $t_4 = \frac{h}{\overline{h}}$. Thus,

$$\begin{aligned} t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_4t_4 &= (t_1 + t_2 + t_3)t_4 + t_2t_3 + t_1t_3 + t_1t_2 \\ &= -(bc + ca + ab)\frac{h}{\overline{h}} + abc(a + b + c) \\ &= -\overline{h}\frac{h}{\overline{h}} + 1 \cdot h = 0 \end{aligned}$$

For any movement the complex angle coefficients of the lines are multiplied by the same constant, thus if the equality

$$t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_4t_4 = 0 \qquad (1)$$

is true for a $\triangle ABC$ which is inscribed in the unit circle centered at the origin O, and for the affixes a, b, c of its vertices A, B, C it is true that $abc = 1$, then this is true for any triangle. Thus, we proved the following theorem.

Theorem. For the complex angle coefficients t_1, t_2, t_3 and t_4 of the sides and the Euler line of an arbitrary triangle the equality (1) holds true. ■

7.5. Theorem. If H is the orthocenter of $\triangle ABC$, A_1, B_1, C_1 are the midpoints of the sides BC, CA, AB, respectively, A_2, B_2, C_2 are foots of the altitudes and A_3, B_3, C_3 are the midpoints of the line segments AH, BH, CH, then the points A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3 are on a same circle.

Proof. The midpoints A_1, B_1, C_1 of the sides BC, CA, AB have the affixes $\frac{b+c}{2}$, $\frac{c+a}{2}$, $\frac{a+b}{2}$. According to consequence 5.7 A_2, B_2, C_2 the foots of the altitudes have the affixes $\frac{h-b^2c^2}{2}$, $\frac{h-a^2c^2}{2}$, $\frac{h-b^2a^2}{2}$, and since A_3, B_3, C_3 are the midpoints of the line segments AH, BH, CH we get that their affixes are $\frac{a+h}{2}$, $\frac{b+h}{2}$, $\frac{c+h}{2}$.

Due to the example II 3.3 we get that O_9 the circumcircle of the $\triangle A_1B_1C_1$ has the affix

$$\varepsilon = \frac{a+b+c}{2} = \frac{h}{2},$$

and its radius is

$$R_1 = \left| \varepsilon - \frac{a+b}{2} \right| = \frac{1}{2}.$$

By a direct check we can conclude that the points A_2, B_2, C_2, A_3, B_3, C_3 are on the same circle centered at O_9 and with radius $R_1 = \frac{1}{2}$, which means that the points A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3 are on a same circle. ■

7.6. Definition. A circle centered at O_9 and with radius $R_1 = \overline{O_9A_1}$, is called *Euler circle*, and the point O_9 is called *Euler point* for $\triangle ABC$.

7.7. Remark. Clearly, for any $\triangle ABC$ the Euler point is a midpoint of the line segment OH, i.e. it is placed on the Euler line, and the radius of the Euler circle is equal to $\frac{R}{2}$, where R is the radius of the circumcircle of the $\triangle ABC$.

7.8. Let L, M, N be the points with the affixes

$$l = b + c,\ \ m = c + a,\ \ n = a + b,$$

respectively. then, the line segments BC and OL have a common midpoint A_1, so the point L is symmetrical to the center O of the circumcircle of the $\triangle ABC$ with respect to the A_1, i.e with respect to the line BC. Since

$$\frac{a+l}{2} = \frac{b+m}{2} = \frac{c+n}{2} = \frac{h}{2}$$

we get that the line segments AL, BM, CL, HO have common midpoint, and that is the Euler point O_9. The equalities

$$h = l + a,\ \ b = l - c,\ \ c = l - b$$

and the fact that $|a| = |b| = |c| = 1$, we conclude that the points H, B, C are on the circle with a center at L and a length radius 1. Similarly, H, C, A are on the circle centered at M and a length radius of 1, and H, B, A are on the circle centered at N and a length radius of 1. The symmetry with respect to the point O_9 implies that the other statements of the following theorem are true.

Theorem. If *L*, *M*, *N* are symmetrical points to the center *O* of the circumcircle of the $\triangle ABC$ to the lines *BC*, *CA*, *AB*, respectively, and *H* is the orthocenter of $\triangle ABC$, then the quadrilaterals *ABCH* and *LMNO* are symmetrical with respect to the Euler point O_9. The triangles *ABC*, *BCH*, *CAH*, *ABH*, *LMN*, *MNO*, *NLO*, *LMO* subsequently have the orthocenters *H*, *A*, *B*, *C*, *O*, *L*, *M*, *N* and the circumcircle with congruent radii and centers at the points *O*, *L*, *M*, *N*, *H*, *A*, *B*, *C*, respectively. ■

7.9. Example. Let *H* be the orthocenter of $\triangle ABC$. Prove that the Euler lines of the triangles *ABC*, *ABH*, *BCH* and *CAH* intersect at a unique point.

Solution. Without loss of generality we can say that $\triangle ABC$ is inscribed into the unit circle. The orthocenter *H* of the triangle has an affix $h = a + b + c$. The point O' with an affix $o' = a + b$ is symmetrical to the center *O* of the circumcenter with respect to the line *AB*. Moreover,

$$\overline{O'A} = |a + b - a| = |a| = 1,$$

$$\overline{O'B} = |a + b - b| = |b| = 1 \text{ and}$$

$$\overline{O'H} = |a + b + c - a - b| = |c| = 1,$$

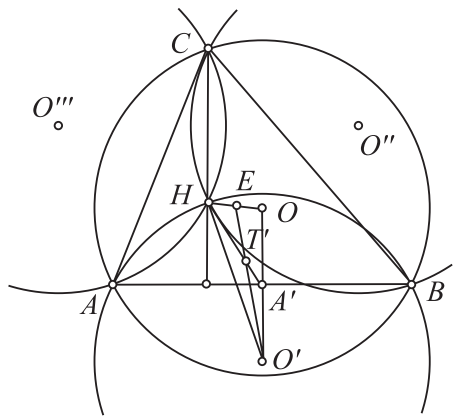

Figure 27

thus O' is the circumcenter of the $\triangle ABH$. Analogously, the points O'' and O''' with the affixes $o'' = b + c$ and $o''' = a + c$ are circumcenters of the triangles *BCH* and *ACH*, respectively. If T' is the centroid of the triangle *ABH*, then its affix is

$$t' = \frac{a+b+(a+b+c)}{3} = \frac{2a+2b+c}{3}.$$

The Euler lines of the triangles *ABH* and *ABC* are the lines $T'O'$ and *OH*, respectively. Thereby, $t' = \frac{(a+b)+(a+b+c)+0}{3}$ we get that T' is the centroid of $\triangle HOO'$, thus the line $T'O'$ intersects the line segment *OH* at point *E* with an affix $e = \frac{a+b+c}{2}$. Therefore, the point *E* is the intersection of the Euler lines of the triangles *ABH* and *ABC*.

Similarly, we can prove that the point *E* is the intersection of the Euler lines of the triangles *BCH* and *ABC*, that is the triangles *CAH* and *ABC*, which means that the Euler lines of the triangles *ABC*, *ABH*, *BCH* and *CAH* intersect in the same point. ■

8. MENELAU'S THEOREM

8.1. If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear vectors, then it exists a real number λ such that $\overrightarrow{AB}=\lambda\overrightarrow{CD}$. In the following consideration we will get that $\lambda=\frac{\overrightarrow{AB}}{\overrightarrow{CD}}$.

Since the equality $\overrightarrow{AB}=\lambda\overrightarrow{CD}$ is equivalent to the equality $\overrightarrow{CD}=\frac{1}{\lambda}\overrightarrow{AB}$ we get that $\frac{\overrightarrow{CD}}{\overrightarrow{AB}}=\frac{1}{\lambda}$.

8.2. Definition. Let the side AB of $\triangle ABC$ be on the line (p). The point P is called *Menelaus point* of the side AB if $P\in(p)$ and $P\not\equiv A,B$. Analogously, we define the Menelaus points of the sides BC and CA of the $\triangle ABC$.

8.3. Theorem (Menelaus). Let D, E and F be the Menelaus\` points of the sides BC, CA and AB of any $\triangle ABC$, respectively. The points D, E and F are collinear if and only if it holds true that

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}}\cdot\frac{\overrightarrow{CE}}{\overrightarrow{EA}}\cdot\frac{\overrightarrow{AF}}{\overrightarrow{FB}}=-1. \qquad (1)$$

Proof 1. Let D, E and F with affixes p, q and r respectively be the Menelaus\` points of the sides BC, CA and AB. If it holds true that

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}}=\lambda,\quad \frac{\overrightarrow{CE}}{\overrightarrow{EA}}=\mu,\quad \frac{\overrightarrow{AF}}{\overrightarrow{FB}}=\nu,$$

then for the affixes p, q and r of the points D, E and F we get that

$$p=\frac{b+\lambda c}{1+\lambda},\quad q=\frac{c+\mu a}{1+\mu},\quad r=\frac{a+\nu b}{1+\nu}. \qquad (2)$$

The points D, E and F are collinear if and only if

$$\frac{p-q}{\overline{p}-\overline{q}}=\frac{r-q}{\overline{r}-\overline{q}}.$$

If in the last equality (2) we substitute the values of p, q and r and the obtained equality we multiply by $(1+\lambda)(1+\mu)(1+\nu)$, we get that

$$(1+\lambda\mu\nu)(a\overline{b}+b\overline{c}+c\overline{a}-b\overline{a}-c\overline{b}-a\overline{c})=0. \qquad (3)$$

Therefore, the points D, E and F are collinear if and only if the equality (3) is satisfied. Lastly, the points D, E and F are collinear if and only if $1+\lambda\mu\nu=0$ (why?), that is if and only the condition (1) is satisfied

Proof 2. Let the condition (1) be satisfied , i.e $\mu=-\frac{1}{\lambda\nu}$. Then

$$p=\frac{b+\lambda c}{1+\lambda},\quad r=\frac{a+\nu b}{1+\nu}\ \text{ and }\ q=\frac{\lambda\nu c-a}{\lambda\nu-1},$$

thus

$$\overrightarrow{DF}=\frac{(1+\lambda)a+(\lambda\nu-1)b-\lambda(1+\nu)c}{(1+\nu)(1+\lambda)},$$

$$\overrightarrow{DE}=\frac{(1+\lambda)a+(\lambda\nu-1)b-\lambda(1+\nu)c}{(1-\lambda\nu)(1+\lambda)},$$

which means that $\frac{\overrightarrow{DE}}{\overrightarrow{DF}} = \frac{1+\lambda}{1-\lambda\nu} \in \mathbf{R}$, since for $1-\lambda\nu = 0$ we get that $FD \parallel AC$, which contradicts the finiteness of the point E. Therefore, $DE \parallel DF$, which implies that the points D, E and F are collinear.

Let the points D, E and F be collinear and let the projections of the points A, B, C on the line ED be the points A', B', C', respectively (see the figure). Then the triangles $BB'D$ and $CC'D$ are directly similar, and thus

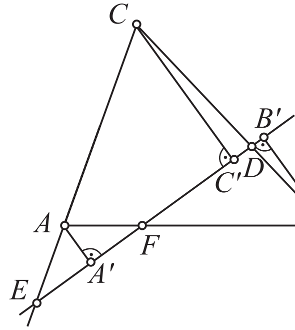

Figure 28

$$\frac{b'-b}{p-b} = \frac{c'-c}{p-c}, \text{ i.e. } \frac{b'-b}{c'-c} = \frac{p-b}{p-c}.$$

Analogously, the direct similarity of the triangles $AA'E$ and $CC'E$ implies that $\frac{a'-a}{c'-c} = \frac{q-a}{q-c}$ and the direct similarity of the triangles $AA'F$ and $BB'F$ implies that $\frac{b'-b}{a'-a} = \frac{r-b}{r-a}$. Finally,

$$\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \cdot \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = \frac{b-p}{p-c} \cdot \frac{c-q}{q-a} \cdot \frac{a-r}{r-b} = \left(-\frac{b'-b}{c'-c}\right)\left(-\frac{c'-c}{a'-a}\right)\left(-\frac{a'-a}{b'-b}\right) - 1 . \blacksquare$$

8.4. Example. Given is the $\triangle ABC$ and the points D and E on the sides BC and CA, respectively, such that $\overline{BD} = \overline{CE} = \overline{AB}$. We draw a line ($l$) through the point D, parallel to AB. If $M = (l) \cap BE$ and $F = CM \cap AB$, then $\overline{AB}^3 = \overline{AE} \cdot \overline{FB} \cdot \overline{CD}$. Prove this!

Solution. Let's consider the $\triangle ACF$ (figure 29). The points E, M and B are Menelaus\` points of the sides AC, CF and AF, respectively, and under a condition, they are collinear. From the Menelaus' theorem , we get that

$$\frac{\overrightarrow{AB}}{\overrightarrow{BF}} \cdot \frac{\overrightarrow{FM}}{\overrightarrow{MC}} \cdot \frac{\overrightarrow{CE}}{\overrightarrow{EA}} = -1,$$

which implies that

$$\frac{\overline{AB}}{\overline{BF}} \cdot \frac{\overline{FM}}{\overline{MC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1.$$

Due to $DM \parallel BF$ we get that $\frac{\overline{FM}}{\overline{MC}} = \frac{\overline{BD}}{\overline{DC}}$. If we substitute in the previous equality, we get that

$$\frac{\overline{AB}}{\overline{BF}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$$

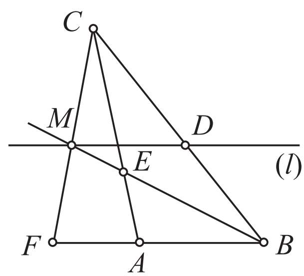

Figure 29

And due to $\overline{BD} = \overline{CE} = \overline{AB}$ we get $\overline{AB}^3 = \overline{AE} \cdot \overline{FB} \cdot \overline{CD}$. $\blacksquare$

9. PASCAL`S AND DESARGUES` THEOREM

9.1. In this section, by using the Menelay's theorem, we are going to prove the Desarg theorem which is a fundamental result in projective geometry. We are also going to prove the Pascal`s theorem for a hexagon inscribed in a circle.

Definition. The triangles ABC and $A'B'C'$ are called *copolar* if the lines AA', BB' and CC' are concurrent.

The triangles ABC and $A'B'C'$ are called *coax* if the points of intersection of the lines BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ lie on the same line.

9.2. Theorem (Desargue). The triangles ABC and $A'B'C'$ are copolar if and only if they are coax.

Proof. Let the triangles ABC and $A'B'C'$ be copolar and let the lines AA', BB' and CC' intersect in the point O. Let's denote the points of intersection of the lines BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ by P, Q, R respectively (figure 30). The Menelaus' theorem, applied to the triangles, BCO, CAO and AOB implies

$$\frac{\overrightarrow{BP}}{\overrightarrow{PC}}\cdot\frac{\overrightarrow{CC'}}{\overrightarrow{C'O}}\cdot\frac{\overrightarrow{OB'}}{\overrightarrow{B'B}}=-1,$$

$$\frac{\overrightarrow{CQ}}{\overrightarrow{QA}}\cdot\frac{\overrightarrow{AA'}}{\overrightarrow{A'O}}\cdot\frac{\overrightarrow{OC'}}{\overrightarrow{C'C}}=-1$$

$$\frac{\overrightarrow{AR}}{\overrightarrow{RB}}\cdot\frac{\overrightarrow{BB'}}{\overrightarrow{B'O}}\cdot\frac{\overrightarrow{OA'}}{\overrightarrow{A'A}}=-1$$

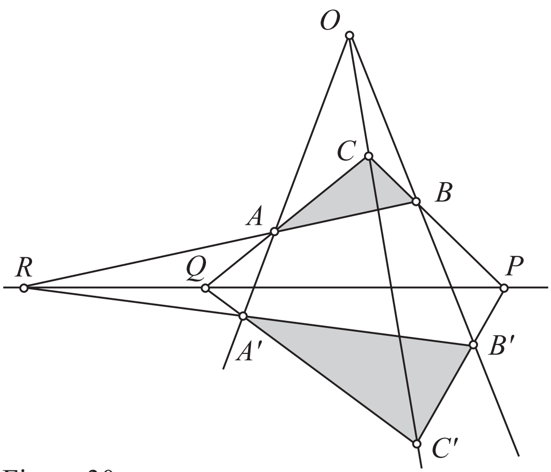

Figure 30

If we multiply the above equalities, we get

$$\frac{\overrightarrow{BP}}{\overrightarrow{PC}}\cdot\frac{\overrightarrow{CQ}}{\overrightarrow{QA}}\cdot\frac{\overrightarrow{AR}}{\overrightarrow{RB}}=-1.$$

Thus, from the Menelaus' theorem we conclude that the points P, Q and R are collinear. Therefore, the triangles ABC and $A'B'C'$ are coax.

Reversely, let's assume that P, Q and R are collinear and let the lines AA' and BB' intersect in the point O. Now, the triangles AQA' and BPB' are copolar, and therefore, coax. According to this, the points O, C and C' are collinear, which means that the coax triangles are copolar. ■

9.3. Theorem (Pascal). Let the hexagon $ABCDEF$, whose opposite sides are not collinear, be inscribed in a circle. Let denote by L, M, N the points of intersection of the three pairs of opposite sides AB and ED, BC and EF, FA and CD, respectively. Then, the points L, M, N are collinear.

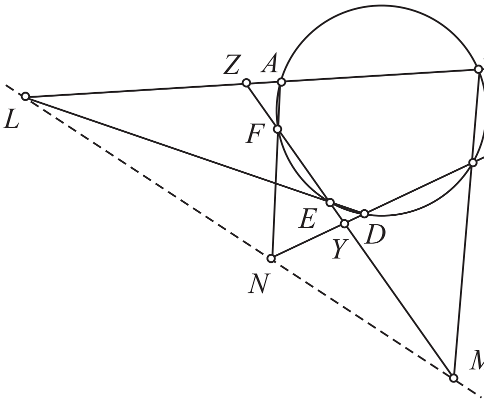

Figure 31

Proof 1. Let X, Y, Z be the points of intersection of AB and CD, CD and EF, EF and AB, respectively (figure 31). The points D, E, L; F, A, N; B, C, M are the Menelaus' points for $\triangle XYZ$, and from the Menelaus' theorem we get that

$$\frac{\overrightarrow{XL}}{\overrightarrow{LZ}} \cdot \frac{\overrightarrow{ZE}}{\overrightarrow{EY}} \cdot \frac{\overrightarrow{YD}}{\overrightarrow{DX}} = -1, \quad \frac{\overrightarrow{XA}}{\overrightarrow{AZ}} \cdot \frac{\overrightarrow{ZF}}{\overrightarrow{FY}} \cdot \frac{\overrightarrow{YN}}{\overrightarrow{NX}} = -1, \quad \frac{\overrightarrow{XB}}{\overrightarrow{BZ}} \cdot \frac{\overrightarrow{ZM}}{\overrightarrow{MY}} \cdot \frac{\overrightarrow{YC}}{\overrightarrow{CX}} = -1.$$

If we multiply the above equalities, we get that

$$\left(\frac{\overrightarrow{XL}}{\overrightarrow{LZ}} \cdot \frac{\overrightarrow{ZM}}{\overrightarrow{MY}} \cdot \frac{\overrightarrow{YN}}{\overrightarrow{NX}}\right) \cdot \frac{\overrightarrow{ZE}}{\overrightarrow{EY}} \cdot \frac{\overrightarrow{YD}}{\overrightarrow{DX}} \cdot \frac{\overrightarrow{XA}}{\overrightarrow{AZ}} \cdot \frac{\overrightarrow{ZF}}{\overrightarrow{FY}} \cdot \frac{\overrightarrow{XB}}{\overrightarrow{BZ}} \cdot \frac{\overrightarrow{YC}}{\overrightarrow{CX}} = -1. \qquad (1)$$

Furthermore, we apply the power of the points X, Y, Z with respect to the circle, then the geometrical interpretation of complex numbers, implies that

$$\overrightarrow{ZE} \cdot \overrightarrow{ZY} = \overrightarrow{AZ} \cdot \overrightarrow{BZ}, \quad \overrightarrow{EY} \cdot \overrightarrow{FY} = \overrightarrow{YD} \cdot \overrightarrow{YC}, \quad \overrightarrow{CX} \cdot \overrightarrow{DX} = \overrightarrow{XA} \cdot \overrightarrow{XB}.$$

If we substitute in (1) we get

$$\frac{\overrightarrow{XL}}{\overrightarrow{LZ}} \cdot \frac{\overrightarrow{ZM}}{\overrightarrow{MY}} \cdot \frac{\overrightarrow{YN}}{\overrightarrow{NX}} = -1,$$

which according to the Menelaus' theorem means that the points L, M and N are collinear.

Proof 2. Without loss of generality, we can say that the hexagon $ABCDEF$ is inscribed in a unit circle. The affixes l, m, n of the points $L = AB \cap DE$, $M = BC \cap FE$ and $N = CD \cap AF$ are

$$\overline{l} = \frac{a+b-(d+e)}{ab-de}, \quad \overline{m} = \frac{b+c-(e+f)}{bc-ef} \quad \text{and} \quad \overline{n} = \frac{c+d-(f+a)}{cd-fa}.$$

Furthermore,

$$\overline{l} - \overline{m} = \frac{(b-e)(bc-cd+de-ef+fa-ab)}{(ab-de)(bc-ef)} \quad \text{and} \quad \overline{m} - \overline{n} = \frac{(c-f)(cd-de+ef-fa+ab-bc)}{(bc-ef)(cd-fa)},$$

So,

$$\frac{\overline{l}-\overline{m}}{\overline{m}-\overline{n}} = \frac{(b-e)(cd-fa)}{(f-c)(ab-de)}.$$

Finally, if we take that for every point of a unit circle it is true that $\overline{x} = \frac{1}{x}$, by applying the properties of the complex numbers we get that

$$\frac{l-m}{m-n}=\frac{\overline{(b-e)(cd-fa)}}{\overline{(f-c)(ab-de)}}=\frac{\left(\frac{1}{b}-\frac{1}{e}\right)\left(\frac{1}{cd}-\frac{1}{fa}\right)}{\left(\frac{1}{f}-\frac{1}{c}\right)\left(\frac{1}{ab}-\frac{1}{de}\right)}=\frac{(e-b)(fa-cd)}{(c-f)(de-ab)}=\frac{\overline{l}-\overline{m}}{\overline{m}-\overline{n}},$$

which means that $\frac{l-m}{m-n}$ is a real number, therefore the consequence 1.4 implies that the points L, M and N are collinear . ■

10. TRIANGULAR COORDINATES

10.1. Lemma. If for the complex numbers a, b and c holds true

$$\lambda a+\mu b+\nu c=0 \tag{1}$$

where

$$\lambda+\mu+\nu=0 \tag{2}$$

and λ, μ, ν are nonzero real numbers, then the points A, B and C whose affixes are a, b and c, respectively, are collinear, and vice versa.

Proof. Truly, from (1) and (2) it follows that

$$c=\frac{a+\frac{\mu}{\lambda}b}{1+\frac{\mu}{\lambda}}, \tag{3}$$

i.e. the point C divides the line segment AB in a ratio $\frac{\mu}{\lambda}$, which means that the points A, B and C are collinear.

Reversely, if the points A, B and C lay on a same line and if the point C divides the line segment AB in a ration of $\frac{\mu}{\lambda}$, then from (3) in

$$\nu=-(\lambda+\mu)$$

we obtain the equalities (1) and (2). ■

10.2. Remark. Equality (2) implies that the numbers λ, μ and ν may not have the same sign, i.e. one of them must have an opposite sign from the other two. The point to which in (1) corresponds this number is between the other two points.

Thus, for example, from the equality

$$3a-b-2c=0$$

according to lemma 10.1 we get that the point whose affix is the complex number a is between the points whose affixes are the complex numbers b and c.

10.3. Lemma. Let A, B and C, with affixes a, b and c, respectively, are three non-collinear points in the plane. Then, to every point in the plane D, with an affix d, correspond three real numbers λ, μ and ν such that

$$\lambda a + \mu b + \nu c = d \tag{4}$$

where

$$\lambda + \mu + \nu = 1 \tag{5}$$

Proof. Let's connect the point *D*, for example, with the point *A* and the intersection of the lines *AD* and *BC* be denoted by D'. Then, the point *D* divides the line segment AD' in a ratio of $\alpha : \lambda = \overline{DD'} : \overline{DA}$, and the numbers α and λ can be chosen so that $\alpha + \lambda = 1$. the lemma 20.1 implies that

$$-d + \lambda a + \alpha d' = 0 \text{ and } -1 + \lambda + \alpha = 0 \tag{6}$$

Furthermore, the point D' divides the line segment *BC* with a ratio

$$\nu : \mu = \overline{CD'} : \overline{D'B}$$

These numbers can be chosen so that

$$\nu + \mu = \alpha .$$

According to the lemma 10.1 we get that

$$-\alpha d' + \mu b + \nu c = 0 \text{ and } -\alpha + \mu + \nu = 0 . \tag{7}$$

Now the equalities (4) are (5) are implied directly by the equalities (6) and (7).

Reversely, for μ and ν it exists a sole point D' on the line *BC* whose affix is determined by

$$d' = \frac{\mu b + \nu c}{\mu + \nu} .$$

Now, on the line AD' there is a unique point *D* such that

$$d = \frac{\lambda a + (\mu + \nu) d'}{\lambda + \mu + \nu} = \lambda a + \mu b + \nu c , \ \lambda + \mu + \nu = 1 ,$$

This means that (4) and (5) determine a unique point *D* in the plane. ■

10.4. Remark. The numbers λ, μ and ν uniquely determine the position of the point *D* with respect to the $\triangle ABC$. Therefore, the ordered triple (λ,μ,ν) is called *triangle coordinates* of the point *D* with respect to $\triangle ABC$. From (5) we get that all three numbers λ, μ and ν cannot be negative, and the position of the point *D* with respect to $\triangle ABC$ is determined by the signs of the numbers λ, μ and ν (figure 32).

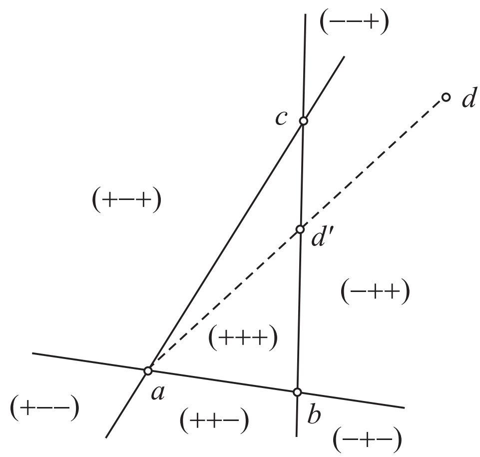

Figure 32

11. CEVA`S AND VAN AUBEL`S THEOREM

11.1. Theorem (Ceva). Let D', D'' and D''' lie on the sides BC, AC and AB of $\triangle ABC$ or their extensions, respectively. The lines AD', BD'' and CD''' intersect in one point if and only if the following equality is satisfied

$$\frac{\overrightarrow{BD'}}{\overrightarrow{D'C}}\cdot\frac{\overrightarrow{CD''}}{\overrightarrow{D''A}}\cdot\frac{\overrightarrow{AD'''}}{\overrightarrow{D'''B}}=1. \qquad (1)$$

Proof. Let the lines AD', BD'' and CD''' intersect in a point Q and let (p) be a line which goes through the point A and is parallel to the line BC. Let $BD''\cap(p)=\{K\}$ and $CD'''\cap(p)=\{L\}$, figure 33. The triangle $D'QB$ is directly similar to the triangle AQK, thus $\frac{q-d'}{b-d'}=\frac{q-a}{k-a}$. The triangle $D'CQ$ is directly similar to the triangle ALQ, thus $\frac{c-d'}{q-d'}=\frac{l-a}{q-a}$. From the last two equalities we get that

$$\frac{b-d'}{c-d'}=\frac{k-a}{l-a}. \qquad (2)$$

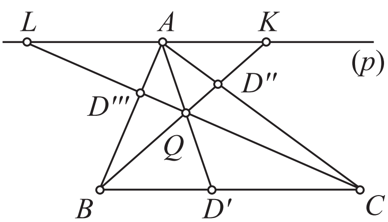

Figure 33

Furthermore, the triangle $CD''B$ is directly similar to the triangle $AD''K$, therefore the equality $\frac{d''-c}{b-c}=\frac{d''-a}{k-a}$ holds true, and it is equivalent to the equality

$$\frac{d''-c}{d''-a}=\frac{b-c}{k-a}, \qquad (3)$$

The triangle BCD''' is directly similar to the triangle ALD''', and therefore the equality $\frac{c-b}{d'''-b}=\frac{l-a}{d'''-a}$ holds true, and is equivalent to the equality

$$\frac{d'''-a}{d'''-b}=\frac{l-a}{c-b}. \qquad (4)$$

Finally, the equalities (2), (3) and (4) imply that

$$\frac{\overrightarrow{BD'}}{\overrightarrow{D'C}}\cdot\frac{\overrightarrow{CD''}}{\overrightarrow{D''A}}\cdot\frac{\overrightarrow{AD'''}}{\overrightarrow{D'''B}}=\frac{d'-b}{c-d'}\cdot\frac{d''-c}{a-d''}\cdot\frac{d'''-a}{b-d'''}=\left(-\frac{k-a}{l-a}\right)\left(-\frac{b-c}{k-a}\right)\left(-\frac{l-a}{c-b}\right)=1,$$

i.e. the equality (1) holds true.

On the other hand, if $BD''\cap CD'''=\{Q\}$ and $AQ\cap BC=\{A'\}$, then, the previously proven, implies that $\frac{\overrightarrow{BA'}}{\overrightarrow{A'C}}\cdot\frac{\overrightarrow{CD''}}{\overrightarrow{D''A}}\cdot\frac{\overrightarrow{AD'''}}{\overrightarrow{D'''B}}=1$ and by assumption that $\frac{\overrightarrow{BD'}}{\overrightarrow{D'C}}\cdot\frac{\overrightarrow{CD''}}{\overrightarrow{D''A}}\cdot\frac{\overrightarrow{AD'''}}{\overrightarrow{D'''B}}=1$ holds true we get that $\frac{\overrightarrow{BD'}}{\overrightarrow{D'C}}=\frac{\overrightarrow{BA'}}{\overrightarrow{A'C}}$. The latter implies that the points A' and D' coincide, i.e. $Q\in AD'$. ■

11.2. Remark. According to the Ceva theorem the medians of $\triangle ABC$ intersect in one point. Namely, if A_1, B_1 and C_1 are the midpoints of the sides BC, CA and AB respectively, then $\overrightarrow{BA_1}=\overrightarrow{A_1C}$, $\overrightarrow{CB_1}=\overrightarrow{B_1A}$ and $\overrightarrow{AC_1}=\overrightarrow{C_1B}$, thus

$$\frac{\overrightarrow{BA_1}}{\overrightarrow{A_1C}}\cdot\frac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}}\cdot\frac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}}=1,$$

which according to the Ceva theorem means that the medians AA_1, BB_1 and CC_1 intersect in one point.

11.3. Definition. The line segments (the lines) AD', BD'' and CD''' from theorem 11.1 are called *Ceva line segments* (lines) for $\triangle ABC$.

11.4. Theorem. The lines which connect the midpoints of the sides of a triangle with the suitable Ceva line segments intersect in a unique point.

Proof. Let in $\triangle ABC$ (figure 34) AD, BE and CF be arbitrary lines which intersect in the point M; A', B' and C' are midpoints of the lines BC, CA and AB respectively and P, K and L are midpoints of the line segments AD, BE and CF respectively. By applying the Ceva theorem to AD, BE and CF, we get the following

$$\frac{\overline{AE}}{\overline{EC}} \cdot \frac{\overline{CD}}{\overline{DB}} \cdot \frac{\overline{BF}}{\overline{FA}} = 1,$$

thus,

$$\frac{\frac{\overline{AE}}{2}}{\frac{\overline{EC}}{2}} \cdot \frac{\frac{\overline{CD}}{2}}{\frac{\overline{DB}}{2}} \cdot \frac{\frac{\overline{BF}}{2}}{\frac{\overline{FA}}{2}} = 1$$

i.e

$$\frac{\overline{C'K}}{\overline{KA'}} \cdot \frac{\overline{B'P}}{\overline{PC'}} \cdot \frac{\overline{A'L}}{\overline{LB'}} = 1.$$

Finally, from the Ceva theorem applied to $\triangle A'B'C'$ we get that the lines $A'P$, $B'K$ and $C'L$ intersect in one and only one point. ■

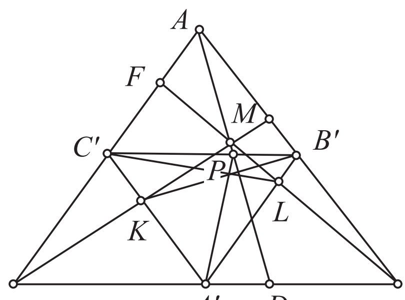

Figure 34

11.5. Consequence. The lines which connect the midpoints of the sides of a triangle and the midpoints of its altitudes intersect in one and only one point.

Proof. It is implied directly from the theorems 5.1 and 11.4. ■

11.6. Theorem (Van Aubel). If A', B', C' are points on the sides BC, CA, AB respectively, of the triangle ABC such that the lines AA', BB', CC' intersect in the point Q, then the following holds true

$$\frac{\overline{AQ}}{\overline{QA'}} = \frac{\overline{AC'}}{\overline{C'B}} + \frac{\overline{AB'}}{\overline{B'C}}.$$

Proof. Let the affixes of points be labeled by a suitable small letter. Let (r) be the line which goes through the point C and is parallel to the line AA', and (s) be the line which goes through the point B and is parallel to the line AA' and let $(r) \cap BB' = \{L\}$ and $(s) \cap CC' = \{K\}$, figure 35.

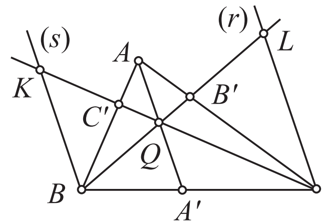

Figure 35

The triangle AQC' is directly similar to the triangle BKC', thus $\frac{q-a}{c'-a}=\frac{k-b}{c'-b}$, i.e. $\frac{c'-a}{c'-b}=\frac{q-a}{k-b}$. Furthermore, the triangle $CB'L$ is directly similar to the triangle $AB'Q$, thus $\frac{b'-c}{l-c}=\frac{b'-a}{q-a}$, i.e $\frac{b'-a}{b'-c}=\frac{q-a}{l-c}$; the triangle BKC is directly similar to the triangle $A'QC$, thus $\frac{k-b}{c-b}=\frac{q-a'}{c-a'}$, i.e. $\frac{c-a'}{c-b}=\frac{q-a'}{k-b}$ and the triangle CLB is directly similar to the triangle $A'QB$, thus $\frac{l-c}{b-c}=\frac{q-a'}{b-a'}$, i.e. $\frac{b-a'}{b-c}=\frac{q-a'}{l-c}$. Now,

$$\frac{\overline{AC'}}{\overline{C'B}}+\frac{\overline{AB'}}{\overline{B'C}}=\frac{|c'-a|}{|b-c'|}+\frac{|b'-a|}{|b'-c|}=\frac{|q-a|}{|k-b|}+\frac{|q-a|}{|l-c|}=|q-a|\left(\frac{1}{|k-b|}+\frac{1}{|l-c|}\right)$$

$$=|q-a|\left(\frac{1}{|q-a'|}\frac{|c-a'|}{|c-b|}+\frac{1}{|q-a'|}\frac{|b-a'|}{|b-c|}\right)=\frac{|q-a|}{|q-a'|}\cdot\frac{|c-a'|+|b-a'|}{|b-c|}$$

$$=\frac{|q-a|}{|q-a'|}=\frac{\overline{AQ}}{\overline{QA'}}.\ \blacksquare$$

11.7. Comment. In the Van Aubel theorem the lines AA', BB', CC' intersect in the point Q, therefore by the Ceva theorem we get that $\frac{\overline{BA'}}{\overline{A'C}}\cdot\frac{\overline{CB'}}{\overline{B'A}}\cdot\frac{\overline{AC'}}{\overline{C'B}}=1$, which means that there exist real numbers m, n, p such that $\frac{\overline{BA'}}{\overline{A'C}}=\frac{p}{n}$, $\frac{\overline{CB'}}{\overline{B'A}}=\frac{m}{p}$, $\frac{\overline{AC'}}{\overline{C'B}}=\frac{n}{m}$. Then, the affix of the point A' is determined by $a'=\frac{nb+pc}{n+p}$ and thereby the Van Aubel theorem holds true that $\frac{\overline{AQ}}{\overline{QA'}}=\frac{\overline{AC'}}{\overline{C'B}}+\frac{\overline{AB'}}{\overline{B'C}}=\frac{n}{m}+\frac{p}{m}=\frac{n+p}{m}$, therefore the affix of the point Q is as following

$$q=\frac{ma+(n+p)a'}{m+n+p}=\frac{ma+nb+pc}{m+n+p}. \tag{5}$$

11.8. Theorem. If A', B', C' are points on the sides BC, CA, AB, respectively, of the triangle ABC such that

$$\frac{\overline{BA'}}{\overline{A'C}}=\frac{p}{n},\ \frac{\overline{CB'}}{\overline{B'A}}=\frac{m}{p},\ \frac{\overline{AC'}}{\overline{C'B}}=\frac{n}{m}.$$

Then the lines AA', BB', CC' intersect in a point Q whose affix is determined in (5).

Proof. Thereby, it holds true that

$$\frac{\overline{BA'}}{\overline{A'C}}=\frac{p}{n},\ \frac{\overline{CB'}}{\overline{B'A}}=\frac{m}{p}\ \text{and}\ \frac{\overline{AC'}}{\overline{C'B}}=\frac{n}{m},$$

we get that

$$a'=\frac{nb+pc}{n+p},\ b'=\frac{pc+ma}{p+m}\ \text{and}\ c'=\frac{ma+nb}{m+n}.$$

Clearly, the first part of the statement implies from the Ceva theorem. We are going to show that the point Q whose affix is given in (5) lies on the line AA'. We have

$$\frac{q-a}{a'-a}=\frac{\frac{ma+nb+pc}{m+n+p}-a}{\frac{nb+pc}{n+p}-a}=\frac{n+p}{m+n+p}\cdot\frac{nb+pc-(n+p)a}{nb+pc-(n+p)a}=\frac{n+p}{m+n+p}\in\mathbf{R},$$

which according to the consequence 1.4 means that the points A, Q and B, Q, B' are collinear. Analogously, we prove that the points are collinear and that the points C, Q, C' are collinear as well. ■

11.9. The last theorem can be used to find the affixes of some important points of a triangle, such as the centroid, the center of the in-circle, the Jargon point(will be discussed later) and so. For example, for the midpoints A', B', C' of the sides BC, CA, AB of the triangle, it holds true that

$$\frac{\overline{BA'}}{\overline{A'C}}=1,\ \frac{\overline{CB'}}{\overline{B'A}}=1,\ \frac{\overline{AC'}}{\overline{C'B}}=1,$$

i.e. $m=n=p=1$, thus, from the theorem 11.8 we get that the medians intersect at a point T with affix $t=\frac{a+b+c}{3}$. ■

12. AREA OF A TRIANGLE

12.1. Let be given a $\triangle ABC$ and let the affixes of the vertices A, B, C be a, b, c respectively. We plot a line through the vertices B and C, whose auto conjugated equation is as following

$$\overline{i(c-b)}z-i(c-b)\overline{z}+i(c\overline{b}-\overline{c}b)=0. \qquad (1)$$

The distance from the point A to the line (1), i.e. the length of the altitude of $\triangle ABC$ plot at the vertex A, is

$$h_{BC}=\frac{|\overline{i(c-b)}a-i(c-b)\overline{a}+i(c\overline{b}-\overline{c}b)|}{2|c-b|},$$

i.e.

$$h_{BC}=\frac{|-(\overline{c}-\overline{b})a+(c-b)\overline{a}-c\overline{b}+\overline{c}b|}{2|c-b|}.$$

Therefore the area of the $\triangle ABC$ is

$$P_{\triangle ABC}=\frac{\overline{BC}\cdot h_{BC}}{2}=\frac{|-(\overline{c}-\overline{b})a+(c-b)\overline{a}-c\overline{b}+\overline{c}b|}{4}. \qquad (2)$$

Since the arbitrary complex numbers u and v the number $u\overline{v}-v\overline{u}$ is an imaginary number, the equality(2) transforms as following

$$P_{\triangle ABC}=\pm i\frac{-(\overline{c}-\overline{b})a+(c-b)\overline{a}-c\overline{b}+\overline{c}b}{4},$$

i.e

$$P_{\triangle ABC}=\pm\frac{i}{4}\begin{vmatrix} a & \overline{a} & 1\\ b & \overline{b} & 1\\ c & \overline{c} & 1\end{vmatrix}. \qquad (3)$$

12.2. Definition. Let be given $\triangle ABC$ and let the affixes of the vertices A, B, C be a, b, c respectively. We shall say that the $\triangle ABC$ is *positively oriented* with respect to the considered coordinate system, if its area, calculated by applying the formula (3), is obtained when we multiply by $+i$, and *negatively oriented* if its area is obtained when the formula (3) is multiplied by $-i$.

12.3. Remark. If $\triangle ABC$ is a right angled triangle, with a right angle in the vertex C, then for the affixes in the vertices A and B it holds true $b=-a$, thus from (3) for the area of the triangle we get that

$$P_{\triangle ABC}=\frac{|c\overline{a}-\overline{c}a|}{2}.$$

12.4. Remark. The affix of the point P', symmetrical to the point P with an affix p, with respect to the line which passes through the points with affixes a and b, can be determined by using the following condition

$$\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ m & \overline{m} & 1 \end{vmatrix}=0,$$

where $m=\frac{p+p'}{2}$. The details are left to the reader as an exercise.

12.5. Definition. We shall say that the n-gon $A_1A_2...A_n$ is *positively oriented* if $\triangle A_1A_2A_3$ is positively oriented.

We shall say that the n-gon $A_1A_2...A_n$ is *negatively oriented* if $\triangle A_1A_2A_3$ is negatively oriented.

12.6. Theorem. If $A_1A_2...A_n$ is a convex polygon whose vertices $A_1, A_2, \ldots, A_n$ have affixes $a_1, a_2, \ldots, a_n$ respectively, and S is its area, then $S=\pm\frac{1}{2}\operatorname{Im}(T\mathbf{a},\mathbf{a})$, where $\mathbf{a}=(a_1,a_2,...,a_n)$ and T is the mapping given in paragraph I 9.

Proof. The area of the polygon is calculated as a sum of the areas of $\triangle A_1A_2A_3$, $\triangle A_1A_3A_4, \ldots, \triangle A_1A_{n-2}A_{n-1}$ and $\triangle A_1A_{n-1}A_n$. Furthermore, these triangles are equivalently oriented, therefore

$$S=\pm\frac{i\begin{vmatrix} a_1 & \overline{a_1} & 1 \\ a_2 & \overline{a_2} & 1 \\ a_3 & \overline{a_3} & 1 \end{vmatrix}}{4}\pm\frac{i\begin{vmatrix} a_1 & \overline{a_1} & 1 \\ a_3 & \overline{a_3} & 1 \\ a_4 & \overline{a_4} & 1 \end{vmatrix}}{4}\pm\frac{i\begin{vmatrix} a_1 & \overline{a_1} & 1 \\ a_4 & \overline{a_4} & 1 \\ a_5 & \overline{a_5} & 1 \end{vmatrix}}{4}\pm\ldots\pm\frac{i\begin{vmatrix} a_1 & \overline{a_1} & 1 \\ a_{n-2} & \overline{a_{n-2}} & 1 \\ a_{n-1} & \overline{a_{n-1}} & 1 \end{vmatrix}}{4}\pm\frac{i\begin{vmatrix} a_1 & \overline{a_1} & 1 \\ a_{n-1} & \overline{a_{n-1}} & 1 \\ a_n & \overline{a_n} & 1 \end{vmatrix}}{4}$$

$$=\pm\frac{i}{4}(a_1\overline{a_2}-a_2\overline{a_1}+a_2\overline{a_3}-a_3\overline{a_2}+a_3\overline{a_4}-a_4\overline{a_3}+\ldots+a_n\overline{a_1}-a_1\overline{a_n})$$

$$=\pm\frac{i}{4}(2i\operatorname{Im}a_1\overline{a_2}+2i\operatorname{Im}a_2\overline{a_3}+2i\operatorname{Im}a_3\overline{a_4}+\ldots+2i\operatorname{Im}a_n\overline{a_1})$$

$$=\pm\frac{i}{4}2i\operatorname{Im}(a_1\overline{a_2}+a_2\overline{a_3}+a_3\overline{a_4}+\ldots+a_n\overline{a_1})$$

$$=\pm\frac{1}{2}\operatorname{Im}(\mathbf{a},T\mathbf{a})=\pm\frac{1}{2}\operatorname{Im}(T\mathbf{a},\mathbf{a}).\ \blacksquare$$

12.7. Remark. The formula for calculating the area of a convex polygon, given in the previous theorem, applies as well when the triangle is not convex.

12.8. Example. Let be given a convex pentagon $A_1A_2A_3A_4A_5$. If we connect the midpoints of its sides consequently, we get a new pentagon. Following the procedure, we get a series of pentagons and let $S_0, S_1, S_2, \ldots$ be their areas. Prove that

$$16S_{n+2} - 12S_{n+1} + S_n = 0.$$

Solution. It is sufficient to prove that

$$16S_2 - 12S_1 + S_0 = 0 \qquad (4)$$

For this purpose we will first calculate S_1 and S_2.

$$\pm 8S_1 = \mathrm{Im}(T\mathbf{a} + T^2\mathbf{a}, \mathbf{a} + T\mathbf{a}) = 2\,\mathrm{Im}(T\mathbf{a}, \mathbf{a}) + \mathrm{Im}(T^2\mathbf{a}, \mathbf{a}) \text{ and}$$

$$\pm 2S_2 = \mathrm{Im}\left(T\left(\tfrac{\mathbf{a}+2T\mathbf{a}+T^2\mathbf{a}}{4}\right), \tfrac{\mathbf{a}+2T\mathbf{a}+T^2\mathbf{a}}{4}\right),$$

i.e.

$$\pm 32S_2 = 5\,\mathrm{Im}(T\mathbf{a}, \mathbf{a}) + 3\,\mathrm{Im}(T^2\mathbf{a}, \mathbf{a}).$$

If we eliminate $\mathrm{Im}(T^2\mathbf{a}, \mathbf{a})$ and take that $\pm S_0 = \mathrm{Im}(T\mathbf{a}, \mathbf{a})$ we get the equality (5). ■

12.9. Example. Prove that if the odd vertices of a n-gon are translated for the same vector, then the area of the new n-gon is equivalent to the area of the given n-gon.

Solution. Let **a** be the oriented n-tuple of the affixes of the n-gon`s vertices. The area of the n-gon is as following

$$S = \pm\tfrac{1}{2}\,\mathrm{Im}(T\mathbf{a}, \mathbf{a}),$$

For the area of the new n-gon we get the following

$$S' = \pm\tfrac{1}{2}\,\mathrm{Im}\big(T(\mathbf{a}+\mathbf{h}), \mathbf{a}+\mathbf{h}\big)$$

where **h** is one of the following ordered n-tuples

$$\mathbf{h} = (\alpha, 0, \alpha, 0, \ldots, \alpha, 0) \text{ or } \mathbf{h} = (\alpha, 0, \alpha, 0, \ldots, \alpha),\ \alpha \in \mathbf{C},$$

depending whether n is an even or an odd number, respectively.

We get that

$$\pm 2S' = \mathrm{Im}\big(T(\mathbf{a}+\mathbf{h}), \mathbf{a}+\mathbf{h}\big) = \mathrm{Im}\{(T\mathbf{a}, \mathbf{a}) + (T\mathbf{a}, \mathbf{h}) + (T\mathbf{h}, \mathbf{a}) + (T\mathbf{h}, \mathbf{h})\}.$$

From

$$\overline{(T\mathbf{a}, \mathbf{h})} = (\mathbf{a}, T\mathbf{h}) = (T\mathbf{a}, T^2\mathbf{h}) = (T\mathbf{a}, \mathbf{h})$$

we get the following

$$\mathrm{Im}\{(T\mathbf{a}, \mathbf{h}) + (\mathbf{a}, T\mathbf{h})\} = 0$$

and if $(T\mathbf{h}, \mathbf{h}) = 0$ we get

$$\pm 2S' = \mathrm{Im}\big(T(\mathbf{a}+\mathbf{h}), \mathbf{a}+\mathbf{h}\big) = \mathrm{Im}(T\mathbf{a}, \mathbf{a}) = \pm 2S.$$

According to this, $S = S'$, which was supposed to be proven. ■

12.10. Let's consider a $\triangle ABC$, whose vertices A, B and C have the affixes a, b and c, respectively. In the example II 3.3 we proved that

$$o=\frac{a\overline{a}(c-b)+b\overline{b}(a-c)+c\overline{c}(b-a)}{\begin{vmatrix} a & \overline{a} & 1\\ b & \overline{b} & 1\\ c & \overline{c} & 1\end{vmatrix}}$$

Is the affix of the center O of the circumcicrle of the $\triangle ABC$. Clearly, the radius of the circumcircle of the $\triangle ABC$ is $R=|a-o|$. According to this, for the radius of the circum radius if of $\triangle ABC$ is as following

$$R=|o-a|=\frac{|a-b|\cdot|b-c|\cdot|a-c|}{\left|\begin{vmatrix} a & \overline{a} & 1\\ b & \overline{b} & 1\\ c & \overline{c} & 1\end{vmatrix}\right|}$$

Since, $\overline{AB}=|a-b|$, $\overline{BC}=|b-c|$, $\overline{CA}=|c-a|$, and also applying 12.1 we get the following formula

$$R=\frac{\overline{AB}\cdot\overline{BC}\cdot\overline{CA}}{4P_{\triangle ABC}}.$$

12.11. Theorem. The ratio between the areas of two similar triangles is equivalent to the ratio of the squares of the respected sides.

Proof. Let the triangles ABC and $A_1B_1C_1$, whose affixes of the vertices are a, b, c and a_1,b_1,c_1 respectively, be similar. There are two possible cases.

a) There is a direct similarity $S(z)=dz+e$ which maps $\triangle ABC$ into $\triangle A_1B_1C_1$ and therefore we get

$$P_{\triangle A_1B_1C_1}=\frac{\left|\begin{vmatrix} a_1 & \overline{a_1} & 1\\ b_1 & \overline{b_1} & 1\\ c_1 & \overline{c_1} & 1\end{vmatrix}\right|}{4}=\frac{\left|\begin{vmatrix} da+e & \overline{da+e} & 1\\ db+e & \overline{db+e} & 1\\ dc+e & \overline{dc+e} & 1\end{vmatrix}\right|}{4}=|d|^2=\frac{\left|\begin{vmatrix} a & \overline{a} & 1\\ b & \overline{b} & 1\\ c & \overline{c} & 1\end{vmatrix}\right|}{4}=|d|^2\,P_{\triangle ABC}.$$

b) There is an indirect similarity $S(z)=d\overline{z}+e$ which maps $\triangle ABC$ to $\triangle A_1B_1C_1$. Analogously as in a) we prove that

$$P_{\triangle A_1B_1C_1}=|d|^2\,P_{\triangle ABC}.$$

The statement of the theorem is implied by the theorems 4.6 and 7.8. ■

12.12. Consequence. The ratio between the areas of two similar n-gons is equivalent to the ratio of the squares of the respected sides.

Proof. Theorems 12.6 and 12.11 directly imply the above given consequence. ■

13. INCIRCLES AND EXCIRCLES OF A TRIANGLE

13.1. Let's consider the $\triangle ABC$ whose vertices A, B and C have affixes a, b and c, respectively. In lemma 10.4 we proved that any point D with an affix d is uniquely determined by the real numbers λ, μ, ν such that

$$\lambda a + \mu b + \nu c = d \tag{1}$$

where

$$\lambda + \mu + \nu = 1\,. \tag{2}$$

Let P, P_1, P_2, P_3 be the areas of the triangles ABC, DBC, DAC and DBA, taken with an appropriate sign depending on the orientation of the triangles (figure 36). From (1) and (2) and the conjugated equation of (1) we obtain the system:

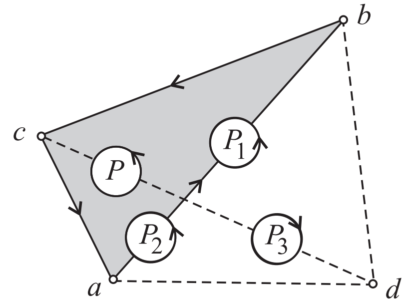

Figure 36

$$\begin{cases} \lambda + \mu + \nu = 1 \\ \lambda a + \mu b + \nu c = d \\ \lambda \bar{a} + \mu \bar{b} + \nu \bar{c} = \bar{d} \end{cases}$$

The solution of the above system is

$$\lambda = \frac{P_1}{P}, \quad \mu = \frac{P_2}{P}, \quad \nu = \frac{P_3}{P}\,. \tag{3}$$

So, we proved the following lemma.

Lemma. The numbers λ, μ and ν which according to the formulae (1) and (2) determine the position of the point D with respect to the $\triangle ABC$ are proportional to the areas of P_1, P_2 and P_3, i.e. they are determined by the relation (3). ■

13.2. Let be given a point I, with an affix z and let the distances between the point I to the sides BC, CA and AB of the $\triangle ABC$ be r_1, r_2 and r_3 respectively. The lemmas 13.1 and 10.4 imply that

$$r_1 = \frac{2\lambda P}{|b-c|}, \quad r_2 = \frac{2\mu P}{|c-a|}, \quad r_3 = \frac{2\nu P}{|a-b|}\,, \tag{4}$$

where the numbers λ, μ and ν are uniquely determined. Clearly, if

$$\lambda = \frac{|b-c|}{|a-b|+|b-c|+|c-a|}, \quad \mu = \frac{|c-a|}{|a-b|+|b-c|+|c-a|}, \quad \nu = \frac{|a-b|}{|a-b|+|b-c|+|c-a|}\,,$$

then

$$r_1 = r_2 = r_3 = r = \frac{2P}{|a-b|+|b-c|+|c-a|}$$

and for the affix of the point I we get that

$$z = \frac{|b-c|\cdot a + |c-a|\cdot b + |a-b|\cdot c}{|a-b|+|b-c|+|c-a|}\,. \tag{5}$$

Since the numbers λ, μ and ν are positive numbers, the remark10.4 implies that the point I is inside the $\triangle ABC$. So, we proved the following lemma.

Lemma. For any triangle, there is a unique point inside of it which is on the same distance from the sides of the triangle. ■

13.3. Remark. Clearly, the circle with a center in I, whose affix is given in (5), and radius

$$r=\frac{2P}{|a-b|+|b-c|+|c-a|}$$

touches the sides of the $\triangle\, ABC$, i.e. it is an incircle of the $\triangle\, ABC$.

13.4. Let A', B', C' be the points in which the circle $K(I,r)$ meets the sides BC, CA, AB of the $\triangle\, ABC$. Then, using the degree of the points A, B, C with respect to the circle $K(I,r)$ we get that

$$\overline{AC'}=\overline{AB'},\ \ \overline{BC'}=\overline{BA'},\ \ \overline{CA'}=\overline{CB'}$$

thus

$$\frac{\overrightarrow{AB'}}{\overrightarrow{B'C}}\cdot\frac{\overrightarrow{CA'}}{\overrightarrow{A'B}}\cdot\frac{\overrightarrow{BC'}}{\overrightarrow{C'A}}=1\,.$$

Now, according to the Ceva theorem we get that the lines AA', BB' and CC' intersect at a point M (figure 37). So, we proved the following lemma.

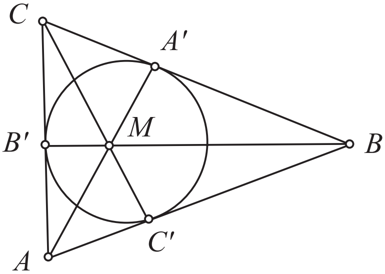

Figure 37

Lemma. The lines which connect the vertices A, B, C of $\triangle\, ABC$ to the points A', B', C' at which the sides meet the incircle $K(I,r)$ intersect in a point M, which is called a *Gergonne point* for $\triangle\, ABC$.

13.5. Remark. Analoguosly, we prove that there exist three circles which are excircle of $\triangle\, ABC$. The affixes of their centers are as following

$$z'=\frac{-|b-c|\cdot a+|c-a|\cdot b+|a-b|\cdot c}{-|a-b|+|b-c|+|c-a|},\ \ z''=\frac{|b-c|\cdot a-|c-a|\cdot b+|a-b|\cdot c}{|a-b|-|b-c|+|c-a|},\ \ z=\frac{|b-c|\cdot a+|c-a|\cdot b-|a-b|\cdot c}{|a-b|+|b-c|-|c-a|},$$

their radii are

$$r'=\frac{2P}{-|a-b|+|b-c|+|c-a|},\ \ r''=\frac{2P}{|a-b|-|b-c|+|c-a|},\ \ r'''=\frac{2P}{|a-b|+|b-c|-|c-a|},$$

(figure 38), respectively.

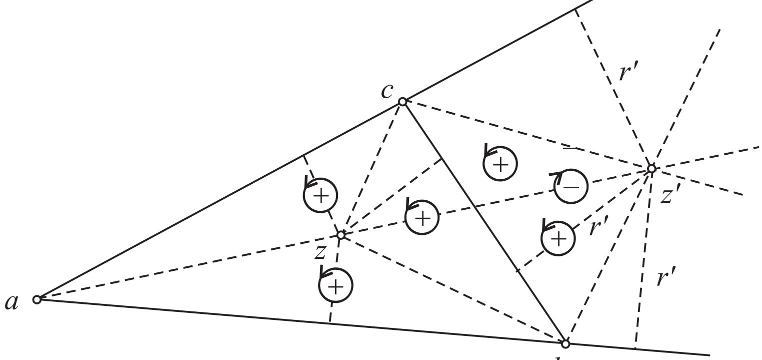

Figure 38

13.6. Lemma. The lines which connect the vertices A, B, C of the $\triangle\, ABC$ to K, F, L the points where the sides of the triangle meet the excircle concur in a point N which is called *Nagel point* for the $\triangle\, ABC$.

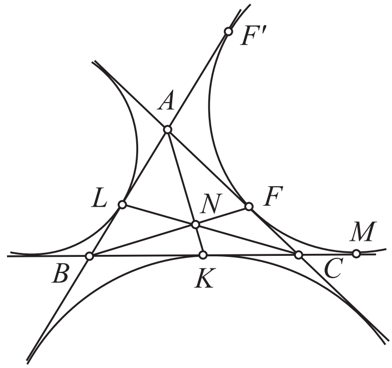

Figure 39

Proof. Firstly we are going to prove that

$$\overline{BF'} = \overline{BM} = \frac{\overline{AB}+\overline{BC}+\overline{CA}}{2}$$

(figure 39). There is

$$\overline{BF'} = \overline{BA} + \overline{AF'} = \overline{BA} + \overline{AF}$$

and

$$\overline{BM} = \overline{BC} + \overline{CM} = \overline{BC} + \overline{CF}\,.$$

If we add the last two equations and take that

$$\overline{AC} = \overline{AF} + \overline{CF}$$

we get the required equality. According to this,

$$\overline{AF} = \overline{AF'} = \overline{BF'} - \overline{BA} = \frac{\overline{AB}+\overline{BC}+\overline{CA}}{2} - \overline{BA} = \frac{-\overline{AB}+\overline{BC}+\overline{CA}}{2}.$$

Analogously to this, we prove that

$$\overline{CK} = \overline{LA} = \frac{\overline{AB}+\overline{BC}-\overline{CA}}{2}\,,\ \overline{BL} = \overline{FC} = \frac{\overline{AB}-\overline{BC}+\overline{CA}}{2}\,,\ \overline{KB} = \frac{-\overline{AB}+\overline{BC}+\overline{CA}}{2}\,.$$

Now, the statement in Lemma is implied from Ceva`s theorem, thereby

$$\frac{\overrightarrow{AF}}{\overrightarrow{FC}} \cdot \frac{\overrightarrow{CK}}{\overrightarrow{KB}} \cdot \frac{\overrightarrow{BL}}{\overrightarrow{LA}} = 1\,. \blacksquare$$

13.7. Remark. Clearly, the lines AI, BI and CI are bisectors of the inside angles of the $\triangle\, ABC$ and their equations are

$$z - a = \frac{|c-a|\cdot(b-a)+|a-b|\cdot(c-a)}{|c-a|\cdot(\overline{b}-\overline{a})+|a-b|\cdot(\overline{c}-\overline{a})}(\overline{z} - \overline{a})\,,$$

$$z - b = \frac{|b-c|\cdot(a-b)+|a-b|\cdot(c-b)}{|b-c|\cdot(\overline{a}-\overline{b})+|a-b|\cdot(\overline{c}-\overline{b})}(\overline{z} - \overline{b}) \text{ and}$$

$$z - c = \frac{|b-c|\cdot(a-c)+|c-a|\cdot(b-c)}{|b-c|\cdot(\overline{a}-\overline{c})+|c-a|\cdot(\overline{b}-\overline{c})}(\overline{z} - \overline{c})\,,$$

respectively. Analogously, we can determine the bisectors of the outside angles of the $\triangle\, ABC$.

13.8. Let A_1 be the point where the bisector AI meets the side BC. Its affix is $a_1 = \frac{b+\lambda c}{1+\lambda}$. Since the points A, I and A_1 are collinear, by applying the consequence II 1.3 we get that

$$\frac{|c-a|\cdot(b-a)+|a-b|\cdot(c-a)}{|c-a|\cdot(\overline{b}-\overline{a})+|a-b|\cdot(\overline{c}-\overline{a})} = \frac{b-a+\lambda(c-a)}{\overline{b}-\overline{a}+\lambda(\overline{c}-\overline{a})}\,.$$

The latter is equivalent to

$$\left(\lambda - \frac{|a-b|}{|c-a|}\right)\left(\frac{\overline{c}-\overline{a}}{c-a} - \frac{\overline{b}-\overline{a}}{b-a}\right) = 0$$

Now, since the points A, B and C are not collinear we get that $\lambda = \frac{|a-b|}{|c-a|}$, i.e. the affix of the point A_1 is

$$a_1 = \frac{|c-a|\cdot b+|a-b|\cdot c}{|c-a|+|a-b|}.$$

Analogously, the affixes of B_1 and C_1 the points where the bisectors BI and CI meet the sides AC and AB are

$$b_1 = \frac{|b-c|\cdot a+|a-b|\cdot c}{|b-c|+|a-b|} \text{ and } c_1 = \frac{|c-a|\cdot b+|b-c|\cdot a}{|c-a|+|b-c|},$$

respectively. Thus

$$\overline{BA_1} = \frac{\overline{AB}\cdot\overline{BC}}{\overline{CA}+\overline{AB}},\ \overline{CA_1} = \frac{\overline{AC}\cdot\overline{BC}}{\overline{CA}+\overline{AB}} \tag{6}$$

i.e.

$$\frac{\overline{BA_1}}{\overline{CA_1}} = \frac{\overline{AB}}{\overline{AC}}. \tag{7}$$

Therefore, we proved the following lemma.

Lemma. If A_1 is the point where AI, the bisector of the inside angle at the vertex A of the $\triangle ABC$, meets the side BC, then the equalities (6) and (7) are satisfied. ■

13.9. Remark. Clearly, the analogous equalities to the equalitity (6) hold true for the bisector BI and CI of the inside angles at the vertices B and C of the $\triangle ABC$. Furthermore, if B_1 and C_1 are the points of where the bisector meets the sides CA and AB respectively, then

$$\frac{\overline{AB_1}}{\overline{CB_1}} = \frac{\overline{BA}}{\overline{BC}} \text{ and } \frac{\overline{AC_1}}{\overline{BC_1}} = \frac{\overline{CA}}{\overline{CB}}.$$

13.10. Theorem (Euler). Let O be I the circumcenter and the incenter of the $\triangle ABC$, and R and r be their radii, respectively. Then,

$$\overline{OI}^2 = R^2 - 2Rr.$$

Proof. Without loss of generality, we can say that the center of the circumscribed circle coincides with the origin. If a, b, c are the affixes of the vertices of the $\triangle ABC$, then $|a| = |b| = |c| = R$ and according to the proof of lemma 13.2 the affix of the incenter is as following

$$z = \frac{|b-c|\cdot a+|c-a|\cdot b+|a-b|\cdot c}{|a-b|+|b-c|+|c-a|}.$$

Therefore,

$$\overline{OI}^2 = \frac{(|b-c|\cdot a+|c-a|\cdot b+|a-b|\cdot c)(|b-c|\cdot \overline{a}+|c-a|\cdot \overline{b}+|a-b|\cdot \overline{c})}{(|a-b|+|b-c|+|c-a|)^2}.$$

Now, the statement in Lemma is directly implied by the operations of complex numbers, the formula for the radius of circumcircle (discussed in 12.10) and the formula for the radius of incircle (discussed in Remark 13.3)

13.11. Remark. Analogously, as in theorem 13.10 it can be proved that $\overline{OI'}^2 = R^2 + 2Rr'$, where I' and r' are the center and the radius of the excircle of the $\triangle$ *ABC*, and *O* and *R* are the center and the radius of the inscribed circle of the $\triangle$ *ABC*.

13.12. Remark. Let the unit circle be inscribed in the $\triangle$ *ABC* whose vertices *A*, *B*, *C* have the affixes *a*, *b*, *c*, respectively, and let it meets the sides *BC*, *CA*, *AB* at the points *P*, *Q*, *R* with affixes *p*, *q*, *r*, respectively. Then, according to the remark II 3.12 d) we get that

$$a = \frac{2qr}{q+r}, \; b = \frac{2rp}{r+p} \text{ and } c = \frac{2pq}{p+q}.$$

Furthermore, the example II 3.3 implies that the circum center of $\triangle$ *ABC* has the affix

$$o = \frac{2pqr(p+q+r)}{(p+q)(q+r)(r+p)},$$

The remark 5.4 implies that the orthocenter of the $\triangle$ *ABC* has an affix

$$h = \frac{2\left(p^2q^2+q^2r^2+r^2p^2+pqr(p+q+r)\right)}{(p+q)(q+r)(r+p)}.$$

13.13. At the end of this part let's note that the following theorem can be proved.

Theorem. Let $\triangle$ *ABC*, whose vertices *A*, *B* and *C* have the affixes *a*, *b* and *c*, respectively, be inscribed in the unit circle. Then, there are complex numbers *u*, *v*, *w* such that $a = u^2$, $b = v^2$, $c = w^2$ and the midpoints of the arcs *AB*, *BC*, *CA* which don't consist of the points *C*, *A*, *B* are points with affixes $-uv, -vw, -wu$, respectively. Therefore, the affix of the incenter *L* of the $\triangle$ *ABC* is $l = -(uv + vw + wu)$. ■

13.14. Example. Let *L* be the incenter of the $\triangle$ *ABC*, and the lines *AL*, *BL*, *CL* meets the circums circle of the $\triangle$ *ABC* at the points A_1, B_1, C_1, respectively. If *R* is the radius of the circumscribed, and *r* the radius of the inscribed circle of the $\triangle$ *ABC* prove that:

a) $\frac{\overline{LA_1}\cdot\overline{LC_1}}{\overline{LB}} = R$, b) $\frac{\overline{LA}\cdot\overline{LB}}{\overline{LC_1}} = 2r$, c) $\frac{P_{\Delta ABC}}{P_{\Delta A_1B_1C_1}} = \frac{2r}{R}$.

Solution. Let the circumscribed circle of the $\triangle$ *ABC* be the unite circle and let *u*, *v*, *w* are the complex numbers as in the theorem 13.13. According to that theorem, we get that $l = -(uv + vw + wu)$ and $a = u^2$, $b = v^2$, $c = w^2$ and the midpoints of the arcs *AB*, *BC*, *CA* which don't consist of the points *C*, *A*, *B* are points with affixes $-uv, -vw, -wu$, respectively. Furthermore, since,

$$\frac{l-(-vw)}{\overline{l}-\overline{(-vw)}} = \frac{-uv-uw}{-\frac{1}{uv}-\frac{1}{uw}} = vwu^2 \text{ and } \frac{a-(-vw)}{\overline{a}-\overline{(-vw)}} = \frac{u^2+vw}{\frac{1}{u^2}+\frac{1}{vw}} = u^2vw$$

we get that the points with affixes *a*, *l* and $-vw$ are collinear, which means that the point A_1 has an affix $a_1 = -vw$. Similarly, the affixes of the points B_1 and C_1 are $b_1 = -uw$ and $c_1 = -uv$, respectively.

a) the statement is implied by the equality

$$\frac{\overline{LA_1}\cdot\overline{LC_1}}{\overline{LB}}=\frac{|l-a_1|\cdot|l-c_1|}{|l-b|}=\frac{|u(v+w)|\cdot|w(u+v)|}{|uv+uw+vw+v^2|}=\frac{|v+w|\cdot|u+v|}{|(u+v)(w+v)|}=1=R\,.$$

b) If z is the affix of the point where the incircle meets the side BC, then z is the affix of the foot of the perpendicular drawn from the point L to the side BC. Thus its affix is

$$z=\tfrac{1}{2}(b+c+l-bc\bar{l})$$

thus

$$r=|\,l-z\,|=\tfrac{1}{2}\left|\frac{(u+v)(v+w)(w+u)}{u}\right|=\tfrac{1}{2}\,|\,(u+v)(v+w)(w+u)\,|\,.$$

Therefore,

$$\begin{aligned}\frac{\overline{LA}\cdot\overline{LB}}{\overline{LC_1}}&=\frac{|(u+v)(u+w)|\cdot|(u+v)(v+w)|}{|w(u+v)|}\\&=|\,(u+v)(v+w)(w+u)\,|=2r.\end{aligned}$$

c) The areas of the triangles are

$$P_{\vartriangle ABC}=\pm\frac{i}{4}\begin{vmatrix}u^2 & 1/u^2 & 1\\ v^2 & 1/v^2 & 1\\ w^2 & 1/w^2 & 1\end{vmatrix}\text{ and }P_{\vartriangle A_1B_1C_1}=\pm\frac{i}{4uvw}\begin{vmatrix}vw & u & 1\\ uw & v & 1\\ uv & w & 1\end{vmatrix},$$

thus

$$\begin{aligned}\left|\frac{P_{\Delta ABC}}{P_{\Delta A_1B_1C_1}}\right|&=\left|\frac{u^4w^2+w^4v^2+v^4u^2-v^4w^2-u^4v^2-w^4u^2}{v^2w+vu^2+uw^2-uv^2-vw^2-wu^2}\right|\\&=\left|\frac{(v^2-u^2)(v^2u^2+w^4-w^2u^2-w^2v^2)}{(u-v)(uv+w^2-wu-wv)}\right|\\&=\left|\frac{(u-v)(u+v)\left[(uv+w^2)^2-(wu+wv)^2\right]}{(u-v)(uv+w^2-wu-wv)}\right|\\&=\left|\frac{(u-v)(u+v)(uv+w^2-wu-wv)(uv+w^2+wu+wv)}{(u-v)(uv+w^2-wu-wv)}\right|\\&=|\,(u+v)(uv+w^2+wu+wv)\,|\\&=|\,(u+v)(v+w)(w+u)\,|=\frac{2r}{R},\end{aligned}$$

which was required to be proven. ■

14. STEWART`S THEOREM

14.1. Let a, b, c be the affixes of the vertices A, B, C of the $\triangle ABC$ and let the point D be on the side BC (figure 40). Then, the affix of D is $d = \frac{b+\lambda c}{1+\lambda}$, for some $\lambda \in (0,1)$. According to this,

$$\overline{AB} = |b-a|,\ \overline{AC} = |c-a|,\ \overline{BC} = |b-c|,$$

$$\overline{BD} = \frac{\lambda|b-c|}{1+\lambda},\ \overline{CD} = \frac{|b-c|}{1+\lambda},\ \overline{AD} = \frac{|b-a+\lambda(c-a)|}{1+\lambda}$$

Thus

$$\overline{AC}^2 \cdot \overline{BD} + \overline{AB}^2 \cdot \overline{CD} - \overline{BC} \cdot \overline{CD} \cdot \overline{BD} = \frac{|b-c|}{1+\lambda}(\lambda|c-a|^2 + |b-a|^2 - \frac{\lambda}{1+\lambda}|b-c|^2)$$

$$= |b-c| \frac{\lambda^2(c-a)(\overline{c}-\overline{a}) + (b-a)(\overline{b}-\overline{a}) + \lambda(c-a)(\overline{c}-\overline{a}) + \lambda(b-a)(\overline{b}-\overline{a}) - \lambda(b-c)(\overline{b}-\overline{c})}{(1+\lambda)^2}$$

$$= |b-c| \frac{\lambda^2(c-a)(\overline{c}-\overline{a}) + (b-a)(\overline{b}-\overline{a}) + \lambda(c-a)(\overline{b}-\overline{a}) + \lambda(b-a)(\overline{c}-\overline{a})}{(1+\lambda)^2}$$

$$= |b-c| \frac{|b-a+\lambda(c-a)|^2}{(1+\lambda)^2} = \overline{AD}^2 \cdot \overline{BC}.$$

So, we proved the following theorem.

Theorem (Stewart). If D is a point on the side BC of the $\triangle ABC$, such that O is between the points B and C, the following equality holds true

$$\overline{AC}^2 \cdot \overline{BD} + \overline{AB}^2 \cdot \overline{CD} - \overline{BC} \cdot \overline{CD} \cdot \overline{BD} = \overline{AD}^2 \cdot \overline{BC}. \ \blacksquare \qquad (1)$$

14.2. Example. Let m_A, m_B, m_C be the lengths of the medians of the $\triangle ABC$ drawn from the vertices A, B, C respectively. Prove that

$$m_A^2 = \frac{\overline{AC}^2 + \overline{AB}^2}{2} - \frac{\overline{BC}^2}{4},\quad m_B^2 = \frac{\overline{BC}^2 + \overline{BA}^2}{2} - \frac{\overline{AC}^2}{4},\quad m_C^2 = \frac{\overline{CA}^2 + \overline{CB}^2}{2} - \frac{\overline{AB}^2}{4}. \ (2)$$

Solution. We shall only prove the first equality in (2). The other two equalities can be proved analogously.

Let A' be the midpoint of the side BC. Since

$$\overline{CA'} = \overline{BA'} = \frac{\overline{BC}}{2} \text{ and } m_A = \overline{AA'},$$

By applying the Stewart theorem, we get the following equality

$$m_A^2 \cdot \overline{BC} = \overline{AC}^2 \cdot \frac{\overline{BC}}{2} + \overline{AB}^2 \cdot \frac{\overline{BC}}{2} - \overline{BC} \cdot \frac{\overline{BC}}{2} \cdot \frac{\overline{BC}}{2}$$

If we divide it by $\overline{BC}$ we get the following equality

$$m_A^2 = \frac{\overline{AC}^2 + \overline{AB}^2}{2} - \frac{\overline{BC}^2}{4}. \ \blacksquare$$

14.3. Example. The lengths of the medians of the $\triangle ABC$ are $m_A = 9$ cm, $m_B = 12$ cm and $m_C = 15$ cm. Calculate the lengths of its sides.

Solution. From the equalities (2) using the length of the medians we can find the length of the sides of the $\triangle ABC$.

$$\overline{AB}^2 = \frac{8(m_A^2+m_b^2)-4m_C^2}{9},\quad \overline{BC}^2 = \frac{8(m_B^2+m_C^2)-4m_A^2}{9},\quad \overline{CA}^2 = \frac{8(m_C^2+m_A^2)-4m_B^2}{9}. \quad (3)$$

Using the equality (3) we get that

$$\overline{AB} = 10 \text{ cm},\ \overline{BC} = 2\sqrt{73} \text{ cm and } CA = 4\sqrt{13} \text{ cm}. \blacksquare$$

14.4. Example. Express the length of the bisectors of the interior angles of the $\triangle ABC$ using the length of its sides.

Solution. Let D be the point where, the bisector l_A of the angle at the vertex A, meets the side BC. In lemma 13.8 we proved that

$$\overline{BD} = \frac{\overline{AB}\cdot\overline{BC}}{\overline{CA}+\overline{AB}} \text{ and } \overline{CD} = \frac{\overline{AC}\cdot\overline{BC}}{\overline{CA}+\overline{AB}}.$$

According to the Stewart theorem, we get that

$$l_A^2 = \overline{AB}\cdot\overline{AC}\left(1-\frac{\overline{BC}^2}{(\overline{AB}+\overline{AC})^2}\right).$$

It can be proved analogously that

$$l_B^2 = \overline{AB}\cdot\overline{BC}\left(1-\frac{\overline{AC}^2}{(\overline{AB}+\overline{BC})^2}\right) \text{ and } l_C^2 = \overline{AC}\cdot\overline{BC}\left(1-\frac{\overline{AB}^2}{(\overline{AC}+\overline{BC})^2}\right). \blacksquare$$

14.5. Definition. The line symmetrical to the median with respect to the bisector of the angle drawn from the same vertex is called *symmedian*.

14.6. Let AA_1, AA' and AA'' be the bisectors of the angle, the median and the symmedian of the $\triangle ABC$, drawn from the vertex A (figure 41). The equations of the median and the symmetry are the following

$$z-a = \frac{b+c-2a}{\overline{b}+\overline{c}-2\overline{a}}(\overline{z}-\overline{a}) \text{ and } z-a = \frac{|c-a|(b-a)+|a-b|(c-a)}{|c-a|(\overline{b}-\overline{a})+|a-b|(\overline{c}-\overline{a})}(\overline{z}-\overline{a}),$$

respectively.

The indirect similarity

$$S(z) = \frac{|c-a|(b-a)+|a-b|(c-a)}{|c-a|(\overline{b}-\overline{a})+|a-b|(\overline{c}-\overline{a})}(\overline{z}-\overline{a})+a \quad (4)$$

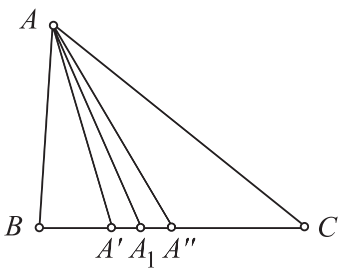

Figure 41

is a line symmetry and the line of symmetry is a bisector of the angle at the vertex A by direct calculations we get that the line with an equation

$$z-a = \frac{|c-a|^2(b-a)+|a-b|^2(c-a)}{|c-a|^2(\overline{b}-\overline{a})+|a-b|^2(\overline{c}-\overline{a})}(\overline{z}-\overline{a}) \quad (5)$$

is an image of the median e under the line symmetry (4). According to this, (5) is the equation of the symmedian drawn from the vertex A. A'' the point where the symmedian and the side BC has the affix $a"=\frac{b+\lambda c}{1+\lambda}$. But, the point $A"$ lies on the line (5), thus the following equality is satisfied

$$\frac{|c-a|^2(b-a)+|a-b|^2(c-a)}{|c-a|^2(\overline{b}-\overline{a})+|a-b|^2(\overline{c}-\overline{a})}=\frac{b-a+\lambda(c-a)}{\overline{b}-\overline{a}+\lambda(\overline{c}-\overline{a})}$$

which is equivalent to the equality

$$\left(\lambda-\frac{|a-b|^2}{|c-a|^2}\right)\left(\frac{\overline{c}-\overline{a}}{c-a}-\frac{\overline{b}-\overline{a}}{b-a}\right)=0$$

and since the points A, B and C are not collinear we get that $\lambda=\frac{|a-b|^2}{|c-a|^2}$, i.e. the affix of the point $A"$ is

$$a"=\frac{|c-a|^2 b+|a-b|^2 c}{|c-a|^2+|a-b|^2}.$$

Analogously to this, for the points of intersection $B"$ and $C"$ of the other symmedians and the sides AC and AB we get that

$$b"=\frac{|b-c|^2 a+|a-b|^2 c}{|b-c|^2+|a-b|^2} \text{ and } c"=\frac{|c-a|^2 b+|b-c|^2 a}{|c-a|^2+|b-c|^2},$$

respectively.

The above stated impies that, thus,

$$\overline{BA"}=\frac{\overline{AB}^2\cdot\overline{BC}}{\overline{AB}^2+\overline{CA}^2} \text{ and } \overline{CA"}=\frac{\overline{AC}^2\cdot\overline{BC}}{\overline{AB}^2+\overline{CA}^2} \tag{6}$$

so

$$\frac{\overline{BA"}}{\overline{CA"}}=\frac{\overline{AB}^2}{\overline{CA}^2}. \tag{7}$$

So we proved the following lemma.

Lemma. If $A"$ is the point of intersection between the symmedian at the vertex A of the $\triangle ABC$ and the side BC, then the equalities (6) and (7) are satisfied. ■

14.7. Consequence. The symmedians $AA"$, $BB"$ and $CC"$ of the $\triangle ABC$ are concurrent.

Proof. The proof is directly implied by lemma 14.6 and the Ceva theorem. ■

14.8. Example. Express the lengths of the symmedians of the $\triangle ABC$ using the lengths of the sides of the triangle.

Solution. By using the equalities (6), and the Stewart theorem for the length of the symmedian $AA"$ we get the following

$$\overline{A"A}^2=\overline{AC}^2\cdot\overline{AB}^2\left(2-\frac{\overline{BC}^2}{(\overline{AC}^2+\overline{AB}^2)^2}\right).$$

Analogously to this, for the symmedians $BB"$ and $CC"$ the following holds true

$$\overline{B"B}^2=\overline{BC}^2\cdot\overline{AB}^2\left(2-\frac{\overline{AC}^2}{(\overline{BC}^2+\overline{AB}^2)^2}\right) \text{ and } \overline{C"C}^2=\overline{AC}^2\cdot\overline{BC}^2\left(2-\frac{\overline{AB}^2}{(\overline{AC}^2+\overline{BC}^2)^2}\right). \blacksquare$$

15. SIMPSON LINE

15.1. Theorem (Simpson). Let D be point on the circumcircle of $\triangle ABC$. Then the foots of the perpendiculars drawn from the point D to the sides of the $\triangle ABC$ are collinear.

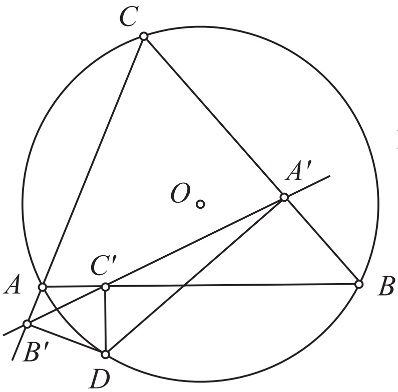

Figure 42

Proof. With no loss of generality, we can say that the $\triangle ABC$ is inscribed in the unit circle. Let a', b', c' be the affixes of A', B', C' the foots of perpendicular drawn from the point D to the sides BC, CA and AB, respectively (figure 42). If we take that

$$\overline{a} = \frac{1}{a},\ \overline{b} = \frac{1}{b},\ \overline{c} = \frac{1}{c} \text{ and } \overline{d} = \frac{1}{d},$$

Then the example 1.9 implies that

$$a' = \frac{1}{2}\left(b + c + d - \frac{bc}{d}\right),\ b' = \frac{1}{2}\left(c + a + d - \frac{ac}{d}\right) \text{ and}$$
$$c' = \frac{1}{2}\left(a + b + d - \frac{ab}{d}\right).$$

Thus,

$$\frac{\overline{a'}-\overline{c'}}{\overline{b'}-\overline{c'}} = \frac{\overline{b+c+d-\frac{bc}{d}-\left(a+b+d-\frac{ab}{d}\right)}}{\overline{c+a+d-\frac{ac}{d}-\left(a+b+d-\frac{ab}{d}\right)}} = \frac{\overline{(c-a)(d-b)}}{\overline{(c-b)(d-a)}} = \frac{(a-c)(b-d)}{(b-c)(a-d)}$$
$$= \frac{(c-a)(b-d)}{(c-b)(d-a)} = \frac{b+c+d-\frac{bc}{d}-\left(a+b+d-\frac{ab}{d}\right)}{c+a+d-\frac{ac}{d}-\left(a+b+d-\frac{ab}{d}\right)} = \frac{a'-c'}{b'-c'},$$

thus the consequence 1.4 implies that the points A', B', C' are collinear. ■

15.2. Definition. The line such that the points A', B', C' (defined as in theorem 15.1) lay on it, is called *Simpson line* of the point D with respect to the $\triangle ABC$.

15.3. We shall give the equation of Simpson line for the point D about the $\triangle ABC$. If Z is an arbitrary point on the Simpson line, then the points Z, A', C' are collinear, therefore $\frac{z-c'}{a'-c'} = \frac{\overline{z}-\overline{c'}}{\overline{a'}-\overline{c'}}$, and for the Simpson line we get that

$$z - \overline{z}\,\frac{a'-c'}{\overline{a'}-\overline{c'}} + \frac{a'\overline{c'}-c'\overline{a'}}{\overline{a'}-\overline{c'}} = 0.$$

Furthermore, according to the proof of theorem 15.1 we get that

$$a'-c' = \frac{1}{2}(c-a)\left(1-\frac{b}{d}\right),\ \overline{a'}-\overline{c'} = \frac{1}{2ac}(a-c)\left(1-\frac{d}{b}\right),$$

therefore $\frac{a'-c'}{\overline{a'}-\overline{c'}} = \frac{acb}{d}$. In order to determine the constant term $m = \frac{a'\overline{c'}-c'\overline{a'}}{\overline{a'}-\overline{c'}}$ we shall take that the point B' is on the Simpson line, i.e.

$$\frac{1}{2}\left(c+a+d-\frac{ac}{d}\right) - \frac{1}{2}\left(\overline{c}+\overline{a}+\overline{d}-\frac{\overline{ac}}{\overline{d}}\right)\frac{acb}{d} + m = 0.$$

So,

$$m = \frac{abc}{2d}(\overline{a}+\overline{b}+\overline{c}+\overline{d}) - \frac{1}{2}(a+b+c+d)$$

thus the equation of Simpson line of the point D with respect to the $\triangle ABC$ is

$$z - \overline{z}\frac{acb}{d} + \frac{abc}{2d}(\overline{a}+\overline{b}+\overline{c}+\overline{d}) - \frac{1}{2}(a+b+c+d) = 0. \qquad (1)$$

15.4. Example. Let the points A, B, C, D lie on a same circle. Prove that the point X where the Simpson line of A for the $\triangle BCD$ meets the Simpson line of B for the $\triangle ACD$ is on the line which passes through the point C and the orthocenter H of the $\triangle ABD$.

Solution. The quadrilateral $ABCD$ is cyclic, so we can say that it is inscribed in a unit circle. Let a', a'', a''' be the affixes of A', A'', A''' the foot of the perpendiculars drawn from the point A to the lines BC, CD, DB, respectively, and b', b'', b''' are the affixes of the food of perpendicular B', B'', B''' of the normal lines drawn from the point B of the lines AC, CD, DA, respectively. So,

$$a' = \frac{1}{2}\left(a+b+c-\frac{bc}{a}\right),\; a'' = \frac{1}{2}\left(a+b+d-\frac{bd}{a}\right),\; a''' = \frac{1}{2}\left(a+c+d-\frac{cd}{a}\right),$$

$$b' = \frac{1}{2}\left(b+a+c-\frac{ac}{b}\right),\; b'' = \frac{1}{2}\left(b+c+d-\frac{cd}{b}\right),\; a''' = \frac{1}{2}\left(b+d+a-\frac{ad}{b}\right).$$

The equation of Simpson line of the point A is: $z - a' = \frac{a''-a'}{\overline{a''}-\overline{a'}}(\overline{z}-\overline{a'})$, i.e.

$$z - a' = \frac{bcd}{a}(\overline{z}-\overline{a'}), \qquad (2)$$

and the equation of Simpson line of the point B is: $z - b' = \frac{b''-b'}{\overline{b''}-\overline{b'}}(\overline{z}-\overline{b'})$, i.e.

$$z - b' = \frac{acd}{b}(\overline{z}-\overline{b'}). \qquad (3)$$

By solving the system of equations (2) and (3) we get the affix x of the point of intersection X, as following

$$x = \frac{1}{2}(a+b+c+d).$$

Moreover, the affix of the orthocenter H of the $\triangle ABD$ is $h = a+c+d$ and since

$$\frac{h-c}{\overline{h}-\overline{c}} = \frac{a+b+d-c}{\overline{a}+\overline{b}+\overline{d}-\overline{c}} = \frac{a+b+c+d-2c}{\overline{a}+\overline{b}+\overline{c}+\overline{d}-2\overline{c}} = \frac{\frac{1}{2}(a+b+c+d)-c}{\frac{1}{2}(\overline{a}+\overline{b}+\overline{c}+\overline{d})-\overline{c}} = \frac{x-c}{\overline{x}-\overline{c}},$$

According to the consequence II 1.3 we get that the points C, H and X are collinear. ■

15.5. Example. Let $l(N,PQR)$ be the SImson line of the point N of the $\triangle PQR$. Let the points A, B, C, D be on a same circle. Prove that the lines $l(A,BCD)$, $l(B,ACD)$, $l(C,ABD)$, $l(D,ABC)$ concur.

Solution. We shall say that the points A, B, C, D lie on a unit circle. According to the example 15.4 the lines $l(A,BCD)$ and $l(B,ACD)$ intersect at the point X with an affix $x = \frac{1}{2}(a+b+c+d)$. The right part of the last equality is symmetrical with respect to the affixes a, b, c, d of the points A, B, C, D, thus the point X is the point of intersection of the Simpsons lines $l(A,BCD)$, $l(B,ACD)$, $l(C,ABD)$, $l(D,ABC)$. ■

15.6. Example. Let $l(P,ABC)$ and $l(Q,ABC)$ be the Simpsons lines of the points P and Q with respect to the $\triangle ABC$ and O be the center of the circum circle of the $\triangle ABC$. Prove that $\angle\big(l(P,ABC),l(Q,ABC)\big)=\frac{1}{2}\angle POQ$.

Solution. We shall say that $\triangle ABC$ is inscribed in the unit circle. 15.3 implies that the equations of the lines $l(P,ABC)$ and $l(Q,ABC)$ are

$$z-\overline{z}\frac{acb}{p}+\frac{abc}{2p}(\overline{a}+\overline{b}+\overline{c}+\overline{p})-\frac{1}{2}(a+b+c+p)=0 \text{ and}$$

$$z-\overline{z}\frac{acb}{q}+\frac{abc}{2q}(\overline{a}+\overline{b}+\overline{c}+\overline{q})-\frac{1}{2}(a+b+c+q)=0,$$

respectively. The complex angle coefficients of the lines $l(P,ABC)$ and $l(Q,ABC)$ are $\eta=\frac{acb}{p}$ and $\eta'=\frac{acb}{q}$, respectively. According to theorem II 1.7 the oriented angle φ between the lines $l(P,ABC)$ and $l(Q,ABC)$ is given by the formula $e^{2i\varphi}=\frac{\eta}{\eta'}=\frac{q}{p}$, which according to I 8.7 means that

$$\angle\big(l(P,ABC),l(Q,ABC)\big)=\frac{1}{2}\angle POQ. \ \blacksquare$$

16. PTOLEMY`S THEOREM

16.1. Lemma. If $z_j,\ j=1,2,3,4$ are the affixes of the consecutive vertices of a cyclic quadrilateral, then

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_2-z_3)}>0 \qquad (1)$$

Proof. With no loss of generality we shall say that the center of the circum circle coincides with the origin, and the radius of the circle is r. Then, $z_j=re^{i\varphi_j}$, $j=1,2,3,4$. Also, we can assume that consecutively of the vertices is equivalent to the condition

$$\varphi_1<\varphi_2<\varphi_3<\varphi_4<\varphi_1+2\pi. \qquad (2)$$

Thus, it holds true that:

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_2-z_3)}=\frac{(e^{i\varphi_1}-e^{i\varphi_2})(e^{i\varphi_3}-e^{i\varphi_4})}{(e^{i\varphi_1}-e^{i\varphi_4})(e^{i\varphi_2}-e^{i\varphi_3})}$$

$$=\frac{\left(e^{i\frac{\varphi_1-\varphi_2}{2}}-e^{-i\frac{\varphi_1-\varphi_2}{2}}\right)\left(e^{i\frac{\varphi_3-\varphi_4}{2}}-e^{-i\frac{\varphi_3-\varphi_4}{2}}\right)}{\left(e^{i\frac{\varphi_1-\varphi_4}{2}}-e^{-i\frac{\varphi_1-\varphi_4}{2}}\right)\left(e^{i\frac{\varphi_2-\varphi_3}{2}}-e^{-i\frac{\varphi_2-\varphi_3}{2}}\right)}$$

$$=\frac{\sin\frac{\varphi_1-\varphi_2}{2}\sin\frac{\varphi_3-\varphi_4}{2}}{\sin\frac{\varphi_1-\varphi_4}{2}\sin\frac{\varphi_2-\varphi_3}{2}}>0$$

therefore according to (2), each argument of sin belongs to the interval $(-\pi,0)$. Thus the inequality (1). is proved. ■

16.2. Theorem (Ptolemy). The product of the lengths of the diagonals of a cyclic quadrilateral is equal to the sum of the products of its lengths of the opposite sides.

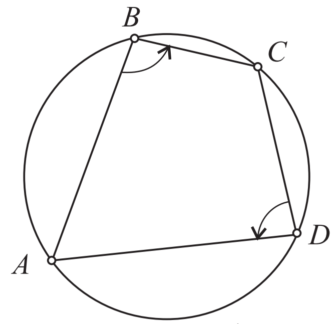

Figure 43

Proof. Let z_j, $j = 1,2,3,4$ be the affixes of the vertices A, B, C, D of a cyclic quadrilateral $ABCD$, figure 43. The statement of the theorem is equal to the equality

$$\overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD},$$

i.e. the equality

$$|z_1 - z_3| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| + |z_1 - z_4| \cdot |z_2 - z_3|. \quad (3)$$

According to the lemma 26.1 we have that

$$\begin{aligned}|(z_1 - z_2)(z_3 - z_4)| + |(z_1 - z_4)(z_2 - z_3)| &= |(z_1 - z_2)(z_3 - z_4) + (z_1 - z_4)(z_2 - z_3)| \\ &= |-z_1 z_4 - z_2 z_3 + z_1 z_2 + z_3 z_4| \\ &= |(z_1 - z_3)(z_2 - z_4)|,\end{aligned}$$

which means that the equality (3) is satisfied. ■

16.3. Theorem. If for the complex numbers p, q, r, s it is true that $\frac{(p-s)(r-q)}{(p-q)(r-s)} \in \mathbf{R}$, then they are the affixes of the consecutive vertices P, Q, R, S of a cyclic quadrilateral (cyclic) or they are collinear.

Proof. In extended complex plane the points Q, R, S, determine a circle (K). A Möbius transformation determined by $f(q) = \infty$, $f(r) = 1$, $f(s) = 0$ maps the circle (K) on the real axis and thereby theorem II 10.8 it is defined by $f(z) = \frac{(z-s)(r-q)}{(z-q)(r-s)}$. The point P is on the circle (K) if and only if its image is on the real axis i.e. if and only if $f(p) = \frac{(p-s)(r-q)}{(p-q)(r-s)} \in \mathbf{R}$. ■

16.4. Example. An equilateral triangle ABC is inscribed in a circle. An arbitrary point M is on the arc $\overset{\frown}{BC}$ on which doesn't belong the point A. Prove that $\overline{BM} + \overline{CM} = \overline{AM}$.

Solution. By applying the Ptolemy`s theorem on the cyclic quadrilateral $ABMC$ we get that

$$\overline{BM} \cdot \overline{CA} + \overline{CM} \cdot \overline{AB} = \overline{BC} \cdot \overline{AM}. \quad (4)$$

But, the triangle ABC is equilateral, $\overline{AB} = \overline{BC} = \overline{CA}$, so thereby (4) holds true we get that

$$\overline{BM} \cdot \overline{AB} + \overline{CM} \cdot \overline{AB} = \overline{AB} \cdot \overline{AM}$$

and if we divide the last equality by $\overline{AB}$ we get the necessary equality. ■

16.5. Example. Given are circles k_1, k_2, k_3, k_4 so required that

$$k_1 \cap k_2 = \{A_1, B_1\},\ k_2 \cap k_3 = \{A_2, B_2\},\ k_3 \cap k_4 = \{A_3, B_3\} \text{ and }$$
$$k_4 \cap k_1 = \{A_4, B_4\}.$$

If the points A_1, A_2, A_3, A_4 are concyclic (they lie on a same circle too) or are collinear, then the points B_1, B_2, B_3, B_4 are concyclic or collinear. Prove it!

Solution. The points

$$A_1, B_1, A_2, B_2;\ A_2, B_2, A_3, B_3;\ A_3, B_3, A_4, B_4 \text{ and } A_4, B_4, A_1, B_1$$

are concyclic, so thereby lemma 25.1 we get that the numbers

$$\frac{(a_1-a_2)(b_2-b_1)}{(a_1-b_1)(b_2-a_2)},\ \frac{(a_2-a_3)(b_3-b_2)}{(a_2-b_2)(b_3-a_3)},\ \frac{(a_3-a_4)(b_4-b_3)}{(a_3-b_3)(b_4-a_4)},\ \frac{(a_4-a_1)(b_1-b_4)}{(a_4-b_4)(b_1-a_1)}$$

are real numbers. The product of the first and the third number divided by the product of the second and the forth number in (1) is equal to

$$\frac{(a_1-a_2)(a_3-a_4)}{(a_2-a_3)(a_4-a_1)} \cdot \frac{(b_2-b_1)(b_4-b_3)}{(b_3-b_2)(b_1-b_4)}$$

and it is a real number too. According to the condition of the example, the points A_1, A_2, A_3, A_4 are concyclic, so from the lemma 16.1 we get that $\frac{(a_1-a_2)(a_3-a_4)}{(a_2-a_3)(a_4-a_1)}$ is a real number. But, that means that the number $\frac{(b_2-b_1)(b_4-b_3)}{(b_3-b_2)(b_1-b_4)}$ is a real number, so the theorem 16.3 we get that the points B_1, B_2, B_3, B_4 are concyclic or collinear. ■

16.6. Lemma (Ptolemy`s inequality). For arbitrary points *A*, *B*, *C*, *D* in a plane, the following inequality holds true

$$\overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD} \geq \overline{AC} \cdot \overline{BD}. \qquad (5)$$

Proof. Let *a*, *b*, *c*, *d* be the affixes of the vertices *A*, *B*, *C*, *D* respectively. Then

$$(a-b)(c-d) + (b-c)(a-d) = (a-c)(b-d)$$

and if we apply the triangle inequality, we get the following inequality

$$|(a-b)(c-d)| + |(b-c)(a-d)| \geq |(a-c)(b-d)|,$$

which is actually the inequality (5) written in terms of the affixes of the vertices of the quadrilateral *ABCD*. ■

17. INNER PRODUCT

17.1. Let the complex numbers $a = |a| e^{i\alpha}$ and $b = |b| e^{i\beta}$ be given. Then

$$\overline{a}b = |a| e^{-i\alpha} |b| e^{i\beta} = |a| \cdot |b| e^{i(\beta-\alpha)} = |a| \cdot |b| (\cos(\beta-\alpha) + i\sin(\beta-\alpha)),$$

so

$$|a| \cdot |b| \cos(\beta-\alpha) = \operatorname{Re} \overline{a}b \text{ and } |a| \cdot |b| \sin(\beta-\alpha) = \operatorname{Im} \overline{a}b.$$

Furthermore, the complex numbers a and b correspond to the vectors $\vec{a}$ and $\vec{b}$ with tale at the origin and held at the points A and B with affixes a and b. Moreover, the scalar product of the vectors $\vec{a}$ and $\vec{b}$ is defined by $\vec{a}\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos\angle(\vec{a},\vec{b})$ and since $\angle(\vec{a},\vec{b})=\pm(\beta-\alpha)$ we get that the scalar product of the vectors $\vec{a}$ and $\vec{b}$ (the complex numbers a and b) holds true that $\vec{a}\vec{b}=\operatorname{Re}\overline{a}b$ ($a\cdot b=\operatorname{Re}\overline{a}b$). The following properties of a scalar product are very simple, and therefore they are left on the reads.

Theorem. The following statements hold true:

1) $a\cdot a=|a|^2$, for each $a\in\mathbf{C}$,
2) $a\cdot b=b\cdot a$, for all $a,b\in\mathbf{C}$,
3) $a\cdot(b+c)=a\cdot b+a\cdot c$, for all $a,b,c\in\mathbf{C}$,
4) $(ka)\cdot b=k(a\cdot b)=a\cdot(kb)$, for all $a,b\in\mathbf{C}$ and for each $k\in\mathbf{R}$,
5) $(ac)\cdot(bc)=|c|^2\,a\cdot b$, for all $a,b\in\mathbf{C}$, and
6) $a\cdot b=0$ if and only if $OA\perp OB$. ■

17.2. Lemma. Let A, B, C, D be four distinct points with affixes a, b, c, d, respectively. Then $AB\perp CD$ if and only if $(b-a)\cdot(d-c)=0$.

Proof. The complex numbers $b-a$ and $d-c$ correspond to the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$. The statement directly implies the statement 6) of the previous theorem. ■

17.3. Example. Let O be the circumcenter of the $\triangle ABC$, C' be the midpoint of the side AB and T be the centroid. Prove that $OT\perp CC'$ if and only if $\overline{BC}^2+\overline{AC}^2=2\overline{AB}^2$.

Solution. With no loss of generality, we shall say that the triangle is inscribed in the unit circle. Then $OT\perp CC'$ if and only if

$$t\cdot(c'-c)=0\quad\Leftrightarrow$$
$$\tfrac{a+b+c}{3}\cdot\left(\tfrac{a+b}{2}-c\right)=0\quad\Leftrightarrow$$
$$(a+b+c)\cdot(a+b-2c)=0\quad\Leftrightarrow$$
$$|a|^2+|b|^2-2|c|^2+2a\cdot b-a\cdot c-b\cdot c=0\quad\Leftrightarrow$$
$$2a\cdot b-a\cdot c-b\cdot c=0.$$

Furthermore,

$$\begin{aligned}\overline{BC}^2+\overline{AC}^2-2\overline{AB}^2&=|c-b|^2+|a-c|^2-2|a-b|^2\\&=(c-b)\cdot(c-b)+(a-c)\cdot(a-c)-2(a-b)\cdot(a-b)\\&=2|c|^2-|a|^2-|b|^2+4a\cdot b-2a\cdot c-2b\cdot c\\&=2(2a\cdot b-a\cdot c-b\cdot c).\end{aligned}$$

Therefore, $OT\perp CC'$ if and only if $2a\cdot b-a\cdot c-b\cdot c=0$ if and only if $\overline{BC}^2+\overline{AC}^2=2\overline{AB}^2$. ■

17.4. Theorem (Apollonius). Let M be a point on the side BC of the triangle ABC such that $\overline{BM}:\overline{MC}=m:n$. Then

$$n\overline{AB}^2+m\overline{AC}^2=m\overline{CM}^2+n\overline{BM}^2+(m+n)\overline{AM}^2 .$$

Proof. Let a, b, c, z be determine the affixes of the points A, B, C, M, respectively. Then $z=\frac{nb+mc}{m+n}$, thus

$$\overline{AB}^2=|b-a|^2=(b-a)\cdot(b-a)=|a|^2-2a\cdot b+|b|^2,$$

$$\overline{AC}^2=|c-a|^2=(c-a)\cdot(c-a)=|a|^2-2a\cdot c+|c|^2,$$

$$\overline{CM}^2=\frac{n^2}{(m+n)^2}|b-c|^2=\frac{n^2}{(m+n)^2}(|b|^2-2b\cdot c+|c|^2),$$

$$\overline{BM}^2=\frac{m^2}{(m+n)^2}|b-c|^2=\frac{m^2}{(m+n)^2}(|b|^2-2b\cdot c+|c|^2),$$

$$\overline{AM}^2=|a|^2+\frac{n^2}{(m+n)^2}|b|^2+\frac{m^2}{(m+n)^2}|c|^2+\frac{2mn}{(m+n)^2}b\cdot c-\frac{2n}{m+n}a\cdot b-\frac{2m}{m+n}a\cdot c.$$

So,

$$\begin{aligned}m\overline{CM}^2+n\overline{BM}^2+(m+n)\overline{AM}^2&=\frac{mn}{m+n}|b|^2+\frac{mn}{m+n}|c|^2-2\frac{mn}{m+n}b\cdot c\\&\quad+(m+n)|a|^2+\frac{n^2}{m+n}|b|^2+\frac{m^2}{m+n}|c|^2\\&\quad+\frac{2mn}{m+n}b\cdot c-2na\cdot b-2ma\cdot c\\&=m(|a|^2-2a\cdot c+|c|^2)+n(|a|^2-2a\cdot b+|b|^2)\\&=n\overline{AB}^2+m\overline{AC}^2 . \blacksquare\end{aligned}$$

17.5. Let's consider distinct points $A_i,\ i=1,2,\ldots,n$ in a plane with affixes $a_i,\ i=1,2,\ldots,n$ and let $k_i,\ i=1,2,\ldots,n$ be nonzero numbers, such that $k_1+k_2+\ldots+k_n=k\neq 0$. A *barycenter* or *centroid* of the system composed of the points $A_i,\ i=1,2,\ldots,n$ with centroids $k_i,\ i=1,2,\ldots,n$ is called the point T with an affix

$$t=\frac{k_1a_1+k_2a_2+\ldots+k_na_n}{k_1+k_2+\ldots+k_n}.$$

If $k_i=1,\ i=1,2,\ldots,n$, then the point T is called the *equibarycenter* or *centroid* of the set of points $A_i,\ i=1,2,\ldots,n$.

17.6. Theorem (Lagrange). Let the points $A_i,\ i=1,2,\ldots,n$ be given and the centroids $k_i,\ i=1,2,\ldots,n$, be non-zero, such that $k_1+k_2+\ldots+k_n=k\neq 0$. If T is the barycenter of the system composed of the points $A_i,\ i=1,2,\ldots,n$ and centroids $k_i,\ i=1,2,\ldots,n$, then for each point M with affix z the following holds true:

$$\sum_{i=1}^{n}k_i\overline{MA}_i^2=k\overline{MT}^2+\sum_{i=1}^{n}k_i\overline{TA}_i^2 . \qquad (1)$$

Proof . We have that:

$$\overline{MA}_i^2 = \big((t-z)+(a_i-t)\big)\cdot\big((t-z)+(a_i-t)\big) = |t-z|^2 + |a_i-t|^2 + 2(a_i-t)\cdot(t-z),$$

thus

$$\sum_{i=1}^{n} k_i \overline{MA}_i^2 = |t-z|^2 \sum_{i=1}^{n} k_i + \sum_{i=1}^{n} k_i |a_i-t|^2 + 2(t-z)\bullet\sum_{i=1}^{n} k_i(a_i-t)$$

$$= k\overline{MT}^2 + \sum_{i=1}^{n} k_i \overline{TA}_i^2 + 2(t-z)\cdot\left(\sum_{i=1}^{n} k_i a_i - t\sum_{i=1}^{n} k_i\right)$$

$$= k\overline{MT}^2 + \sum_{i=1}^{n} k_i \overline{TA}_i^2 + 2(t-z)\cdot 0 = k\overline{MT}^2 + \sum_{i=1}^{n} k_i \overline{TA}_i^2,$$

i.e. the equality (1) holds true . ■

17.7. Consequence (Leibnez). Let the points A_i, $i=1,2,...,n$ be given and let T be the centroid. Then for any point M in the plane the following holds true

.
$$\sum_{i=1}^{n} \overline{MA}_i^2 = n\overline{MT}^2 + \sum_{i=1}^{n} \overline{TA}_i^2 \qquad (2)$$

Proof. The equality (2) is implied directly from the equality (1) for $k_i = 1$, $i=1,2,...,n$. ■

17.8. Comment a) If the points A_i, $i=1,2,...,n$ be on a circle centered at O and with radius R and if the point M corresponds with the center of the circle, we get the formula

$$nR^2 = n\overline{OT}^2 + \sum_{i=1}^{n} \overline{TA}_i^2 .$$

b) If A_i, $i=1,2,...,n$ are the vertices of a regular n-gon inscribed in the circle $|z|=R$, then its centroid shall be the origin (why?). Now, according to the Leibniz theorem we get that

$$\sum_{i=1}^{n} \overline{MA}_i^2 = n\overline{MO}^2 + nR^2 ,$$

and if the point M is on the circumcircle of this polygon, then the previous formula implies the following

$$\sum_{i=1}^{n} \overline{MA}_i^2 = nR^2 + nR^2 = 2nR^2 .$$

IV CHAPTER
EXAMPLES AND EXERCISES

1. EXAMPLES (CHAPTER I)

1. Without any transformation in a trigonometric form, find the set of the second roots of the complex number $z = a + ib$. Hence, find the following

$$\sqrt[4]{-7+24i}.$$

Solution. Since $(x+iy)^2 = a+ib$ holds true, we obtain the following system of equations

$$x^2 - y^2 = a,\ \ 2xy = b,$$

which implies

$$x + iy = \pm\left(\sqrt{\tfrac{a+\sqrt{a^2+b^2}}{2}} + i\sqrt{\tfrac{-a+\sqrt{a^2+b^2}}{2}}\right).$$

The fourth roots of $-7+24i$ are the following $2+i, -2-i, 1-2i, -1+2i$. ■

2. Determine the set of points z in the complex plane for which it exists a real number c such that it satisfies the following $z = \frac{c-i}{2c-i}$.

Solution. Let $z = x+iy$. Since $z = \frac{c-i}{2c-i}$ we get that

$$c = \frac{zi-i}{2z-1} = \frac{-y+(x-1)i}{2x-1+2iy}\cdot\frac{2x-1-2iy}{2x-1-2iy} = \frac{(1-2x)y+2(x-1)y+((x-1)(2x-1)+2y^2)\,i}{(2x-1)^2+4y^2}$$

Now, $(2x-1)^2 + 4y^2 \neq 0$, thus $x \neq \frac{1}{2}$ and $y \neq 0$. Furthermore, it should be

$$(x-1)(2x-1) + 2y^2 = 0, \text{ i.e. } \left(x-\tfrac{3}{4}\right)^2 + y^2 = \tfrac{1}{16}.$$

So, the required set of points is the following

$$S = \left\{z = x+iy \mid \left(x-\tfrac{3}{4}\right)^2 + y^2 = \tfrac{1}{16}, (x,y) \neq \left(\tfrac{1}{2}, 0\right)\right\}$$
$$= \left\{z \mid |z-a| = \tfrac{1}{4}, a = \tfrac{3}{4}, z \neq \tfrac{1}{2}\right\}. \ ■$$

3. Let a, b, c be complex numbers such that they satisfy the following $|a| = |b| = |c| = 1$. Prove the following

$$|ab+bc+ca| = |a+b+c|.$$

Solution. Since the condition of the given problem, it is true that $a\bar{a} = b\bar{b} = c\bar{c} = 1$, thus $\bar{a} = \frac{1}{a}$, $\bar{b} = \frac{1}{b}$, $\bar{c} = \frac{1}{c}$. Hence,

$$|ab+bc+ca| = \left|abc\left(\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c}\right)\right| = |a|\cdot|b|\cdot|c|\cdot|\bar{a}+\bar{b}+\bar{c}|$$
$$= |\overline{a+b+c}| = |a+b+c|. \ ■$$

4. If z and w are complex numbers such that they satisfy the following $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$, then $\left|\frac{z-w}{\bar{z}+w}\right| < 1$. Prove that!

Solution. *I solution.* Let $z = x + iy$, $w = a + ib$. Then:

$$\left|\frac{z-w}{\bar{z}+w}\right| = \frac{|z-w|}{|\bar{z}+w|} = \frac{|(x-a)+(y-b)i|}{|(x+a)+(-y+b)i|} = \frac{\sqrt{(x-a)^2+(y-b)^2}}{\sqrt{(x+a)^2+(b-y)^2}}.$$

If $x > 0$ and $a > 0$, then $(x-a)^2 < (x+a)^2$, and $(y-b)^2 = (b-y)^2$, the radicand of the numerator is always smaller then the radicand of the denominator.

II solution. Since $\operatorname{Re} z > 0$, $\operatorname{Re} w > 0$ and the properties of complex numbers, we have that

$$\begin{aligned} |z-w|^2 - |\bar{z}+w|^2 &= (z-w)(\bar{z}-\bar{w}) - (\bar{z}+w)(z+\bar{w}) \\ &= z\bar{z} - \bar{z}w - z\bar{w} + \bar{w}w - z\bar{z} - \bar{z}\bar{w} - zw - w\bar{w} \\ &= -[z(w+\bar{w}) + \bar{z}(w+\bar{w})] \\ &= -(w+\bar{w})(z+\bar{z}) = -4\operatorname{Re} z \cdot \operatorname{Re} w < 0, \end{aligned}$$

i.e. $|z-w|^2 < |\bar{z}+w|^2$, which implies that $\left|\frac{z-w}{\bar{z}+w}\right| < 1$. ■

5. Prove the following inequality

$$\sqrt{(a_1+a_2+\ldots+a_n)^2+(b_1+b_2+\ldots+b_n)^2} \le \sum_{i=1}^{n}\sqrt{a_i^2+b_i^2}$$

for $a_i, b_i \in \mathbf{R}$, $i = 1,2,\ldots,n$.

Solution. Let consider complex numbers $z_i = a_i + ib_i$, for $i = 1,2,\ldots,n$. By substitution in the inequality

$$|z_1 + z_2 + \ldots + z_n| \le |z_1| + |z_2| + \ldots + |z_n|$$

we obtain the required inequality. ■

5. Find the smallest possible value of the following expression $\left|z - \frac{1}{z}\right|$, $z \in \mathbf{C}$, if given $|z| = 2$.

Solution. If $|z| = 2$, then

$$\left|z - \frac{1}{z}\right| = \left|\frac{z^2-1}{z}\right| = \frac{|z^2-1|}{|z|} = \frac{|z^2-1|}{2} \ge \frac{|z^2|-1}{2} = \frac{|z|^2-1}{2} = \frac{3}{2}.$$

The above stated implies that the required minimal value is equal to $\frac{3}{2}$ and it is achieved for $z^2 = 4$, i.e. $z = \pm 2$. ■

6. Determine and show in a complex plane the set

$$\left\{z = \frac{3t+i}{t-i} : t \in \mathbf{R}\right\}.$$

Solution. Let

$$z = \frac{3t+i}{t-i} = \frac{3t^2-1}{t^2+1} + i\frac{4t}{t^2+1} = x + iy.$$

Therefore, $x=\frac{3t^2-1}{t^2+1}$, $y=\frac{4t}{t^2+1}$. Hence, $t=\frac{y}{3-x}$, that is

$$(x-1)^2+y^2=4,\ x\neq 3.$$

So, the required set is a circle centered at (1,0) and radius 2, except the point (3,0). ■

7. Solve the equation

$$2(1+i)z^2-4(2-i)z-5-3i=0.$$

Solution. By solving the given quadratic equation for z we obtain the following

$$z=\frac{4(2-i)\pm\sqrt{16(2-i)^2+8(1+i)(5+3i)}}{4(1+i)},$$

hence, $z_1=\frac{4-i}{1+i}=\frac{3-5i}{2}$ and $z_2=\frac{-i}{1+i}=-\frac{1+i}{2}$. ■

8. Determine all complex numbers z such that they satisfy the following

$$|z|=\frac{1}{|z|}=|z-1|.$$

Solution. Firstly, let's notice that the given expressions are defined for $z\neq 0$. Since $|z|=\frac{1}{|z|}$, we get that $|z|=1$ and $|z|=|z-1|$, imply that $|z-1|=1$. If $z=x+iy$, then the above stated conditions imply that

$$x^2+y^2=1 \text{ and } (x-1)^2+y^2=1,$$

that is $x^2=(x-1)^2$, thus $x=\frac{1}{2}$, and $y=\frac{\sqrt{3}}{2}$ or $y=-\frac{\sqrt{3}}{2}$.

Finally, the solutions of the given equation are: $\frac{1}{2}+i\frac{\sqrt{3}}{2}$ and $\frac{1}{2}-i\frac{\sqrt{3}}{2}$. ■

9. Solve the equation in a field of complex numbers

$$(x-3)^4+(x-4)^4=(2x-7)^4.$$

Solution. Let $y=x-\frac{7}{2}$. The given equation can be written as following

$$112y^4-24y^2-1=0,$$

thus $y^2=\frac{1}{4}$ and $y^2=-\frac{1}{28}$, i.e.

$$y\in\left\{\frac{1}{2},-\frac{1}{2},i\sqrt{\frac{1}{28}},-i\sqrt{\frac{1}{28}}\right\},$$

thus

$$x\in\left\{3,4,\frac{7}{2}+i\sqrt{\frac{1}{28}},\frac{7}{2}-i\sqrt{\frac{1}{28}}\right\}. \blacksquare$$

10. Find all complex numbers n, so that the number $z=\left(\frac{3+i}{2-i}\right)^n$ is a real number.

Solution. Since

$$z=\left(\frac{3+i}{2-i}\cdot\frac{2+i}{2+i}\right)^n=(1+i)^n,$$

we have that $z^2=(2i)^n$, so the number z is real if and only if $z^2\geq 0$ holds true, and that it is true if and only if n is factored by 4. ■

11. Given a complex number u. Find all complex numbers z, such that $a = \frac{u-\bar{u}z}{1-z}$ is a real number.

Solution. A number a is a real number if and only if $a = \bar{a}$. Thus, $\frac{u-\bar{u}z}{1-z}$ is a real number if and only if $\frac{u-\bar{u}z}{1-z} = \frac{\bar{u}-\bar{z}u}{1-\bar{z}}$, i.e. if and only if

$$(\bar{u}-u)(1-z\bar{z}) = 0 .$$

Hence, if u is a real number, then a solution is each complex number $z \neq 1$, and if u is not a real number, then a solution of the given problem is each complex number $z \neq 1$ such that $z\bar{z} = 1$, i.e. a solution is each complex number z such that $|z| = 1,\ z \neq 1$. ■

12. Let $a_1,...,a_n$ be given complex numbers, such that

$$|a_1| = ... = |a_n| = 1 \text{ and } a_1 + a_2 + ... + a_n = 0 .$$

Prove that for each complex number z

$$|a_1 - z| + |a_2 - z| + ... + |a_n - z| \geq n \text{ holds true.}$$

Solution. Since $|a_i| = |\overline{a_i}|$, for each $i = 1,2,...,n$ and $\sum_{i=1}^{n} \overline{a_i} = 0$, we get that

$$n = \sum_{i=1}^{n} |a_i|^2 = \sum_{i=1}^{n} a_i \overline{a_i} = \sum_{i=1}^{n} a_i \overline{a_i} - z\sum_{i=1}^{n} \overline{a_i} = \sum_{i=1}^{n} (a_i \overline{a_i} - z\overline{a_i})$$

$$= \left| \sum_{i=1}^{n} (a_i - z)\overline{a_i} \right| \leq \sum_{i=1}^{n} |a_i - z| \cdot |a_i| = \sum_{i=1}^{n} |a_i - z| . \ ■$$

13. Given complex numbers $z_1, z_2, ..., z_{2n+1}$ such that $|z_i| = 1$ and $\operatorname{Im} z_i \geq 0$, for $i = 1,2,...,2n+1$. Prove that the following holds true

$$\left| \sum_{i=1}^{2n+1} z_i \right| \geq 1 .$$

Solution. The inequality can be proven by the principle of mathematical induction. Clearly, the inequality holds true for $n = 0$. Let assume that it holds true for all $2n-1$ complex numbers which satisfy the condition of the given problem.

Let $z_1, z_2, ..., z_{2n+1}$ be complex numbers which satisfy the conditions of the problem. Without loss of the generality we assume

$$\arg z_1 \leq \arg z_2 \leq ... \leq \arg z_{2n+1} .$$

Let position a new coordinate system in a complex plane so that the imaginary axis is a bisector of $\angle z_1 O z_2$, and the real axis passes through the point O(0,0). In a so-defined coordinate system the points are denoted by

$$z_k = x_k + iy_k ,\ k = 1,2,...,2n+1 .$$

We have

$$y_k \geq 0 \text{ and } x_1 = -x_{2n+1},\ y_1 = y_{2n+1} ,$$

The above stated and the inductive assumption imply that:

$$\left|\sum_{i=1}^{2n+1} z_i\right| = (x_1 + x_2 + ... + x_{2n} + x_{2n+1})^2 + (y_1 + y_2 + ... + y_{2n} + y_{2n+1})^2$$

$$= (x_2 + ... + x_{2n})^2 + (y_1 + y_2 + ... + y_{2n} + y_{2n+1})^2$$

$$\geq (x_2 + ... + x_{2n})^2 + (y_2 + ... + y_{2n})^2$$

$$= | z_2 + ... + z_{2n} |^2 \geq 1.$$

Finally, by applying the principle of mathematical induction we get that each odd number of complex numbers satisfies the conditions in the given problem. ■

14. Let $a_0, a_1, ..., a_n$ be complex numbers such that if $z \in \mathbf{C}$, $|z| \leq 1$, then

$$| a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 | \leq 1.$$

Prove the following

$$| a_k | \leq 1 \text{ and } | a_0 + a_1 + ... + a_n - (n+1) a_k | \leq n, \text{ for each } k = 0,1,2,...,n.$$

Solution. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

and w_i, $i = 0,1,...,n$ be the $(n+1)$-th roots of the one. But,

$$\sum_{i=0}^{n} w_i^k = 0, \quad \text{if } k \text{ is not factored by } n+1 \text{ and}$$

$$\sum_{i=0}^{n} w_i^k = n+1 \text{ if } k \text{ is factored by } n+1.$$

So,

$$\sum_{i=0}^{n} w_i^k P(w_i) = (n+1) a_{n-k}, \text{ for each } k \in \{0,1,...,n\}.$$

Hence,

$$(n+1) | a_{n-k} | = \left| \sum_{i=0}^{n} w_i^k P(w_i) \right| \leq \sum_{i=0}^{n} | w_i^k P(w_i) | = \sum_{i=0}^{n} | P(w_i) | \leq \underbrace{1 + 1 + ... + 1}_{n+1} = n+1,$$

The last implies that $| a_{n-k} | \leq 1$, for each $k \in \{0,1,...,n\}$.

For the second part of the statement it is true that

$$\sum_{i=1}^{n} w_i^k P(w_i) = \sum_{i=0}^{n} w_i^k P(w_i) - P(1) = (n+1) a_{n-k} - \sum_{i=1}^{n} a_i \text{ and}$$

$$\left| \sum_{i=1}^{n} w_i^k P(w_i) \right| \leq \sum_{i=1}^{n} | w_i^k P(w_i) | = \sum_{i=1}^{n} | P(w_i) | \leq n$$

thus,

$$\left| (n+1) a_{n-k} - \sum_{i=1}^{n} a_i \right| \leq n, \text{ for each } k \in \{0,1,...,n\}. \blacksquare$$

15. Let n be a positive integer. Prove that the polynomial

$$P(z) = z^{n+1} - z^n - 1$$

has a root w such that $|w| = 1$ if and only if $6 \mid (n+2)$.

Solution. Let $|w| = 1$ and $w^{n+1} - w^n - 1 = 0$. Then $w^n(w-1) = 1$ and since $|w| = 1$ we get that $|w-1| = 1$. Hence, w is one of the intersection of the circles $|z| = 1$ and $|z-1| = 1$, that is $w = \cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}$. Moreover, $w - 1 = \cos\frac{2\pi}{3} \pm i\sin\frac{2\pi}{3} = w^2$. Finally,

$$1 = w^n(w-1) = w^{n+2} = \cos\tfrac{(n+2)\pi}{3} \pm i\sin\tfrac{(n+2)\pi}{3},$$

The last means that $\frac{(n+2)\pi}{3} = 2k\pi,$ for some $k \in \mathbf{N}$, thus $n + 2 = 6k,$ for some $k \in \mathbf{N}$, i.e. $6 \mid (n+2)$.

Conversely, if $6 \mid (n+2)$, i.e. $n + 2 = 6k,$ for some $k \in \mathbf{N}$, then for $w = \cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}$ holds true that $|w| = 1,$ $w^2 = w - 1$ and $w^{n+2} = 1$, thus

$$w^{n+1} - w^n - 1 = w^n(w-1) - 1 = w^n w^2 - 1 = w^{n+2} - 1 = 0. \ \blacksquare$$

16. Let $z_1, z_2, ..., z_n$ be arbitrary complex numbers. Prove that positive integers $i_1, ..., i_k$ can be chosen such that they satisfy the followings $1 \le i_1 < ... < i_k \le n$ and

$$|z_{i_1} + z_{i_2} + ... + z_{i_k}| \ge \tfrac{2}{4\sqrt{2}}(|z_1| + |z_2| + ... + |z_n|).$$

Solution. Let $z_j = x_j + iy_j,$ $x_j, y_j \in \mathbf{R},$ $j = 1, ..., n$. Lets

$$S_1 = \{j \mid x_j \ge 0, y_j \ge 0\}, \ S_2 = \{j \mid x_j < 0, y_j \ge 0\},$$
$$S_3 = \{j \mid x_j < 0, y_j < 0\}, \ S_4 = \{j \mid x_j \ge 0, y_j < 0\}.$$

Then,

$$\sum_{j=1}^{n} |z_j| = \sum_{j=1}^{4} \sum_{j \in S_k} |z_j|$$

by applying the principle of Dirichlet, we get that for some $k \in \{1,2,3,4\}$ the following inequality holds true

$$\sum_{j \in S_k} |z_j| \ge \tfrac{1}{4} \sum_{j=1}^{n} |z_j|.$$

For such a number k we get the following

$$\begin{aligned}
\tfrac{1}{4\sqrt{2}} \sum_{j=1}^{n} |z_j| \le \tfrac{1}{\sqrt{2}} \sum_{j \in S_k} |z_j| = \tfrac{1}{\sqrt{2}} \sum_{j \in S_k} |x_j + iy_j| \le \tfrac{1}{\sqrt{2}} \sum_{j \in S_k} (|x_j| + |y_j|) \\
= \tfrac{1}{\sqrt{2}}\left(\left| \sum_{j \in S_k} x_j \right| + \left| \sum_{j \in S_k} y_j \right| \right) \\
\le \tfrac{2}{\sqrt{2}} \sqrt{\tfrac{1}{2}\left(\left| \sum_{j \in S_k} x_j \right|^2 + \left| \sum_{j \in S_k} y_j \right|^2 \right)} = \left| \sum_{j \in S_k} z_j \right|. \ \blacksquare
\end{aligned}$$

17. Let $z_1, z_2, ..., z_n$ be complex numbers such that

$$|z_1|+|z_2|+...+|z_n|=1.$$

Prove that it exists a set $S \subseteq \{z_1, z_2, ..., z_n\}$ such that

$$\left|\sum_{z_i \in S} z_i\right| \geq \frac{1}{6}.$$

Solution. Let $z_k = x_k + y_k$, $k=1,2,...,n$. Then

$$1=\sum_{k=1}^{n}|z_k|=\sum_{k=1}^{n}\sqrt{x_k^2+y_k^2} \leq \sum_{k=1}^{n}(|x_k|+|y_k|)$$

$$=\sum_{x_k \geq 0}|x_k|+\sum_{x_k<0}|x_k|+\sum_{y_k \geq 0}|y_k|+\sum_{y_k<0}|y_k|.$$

by applying the principle of Dirichlet, we get that at least one of the four sums of the right side of the equality is equal or greater to $\frac{1}{4}$. Let assume that $\sum\limits_{x_k<0}|x_k| \geq \frac{1}{4}$.So we get that

$$\left|\sum_{x_k<0} z_i\right| \geq \left|\sum_{x_k<0} x_k\right| = \sum_{x_k<0}|x_k| \geq \frac{1}{4} > \frac{1}{6}. \blacksquare$$

18. Let a, b, c be any complex numbers and $w=\frac{-1+i\sqrt{3}}{2}$. Prove that the following holds true

$$a^3+b^3+c^3-3abc=(a+b+c)(a+bw+cw^2)(a+bw^2+cw).$$

Solution. It can be directly checked that

$$w^3=1,\ w^4=w,\ w+w^2=-1,$$

thus,

$$(a+b+c)(a+bw+cw^2)(a+bw^2+cw)=$$

$$=(a+b+c)\left(a^2+b^2+c^2+w(ab+bc+ca)+w^2(ab+bc+ca)\right)$$

$$=(a+b+c)\left(a^2+b^2+c^2+(w+w^2)(ab+bc+ca)\right)$$

$$=(a+b+c)\left(a^2+b^2+c^2-(ab+bc+ca)\right)=a^3+b^3+c^3-3abc,$$

which was required to be proven. $\blacksquare$

19. Let a and b be positive real numbers. Determine the minimum of the expression $\left|\frac{x+y}{1+x\overline{y}}\right|$, if x and y are complex numbers, so that $|x|=a,\ |y|=b$.

Solution. We have

$$\left|\frac{x+y}{1+x\overline{y}}\right|^2=\frac{x+y}{1+x\overline{y}}\cdot\frac{\overline{x}+\overline{y}}{1+\overline{x}y}=\frac{|x|^2+|y|^2+2\operatorname{Re}x\overline{y}}{1+|x\overline{y}|^2+2\operatorname{Re}x\overline{y}}$$

$$=1+\frac{|x|^2+|y|^2-1-|x\overline{y}|^2}{1+|x\overline{y}|^2+2\operatorname{Re}x\overline{y}}=1-\frac{(a^2-1)(b^2-1)}{1+|x\overline{y}|^2+2\operatorname{Re}x\overline{y}},$$

whereby

$$\min\{\operatorname{Re} x\overline{y} : |x| = a, |y| = b\} = -ab,$$
$$\max\{\operatorname{Re} x\overline{y} : |x| = a, |y| = b\} = ab.$$
If at least one of the numbers a and b is equal to 1, then
$$\min_{|x|=a, |y|=b} \left|\frac{x+y}{1+x\overline{y}}\right| = 1$$
If both of the numbers a and b are either greater or less than 1, then
$$\min_{|x|=a, |y|=b} \left|\frac{x+y}{1+x\overline{y}}\right| = \sqrt{1-\frac{(a^2-1)(b^2-1)}{1+a^2b^2-2ab}} = \left|\frac{a-b}{1-ab}\right|$$
If one of the numbers a and b is greater than 1, and the other is less than 1, then
$$\min_{|x|=a, |y|=b} \left|\frac{x+y}{1+x\overline{y}}\right| = \sqrt{1-\frac{(a^2-1)(b^2-1)}{1+a^2b^2+2ab}} = \left|\frac{a+b}{1+ab}\right|. \blacksquare$$

20. Compute $z = \frac{(\sqrt{2}+\sqrt{2}i)^{10}}{(\sqrt{3}+i)^8}$.

Solution. The numbers
$$z_1 = \sqrt{3}+1 \text{ and } z_2 = \sqrt{2}+\sqrt{2}i$$
shall be rewritten using Euler's formulas. So,
$$r_1 = \sqrt{\sqrt{3}^2+1^2} = 2, \ r_1 = \sqrt{\sqrt{2}^2+\sqrt{2}^2} = 2,$$
$$\varphi_1 = \operatorname{arctg}\frac{1}{\sqrt{3}} = \frac{\pi}{6}, \ \varphi_2 = \operatorname{arctg} 1 = \frac{\pi}{4},$$
thus $z_1 = 2e^{i\frac{\pi}{6}}$ and $z_2 = 2e^{i\frac{\pi}{4}}$. Therefore,
$$z = \frac{(\sqrt{2}+\sqrt{2}i)^{10}}{(\sqrt{3}+i)^8} = \frac{\left(2e^{i\frac{\pi}{4}}\right)^{10}}{\left(2e^{i\frac{\pi}{6}}\right)^8} = \frac{2^{10}e^{i\frac{10\pi}{4}}}{2^8e^{i\frac{8\pi}{6}}}$$
$$= \frac{2^2e^{i\frac{5\pi}{2}}}{e^{i\frac{4\pi}{3}}} = 4e^{i\left(\frac{5\pi}{2}-\frac{4\pi}{3}\right)} = 4e^{i\frac{7\pi}{6}}$$
$$= 4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -2\sqrt{3}-2i. \blacksquare$$

21. Let z_1, z_2, z_3 be distinct complex numbers with equal modulus. If the numbers $z_1+z_2z_3$, $z_2+z_3z_1$ and $z_3+z_1z_2$ are real, then the equality $z_1z_2z_3 = 1$. holds true. Prove it!

Solution. Let $z_k = r(\cos\varphi_k + i\sin\varphi_k)$, $k = 1,2,3$. By direct computations we get that
$$z_1z_2z_3 = r^3(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 + i\sin\varphi_2)(\cos\varphi_3 + i\sin\varphi_3)$$
$$= r^3(\cos\varphi + i\sin\varphi),$$
where $\varphi = \varphi_1+\varphi_2+\varphi_3$.

The condition of the given problem therefore

$$\sin\varphi_k + r\sin(\varphi-\varphi_k) = 0, \quad k = 1,2,3,$$

that is

$$\sin\varphi_k(1-r\cos\varphi) + \cos\varphi_k \cdot r\sin\varphi = 0, \quad k = 1,2,3.$$

Let assume that $1-r\cos\varphi \neq 0.$ Then

$$\operatorname{tg}\varphi_k = \frac{r\sin\varphi}{r\cos\varphi-1}, \, k = 1,2,3,$$

which is not possible thereby $\varphi_k \in [0,2\pi)$. These numbers differ each other, hence $1-r\cos\varphi = 0$ and $\sin\varphi = 0,$ thus $\cos\varphi = 1,$ $r = 1.$ Finally,

$$z_1z_2z_3 = r^3(\cos\varphi + i\sin\varphi) = 1. \ \blacksquare$$

22. Determine the maximal value of $|z|$ if given that the complex number z satisfies the following condition $\left|z+\frac{1}{z}\right| = 1$?

Solution. Let $z = re^{i\varphi}$, where $e^{i\varphi} = \cos\varphi + i\sin\varphi$. So,

$$\left|z+\frac{1}{z}\right|^2 = \left(re^{i\varphi} + \frac{1}{r}e^{-i\varphi}\right)\left(re^{-i\varphi} + \frac{1}{r}e^{i\varphi}\right)$$

$$= r^2 + \frac{1}{r^2} + e^{2i\varphi} + e^{-2i\varphi} = r^2 + \frac{1}{r^2} + 2\cos 2\varphi.$$

The equation

$$r^2 + \frac{1}{r^2} + 2\cos 2\varphi = 1$$

implies

$$r^2 = \frac{-2\cos 2\varphi + 1 \pm \sqrt{(2\cos 2\varphi - 1)^2 - 4}}{2}.$$

This implies that r^2, as well as r will be maximal if

$$\cos 2\varphi = -1.$$

Then

$$r_{\max} = \sqrt{\frac{-2(-1)+1+\sqrt{(2(-1)-1)^2-4}}{2}} = \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2}. \ \blacksquare$$

23. Prove that $z + \frac{1}{z} = 2\cos\theta$ implies $z^m + \frac{1}{z^m} = 2\cos m\theta$.

Solution. The equation $z + \frac{1}{z} = 2\cos\theta$ implies $z^2 - 2z\cos\theta + 1 = 0$. The solutions of the last quadratic equations are $z = \cos\theta \pm i\sin\theta$. Now $\frac{1}{z} = \cos\theta \mp i\sin\theta$. Therefore,

$$z^m = \cos m\theta \pm i\sin m\theta \ \text{ and } \ \frac{1}{z^m} = \cos\theta \mp i\sin m\theta.$$

By adding the last two equalities we get

$$z^m + \frac{1}{z^m} = 2\cos m\theta. \ \blacksquare$$

24. Let $t \in \mathbf{R}$ and $z = \frac{1+it}{1-it}$. Prove that

$$z^n + \overline{z^n} = 2\cos(2n\operatorname{arctg} t).$$

Solution. We express z as following $z=\frac{1+it}{1-it}=\frac{1-t^2}{1+t^2}-i\frac{2t}{1+t^2}$. If $t=\operatorname{tg}\frac{x}{2}$, i.e. $x=2\operatorname{arctg}t$, then $\cos x=\frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$ and $\sin x=\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}=\frac{2t}{1+t^2}$, thus

$$\begin{aligned} z^n+\overline{z^n} &= (\cos x+i\sin x)^n+(\cos x-i\sin x)^n \\ &= \cos nx-i\sin nx+conx-i\sin nx \\ &= 2\cos nx=2\cos(2n\operatorname{arctg}t)\text{. ■} \end{aligned}$$

25. Solve the eqaution:

$$\frac{iz^6+8}{8i-z^6}=\sqrt{3}\,.$$

Solution. If $8i-z^6=0$, then the fraction which is on the left side of the equation is not defined. If $8i-z^6\neq 0$, i.e. $z^6\neq 8i$, then we multiply by $8i-z^6$ the both sides of the equation and after grouping the terms which consist of z on the left side, and the other ones on the right side of the equation we get the following

$$z^6\left(\sqrt{3}+i\right)=-8+8\sqrt{3}i\,.$$

Thus,

$$z^6=\frac{-8+8\sqrt{3}i}{\sqrt{3}+i}=\frac{\left(-8+8\sqrt{3}i\right)\left(\sqrt{3}-i\right)}{\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right)}=\frac{8\left(-1+i\sqrt{3}\right)\left(\sqrt{3}-i\right)}{4}=8i\,.$$

But, we previously found that $z^6\neq 8i$, so, we find that the given equation has no solution. ■

26. On the field of complex numbers find the solution of the given system

$$\begin{cases} x^4+6x^2y^2+y^4=5 \\ x^3y+xy^3=1 \end{cases}$$

Solution. The given system is equivalent to the system whose second equation is multiplied by 4, that is equivalent to the following system

$$\begin{cases} x^4+6x^2y^2+y^4=5 \\ 4x^3y+4xy^3=4 \end{cases}$$

Thus, it is equivalent to the system which consists of the sum and the difference of these two equations, that is equivalent to the following system

$$\begin{cases} (x+y)^4=9 \\ (x-y)^4=1 \end{cases}$$

Since the above equations we get that

$$x=\frac{\alpha\cdot\sqrt{3}+\beta}{2},\ y=\frac{\alpha\cdot\sqrt{3}-\beta}{2},\ \alpha,\beta\in\{1,-1,i,-i\},$$

that is, 16 solutions. It is necessary to percieve that these solutions differed from each other. Namely, if

$$\frac{\alpha_1\cdot\sqrt{3}+\beta_1}{2}=\frac{\alpha_2\cdot\sqrt{3}+\beta_2}{2},$$

then

$$(\alpha_1-\alpha_2)\sqrt{3}=(\beta_2-\beta_1),$$

so $\alpha_1=\alpha_2$, which means that $\beta_1=\beta_2$, i.e.that is the same solution, since if it is conversely $\sqrt{3}$ can not be expressed as $r+is$, where $r,s\in\mathbf{Q}$, therefore $\sqrt{3}$ is an irrational number. ■

27. Solve the system in the field of complex numbers

$$\begin{cases} z^{19}w^{25}=1, \\ z^5w^7=1, \\ z^4+w^4=2. \end{cases}$$

Solution. If we cube the second equation of the system we get that $z^{15}w^{21}=1$. Since $z^{19}w^{25}=1$ and $z^{15}w^{21}=1$ it is true that $z^4w^4=1$. But, $z^4+w^4=2$, and the Vieta's formulas we get that z^4 and w^4 are solutions of the quadratic equation $t^2-2t+1=0$, i.e. $z^4=1$ and $w^4=1$. Moreover, $z^5w^8=w$ implies $zz^4(w^4)^2=w$, i.e. $z=w$. Finally, solutions of the given system might be only the ordered pairs $(1,1),\ (-1,-1),\ (i,i),\ (-i,-i)$. With direct check it is easy to be proven that these pairs are truly solutions of the given system. ■

28. If given that $w_0,w_1,...,w_{n-1}$ are the n-th roots of 1, calculate the following sums:

a) $\sum\limits_{k=1}^{n} kw_{k-1}$ b) $\sum\limits_{k=1}^{n} k^2w_{k-1}$ c) $\sum\limits_{k=1}^{n} k^3w_{k-1}$

Guidelines. a) The n-th roots of 1 can be written as following $w_k=w^{k-1}$, where $w=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$. The required sum can be written as

$$S=\sum_{k=1}^{n} kw_{k-1}=\sum_{k=1}^{n} kw^{k-1}$$

Since $1-w\neq 0$, the following equalities hold true

$$S=\frac{S-Sw}{1-w}=\frac{1+2w+3w^2+...+nw^{n-1}-w-2w^2-3w^3-...-nw^n}{1-w}$$

$$=\frac{1+w+...+w^{n-1}-nw^n}{1-w}=\frac{\frac{1-w^n}{1-w}-nw^n}{1-w}=\frac{1-(n+1)w^n+nw^{n+1}}{(1-w)^2}.$$

Finally, the required sum can be found by substituting for w as following

$$w=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$$

and by using the De Moivre's formula.

The examples b) and c) can be solved analogously as already explaned example a). ■

29. Calculate the sums:

a) $\sum\limits_{k=1}^{n} k\cos\frac{2k\pi}{n}$, b) $\sum\limits_{k=1}^{n} k\sin\frac{2k\pi}{n}$

Guidelines. Let

$$A=\sum_{k=1}^{n} k\cos\frac{2k\pi}{n},\ B=\sum_{k=1}^{n} k\sin\frac{2k\pi}{n}$$

calculate $S=1+A+iB$. Then, apply the example 28 a) and further find

$$A=\mathrm{Re}(S-1) \text{ and } B=\mathrm{Im}(S-1)\,.\ ■$$

30. Prove that

a) $\sum\limits_{k=0}^{n} C_n^k \cos(k+1)\alpha = 2^n\cos^n\frac{\alpha}{2}\cos\frac{n+2}{2}\alpha$

b) $\sum\limits_{k=0}^{n} C_n^k \sin(k+1)\alpha = 2^n\cos^n\frac{\alpha}{2}\sin\frac{n+2}{2}\alpha$.

Guidelines. Let $z=\cos\alpha+i\sin\alpha$, hence apply the binomial formula for $(1+z)^n$. ■

31. Prove that

$$s=\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}=\frac{1}{2}\,.$$

Solution. Firstly, let declare that $z=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}$ implies

$$z^n=-1,\ z^{2n}=1\,. \tag{1}$$

Let $w=\cos\varphi+i\sin\varphi$. Hence $\overline{w}=\cos\varphi-i\sin\varphi=\frac{1}{w}$, thus

$$\cos\varphi=\frac{1}{2}\left(w+\frac{1}{w}\right)=\frac{w^2+1}{2w},\ \sin\varphi=\frac{1}{2i}\left(w-\frac{1}{w}\right)=\frac{w^2-1}{2iw}\,.$$

Further, by applying De Moivre's formula we get

$$w^k=\cos k\varphi+i\sin k\varphi \text{ and } \overline{w}^k=\cos k\varphi-i\sin k\varphi=\frac{1}{w^k}\,,$$

thus

$$\cos k\varphi=\frac{w^{2k}+1}{2w^k},\ \sin k\varphi=\frac{w^{2k}-1}{2iw^k}\,. \tag{2}$$

Let $z=\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}$. By taking for $k=1,2,3$ in (2) and hence by applying (1) we gat that

$$s=\frac{z^2+1}{2z}-\frac{z^4+1}{2z^2}+\frac{z^6+1}{2z^3}=\frac{(z^6-z^5+z^4-z^3+z^2-z+1)+z^3}{2z^3}=\frac{\frac{z^7+1}{z+1}+z^3}{2z^3}=\frac{z^3}{2z^3}=\frac{1}{2}\,.\ ■$$

32. Compute

$$p=\cos\frac{\pi}{7}\cos\frac{4\pi}{7}\cos\frac{5\pi}{7}\,.$$

Solution. Let $z=\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}$. Then (2) and (1) in the example 31 therefore

$$\cos\frac{\pi}{7}=\frac{z^2+1}{2z},\ \cos\frac{2\pi}{7}=\frac{z^4+1}{2z^2},\ \cos\frac{3\pi}{7}=\frac{z^6+1}{2z^3},$$

thus

$$p=\cos\frac{\pi}{7}\cos\frac{4\pi}{7}\cos\frac{5\pi}{7}=\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{2\pi}{7}=\frac{(z^2+1)(z^4+1)(z^6+1)}{8z^6}$$

$$=\frac{z^{12}+z^{10}+z^8+2z^6+z^4+z^2+1}{8z^6}=\frac{-z^5-z^3-z+z^6+z^4+z^2+1+z^6}{8z^6}$$

$$=\frac{z^6-z^5+z^4-z^3+z^2-z+1+z^6}{8z^6}=\frac{\frac{z^7+1}{z+1}+z^6}{8z^6}=\frac{1}{8}.\ \blacksquare$$

33. Let α, β and γ be angles of any triangle. Then,

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=1,$$

holds. Prove it!

Solution. Let the left side of the identity be denoted by S and let $z=\cos\alpha+i\sin\alpha,\ w=\cos\beta+i\sin\beta$. Then, the trigonometric entry of complex number implies $zw=\cos(\alpha+\beta)+i\sin(\alpha+\beta)$, and since the equalities (2) in the example 31 we get

$$\begin{aligned}\cos^2\alpha+\cos^2\beta+\cos^2\gamma&=\cos^2\alpha+\cos^2\beta+\cos^2(\pi-(\alpha+\beta))\\&=\cos^2\alpha+\cos^2\beta+\cos^2(\alpha+\beta)\\&=\left(\frac{z^2+1}{2z}\right)^2+\left(\frac{w^2+1}{2w}\right)^2+\left(\frac{z^2w^2+1}{2zw}\right)^2\\&=\frac{w^2z^4+z^2w^4+6z^2w^2+z^2+w^2+z^4w^4+1}{4z^2w^2},\end{aligned}$$

and

$$\begin{aligned}2\cos\alpha\cos\beta\cos\gamma&=-2\cos\alpha\cos\beta\cos(\alpha+\beta)=-2\frac{z^2+1}{2z}\frac{w^2+1}{2w}\frac{z^2w^2+1}{2zw}\\&=\frac{-w^2z^4-z^2w^4-2z^2w^2-z^2-w^2-z^4w^4}{4z^2w^2}.\end{aligned}$$

Finally, $S=\frac{4z^2w^2}{4z^2w^2}=1$. ■

34. Solve the equation

$$\cos^2 x+\cos^2 2x+\cos^2 3x=1.$$

Solution. By using that

$$\cos^2 t=\frac{1+\cos 2t}{2},$$

the given equation can be transformed as following

$$\frac{3}{2}+\frac{1}{2}(\cos 2x+\cos 4x+\cos 6x)=1,$$

i.e.

$$\cos 2x+\cos 4x+\cos 6x=-1\ .$$

Let $z=\cos x+i\sin x$. If we apply the equality (2) of example 31, we obtain the following

$$\frac{z^4+1}{2z^2}+\frac{z^8+1}{2z^4}+\frac{z^{12}+1}{2z^6}=-1,$$

and by using the well known mathematical transformation we get

$$(1+z^2+z^4+^6+z^8+z^{10}+z^{12})+z^6=0,$$

$$\frac{z^{14}-1}{z^2-1}+z^6=0,$$

$$(z^8-1)(z^6+1)=0.$$

In the above transformation we used the fact that $z^2\neq 1$, thereby $x=0$ and $x=\pi$ are not the solution of the given equation. Since $(z^8-1)(z^6-1)=0$, we get that $z^6+1=0$ or $z^2+1=0$ or $z^4+1=0$. Furthermore, $z^6+1=0$ therefore $\cos 6x=-1$, i.e. $x=\frac{\pi}{6}+\frac{k\pi}{3},\ k\in\mathbf{Z}$.

Similarly one can obtain the other solutions

$$x=\frac{\pi}{4}+\frac{k\pi}{2},\ k\in\mathbf{Z}\ \text{and}\ x=\frac{\pi}{2}+k\pi,\ k\in\mathbf{Z}.\ \blacksquare$$

35. Prove the identity

$$\frac{1}{\sin 2x}+\frac{1}{\sin 4x}+\ldots+\frac{1}{\sin 2^n x}=\operatorname{ctg} x-\operatorname{ctg} 2^n x,$$

for $n\in\mathbf{N},\ x\neq\frac{\lambda\pi}{2^k},\ k=0,1,2,\ldots,n,\ \lambda\in\mathbf{Z}$

Solution. Let $z=\cos x+i\sin x$. The equality (2) in the example 31 implies

$$\operatorname{tg} k\varphi=\frac{w^{2k}-1}{i(w^{2k}+1)},\ \operatorname{ctg} k\varphi=\frac{i(w^{2k}-1)}{w^{2k}-1}. \tag{1}$$

Further, thereby (1) for $n=2^s$ it is true that

$$\begin{aligned}\frac{1}{\sin 2^s x}&=\frac{2iz^{2^s}}{z^{2^{s+1}}-1}=i\frac{z^{2^{s+1}}+2z^{2^s}+1-z^{2^{s+1}}-1}{z^{2^{s+1}}-1}=i\frac{(z^{2^s}+1)^2-(z^{2^{s+1}}+1)}{z^{2^{s+1}}-1}\\&=i\frac{z^{2^s}+1}{z^{2^s}-1}-i\frac{z^{2^{s+1}}+1}{z^{2^{s+1}}-1}=\operatorname{ctg} 2^{s-1}x-\operatorname{ctg} 2^s x.\end{aligned} \tag{2}$$

Finally, if in (2) consequently we set $s=1,2,4,\ldots,2^n$ and add the such obtained equalities we get that

$$\begin{aligned}&\frac{1}{\sin 2x}+\frac{1}{\sin 4x}+\ldots+\frac{1}{\sin 2^n x}=\\&=(\operatorname{ctg} x-\operatorname{ctg} 2x)+(\operatorname{ctg} 2x-\operatorname{ctg} 4x)+\ldots+(\operatorname{ctg} 2^{n-1}x-\operatorname{ctg} 2^n x)\\&=\operatorname{ctg} x-\operatorname{ctg} 2^n x,\end{aligned}$$

which was supposed to be proven. $\blacksquare$

36. Let α, β and γ be angles of any triangle. The triangle is right angled triangle if and only if

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma=-1.$$

Prove it!

Guidelines. Firstly, prove that for any triangle

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1 - 4\cos\alpha\cos\beta\cos\gamma$$

holds. ■

37. Find the sum:

$$S_n = 1 + 2\cos x + 2^2 \cos 2x + ... + 2^n \cos nx .$$

Solution. Let

$$T_n = i(\sin x + 2^2 \sin 2x + ... + 2^n \sin nx) \text{ and } z = 2(\cos x + i\sin x) .$$

Then

$$S_n + T_n = 1 + 2(\cos x + i\sin x) + 2^2(\cos 2x + i\sin 2x) + ... + 2^n(\cos nx + i\sin nx)$$

$$= 1 + z + z^2 + ... + z^n = \frac{z^{n+1}-1}{z-1} = \frac{2^{n+1}[\cos(n+1)x + i\sin(n+1)x] - 1}{2(\cos x - i\sin x)},$$

thus

$$S_n = \mathrm{Re}\left[\frac{2^{n+1}[\cos(n+1)x + i\sin(n+1)x] - 1}{2(\cos x - i\sin x)}\right] = \frac{2^{n+2}\cos nx - 2^{n+1}\cos(n+1)x - 2\cos x + 1}{5 - 4\cos x} . \ ■$$

38. Let $a_1, a_2, ..., a_n$ be real numbers such that for each real number x is true that

$$1 + a_1 \cos x + a_2 \cos 2x + ... + a_n \cos nx \geq 0. \qquad (1)$$

Prove that

$$a_1 + a_2 + ... + a_n \leq n . \qquad (2)$$

Solution. Let $\varphi = \frac{2\pi}{n+1}$. Then for $z = e^{i\varphi}$ we get

$$1 + z^k + z^{2k} + ... + z^{nk} = \frac{1 - z^{(n+1)k}}{1 - z^k} = \frac{1 - \cos\frac{2(n+1)k\pi}{n+1} - i\sin\frac{2(n+1)k\pi}{n+1}}{1 - \cos\frac{2k\pi}{n+1} - i\sin\frac{2k\pi}{n+1}} = 0,$$

for $k = 1, 2, ..., n$, therefore

$$1 + \cos k\varphi + \cos 2k\varphi + ... + \cos nk\varphi = 0, \qquad (3)$$

for $k = 1, 2, ..., n$. If in the inequality (1) we set consecutive

$$x = \varphi,\ x = 2\varphi, ...,\ x = n\varphi,$$

we get n inequalities, which after adding, by using the (3) give the inequality $n - a_1 - a_2 - ... - a_n \geq 0$, which is equivalent to the inequality (2). ■

2. EXERCISES (CHAPTER 1)

1. Compute the following

a) i^n, $n \in \mathbf{Z}$, b) $(1+i)^n + (1-i)^n$, $n \in \mathbf{N}$, and

c) $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^n$, $n \in \mathbf{N}$.

2. Find the exact value of the following expression

$$\frac{(1+i)^{2000}}{(1-i)^{2000} - (1+i)^{2000}}.$$

3. Determine the modulus of a complex number

$$\frac{x^2 - y^2 + 2xyi}{xy\sqrt{2} + i\sqrt{x^4 + y^4}}.$$

4. Let $A = \{z_1, z_2, ..., z_{1996}\}$ be a set of complex numbers and let for each $i = 1,2,...,1996$ hold true that $A = \{z_1 z_i, z_2 z_i, ..., z_{1996} z_i\}$.

a) For each $i = 1,2,...,1996$ it holds true that $|z_i| = 1$. Prove it!

b) If $z \in A$ then $\bar{z} \in A$. Prove it!

5. Let $*$ be an operation in a field of complex numbers defined as following: $z * z_1 = zz_1 + i(z + z_1) - (1+i)$. Prove that $*$ is commutative and associative. Determine the *identity element e*, i.e. determine an element $e \in \mathbf{C}$ such that $e * z = z * e = z$ is satisfied for each $z \in \mathbf{C}$. Prove that there exists a unique complex number which does not have its inverse ($z' \in \mathbf{C}$ is an inverse element of $z \in \mathbf{C}$ if $zz' = z'z = e$).

6. Prove that

$$|b_0|^2 + |b_1|^2 + |b_2|^2 = 3(a_0^2 + a_1^2 + a_2^2),$$

for

$$b_k = a_0 + a_1 w^k + a_2 w^k, \quad k = 0,1,2; \quad a_0, a_1, a_2 \in \mathbf{R} \text{ and } w = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

7. Prove that $x + iy = (s + it)^n$, $x, ys, t \in \mathbf{R}$ implies that

$$(x^2 + y^2) = (s^2 + t^2)^n.$$

8. Graphically show the set of complex numbers z such that holds $\left|\frac{z-3}{z+3}\right| = 2$.

9. Solve the equations:

a) $|z| + z = 4 - i$, and b) $|z+i| + |z-i| = 2$.

10. Let $a_1, a_2 ..., a_n$ be given complex numbers, such that $|a_1| = ... = |a_n| = r$.

Let T_n^s be the sum of each products of s pairwise distinct numbers. For example

$$T_n^2 = a_1a_2 + a_1a_3 + ... + a_1a_n + a_2a_3 + ... + a_{n-1}a_n.$$

Prove the following

$$\frac{|T_n^{n-s}|}{|T_n^s|} = r^s, \ s = 1,2,3,...,n-1.$$

11. Let all roots of a polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$$

be on the circle $|z| = r$. Prove the following

$$|a_{n-i}| = r^i |a_i|, \ i = 0,1,2,...,n.$$

12. Prove the following:

a) If $|\alpha| < 1$, then $\left|\frac{z-\alpha}{1-z\overline{\alpha}}\right| \le 1$ if and only if $|z| < 1$.

b) If $|\alpha| > 1$, then $\left|\frac{z-\alpha}{1-z\overline{\alpha}}\right| \ge 1$ if and only if $|z| < 1$.

13. Express in trigonometric form the following complex numbers:

a) $-\sqrt{2}$, b) $-1+i$, c) $2-i\sqrt{3}$

d) $1+\cos\alpha + i\sin\alpha$, e) $\sin\alpha + i(1+\cos\alpha)$.

14. Find the exact value of $(1+w)^n$, if given that $n \in \mathbf{Z}$ and $w = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

15. Compute:

a) $\sqrt{2i}$, b) $\sqrt{-8i}$, c) $\sqrt{4+3i}$,

d) $\sqrt[3]{-1+i}$, e) $\sqrt[4]{-2\sqrt{3}-2i}$.

16. $\sqrt{1+i\sqrt{3}} + \sqrt{1-i\sqrt{3}} = \sqrt{6}$. Prove it!

17. If $\left(\frac{a+i}{a-i}\right)^n = 1$, then $a = \operatorname{ctg}\frac{k\pi}{n}$, $k = 0,1,2,...,n-1$. Prove it!

18. Solve the following equations, for $n \in \mathbf{N}$.

a) $(x+1)^n - (x-1)^n = 0$,

b) $(x+5i)^n - (x-5i)^n = 0$,

c) $(x+3i)^n + i(x-3i)^n = 0$,

d) $\left(\frac{i-x}{i+x}\right)^n = \frac{\text{ctg}\alpha+i}{\text{ctg}\pm\text{-}i}, \alpha \in \mathbf{R}$, and

e) $(x+\alpha i)^n + (\cos\theta + i\sin\theta)(x-\alpha i)^n = 0,\ \theta \neq 2k\pi,\ \alpha \neq 0,\ \alpha, \theta \in \mathbf{R}$.

19. Prove the following identities:

a) $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \ldots + \cos\frac{9\pi}{11} = \frac{1}{2}$,

b) $\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \ldots + \cos\frac{10\pi}{11} = -\frac{1}{2}$.

20. Find the sums:

a) $1 - C_n^2 + C_n^4 - C_n^6 + \ldots$, and

b) $C_n^1 - C_n^3 + C_n^5 - C_n^7 + \ldots$.

21. Find the sum:

$$1 + C_n^4 + C_n^8 + C_n^{12} + \ldots + C_n^{4k} + \ldots.$$

22. Prove the equality:

$$1 + C_n^3 + C_n^6 + C_n^9 + \ldots = \frac{2}{3}\left(2^{n-1} + \cos\frac{\pi n}{3}\right).$$

23. Prove that for $m = 2, 3, 4, \ldots$ it is true that:

$$\sin\frac{\pi}{m}\sin\frac{2\pi}{m}\sin\frac{3\pi}{m}\ldots\sin\frac{(m-1)\pi}{m} = \frac{m}{2^{m-1}}.$$

24. Prove the identity:

$$\sum_{k=0}^{n} (-1)^k \cos^n \frac{\pi k}{n} = \frac{n}{2^{n-1}}.$$

25. Let

$$F_n = a^n \sin nA + b^n \sin nB + c^n \sin nC,$$

where $a, b, c, A, B, C \in \mathbf{R}$ and $A + B + C = 2k\pi$, for some $k \in \mathbf{Z}$. Then, $F_1 = F_2 = 0$ implies $F_n = 0$ for each $n \in \mathbf{N}$. Prove it!

26. Let $x \neq 2k\pi$, $x \in \mathbf{R}$, $k \in \mathbf{Z}$ and $n \in \mathbf{N}$. Find the sum

$$1 + 2\cos x + 3\cos 2x + 4\cos 3x + \ldots + (n+1)\cos nx.$$

27. Let $x, y \in \mathbf{R}$ and $n \in \mathbf{N}$. Calculate the sums:

a) $\cos x + \cos(x+2y) + \ldots + \cos(x+2ny)$, and

b) $\sin x + \sin(x+2y) + \ldots + \sin(x+2ky)$.

28. If given that $\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$, then calculate $\cos\frac{\pi}{12}$.

29. Given that $\operatorname{tg}\frac{\pi}{8}=\sqrt{2}-1$. Prove the following: $\operatorname{tg}\frac{\pi}{24}=\sqrt{2}+\sqrt{6}-\sqrt{3}-2$.

30. Prove the following:

a) $\operatorname{tg}\frac{\pi}{5}=\sqrt{5-2\sqrt{5}}$,

b) $\operatorname{tg}\frac{2\pi}{5}=\sqrt{5+2\sqrt{5}}$,

c) $\operatorname{tg}\frac{3\pi}{5}=-\sqrt{5+2\sqrt{5}}$,

d) $\operatorname{tg}\frac{4\pi}{5}=-\sqrt{5-2\sqrt{5}}$

31. Calculate:

a) $\sin\frac{\pi}{21}\sin\frac{8\pi}{21}\sin\frac{\pi}{7}$,

b) $\operatorname{tg}\frac{\pi}{7}\operatorname{tg}\frac{2\pi}{7}\operatorname{tg}\frac{3\pi}{7}$,

c) $\sin\frac{\pi}{14}\sin\frac{2\pi}{14}\sin\frac{3\pi}{14}\sin\frac{4\pi}{14}\sin\frac{5\pi}{14}\sin\frac{6\pi}{14}$.

32. Prove the identities:

a) $\sin^3 x=\frac{3}{4}\sin x-\frac{1}{4}\sin 3x$, and

b) $\cos^4 x=\frac{1}{8}\cos 4x+\frac{1}{2}\cos 2x+\frac{3}{8}$.

33. Prove the identities:

a) $\cos 5x=16\cos^5 x-20\cos^3 x+5\cos x$, and

b) $\sin 5x=\sin x(16\cos^4 x-12\cos^2 x+1)$.

34. A complex number a is called an *algebraic*, if it is a root of a polynomial

$$P(z)=a_0z^n+a_1z^{n-1}+...+a_{n-1}+a_n$$

with integer coeficients. The numbers wich are not algebraic are called as transcendental numbers. For example π and e are transcendental numbers. But, there exist numbers, such as $e\pi$ and $e+\pi$, for which it is still not found either they are algebraic or trancedental numbers.

Prove that the numbers

a) $\sqrt{3}+\sqrt{2}$ and

b) $\sqrt[3]{4}-2i$

are algebraic numbers.

35. Solve the eqation:

$$x^n-nax^{n-1}-C_n^2a^2x^{n-2}-C_n^3a^3x^{n-3}...-a^n=0, a\neq 0.$$

36. Let $n \geq 2$ be a positive integer. Find all the solutions x_0 of the equation $x^n - x^{n-2} - x + 2 = 0$ such that $|x_0| = 1$.

37. Find the solutions for x:

$$\cos a + C_n^1 x \cos(a+b) + C_n^2 x^2 \cos(a+2b) + ... + C_n^n x^n \cos(a+nb) = 0 .$$

38. Solve the equations:

a) $\sin 2x + \cos 2x = \sqrt{2}$, and

б) $\mathrm{tg}^2 x + \mathrm{ctg}^2 x = 6$.

39. Let $f(x,y)$ be a rational function with real coefficients. If the function f is symmetric, i.e. if $f(x,y) = f(y,x)$, then $f(a,\bar{a}) \in \mathbf{R}$, for each $a \in \mathbf{C}$. Prove it!

40. Let $[t]$ be the maximal integer which is not greater than t. Let $z = x + iy$. Prove the following:

a) $\operatorname{Re} z^n = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}$,

b) $\operatorname{Im} z^n = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}$,

c) $\operatorname{Re} z^n + \operatorname{Im} z^n = \sum_{k=0}^{n} (-1)^{[k/2]} \binom{n}{k} x^{n-k} y^k$.

Find the proper expression for $\operatorname{Re} z^n - \operatorname{Im} z^n$?

41. Each zero of the following polynomial is in the field of complex numbers.

a) $\operatorname{Re}(x+i)^n$,

b) $\operatorname{Im}(x+i)^n$,

c) $\operatorname{Re}(x+i)^n \pm \operatorname{Im}(x+i)^n$

For each of the above polynomial prove the given statement and find the zeros.

3. EXAMPLES (CHAPTER II, CHAPTER III)

1. Construct a trapezoid if given its bases and diagonals.

Solution. Let's suppose that the given problem is solved and let $ABCD$ be the required trapezoid (figure 1). If a, b, c, d are the affixes of the points A, B, C, D respectively, then $\overrightarrow{DC} = c - d$. Let's consider the translation

$$S(z) = z + c - d$$

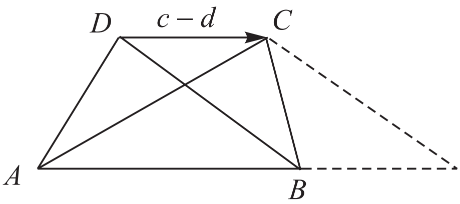

Figure 1

and let the point B_1 be the image of B under this translation. Further, $S(d) = d + c - d = c$ therefore, the point C is the image of D under this translation, thus $\overline{CB}_1 = \overline{DB}$. Therefore, all three sides of the triangle AB_1C are already determined ($\overline{AB}_1 = \overline{AB} + \overline{DC}$, $\overline{AC}$ and $\overline{CB}_1 = \overline{DB}$), so it might be constructed. Since the base $\overline{AB}$ is already determined, the point B can be found, and the point D is the inverse image of C. ■

2. Construct a circle such that it passes through a given point and tangents two parallel lines.

Solution. Let's assume that the given problem is solved (figure 2). If a is a vector parallel to the line (p), then the translation $S(z) = z + a$ maps the circle (K) to a circle (K'), such that it tangents the lines (p) and (q), but it does not pass through A. So, we have to construct a circle (K') such that it tangents the lines (p) and (q) and by the translation we map that circle to the required circle

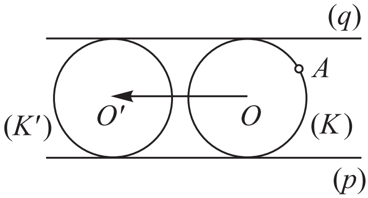

Figure 2

The number of solutions depends on the relationship between the point A and the lines (p) and (q). Namely:

- If the point A is placed between the lines (p) and (q), then the given problem has two solutions;
- If the point A is placed on any of the lines, then the given problem has a unique solution, and
- In any other case the given problem has no solution. ■

3. Given the circles $K'(o', R')$ and $K''(o'', R'')$ and the line (p). Construct a line (q) parallel to (p), so that the circles (K') and (K'') intersept congruent line segments on the line (q).

Solution. Let's assume that the given problem is solved (figure 3). Let's plot a line (a) perpendicular to (p) and a line $O'O$ perpendicular to (a), i.e. parallel to (p). Hence,

$$\overrightarrow{A'A''} = \overrightarrow{B'B''} = \overrightarrow{O'O}.$$

Due to this, if

$$S(z) = z + o - o'$$

is translation for vector $\overrightarrow{O'O}$, then $S(a') = a''$, $S(b') = b''$ and $S(o') = o$, and the circle (K') maps to a circle $K(o, |o - a''|)$ and further $A''(a'')$ and $B''(b'')$ are midpoints of the circles (K) and (K'').

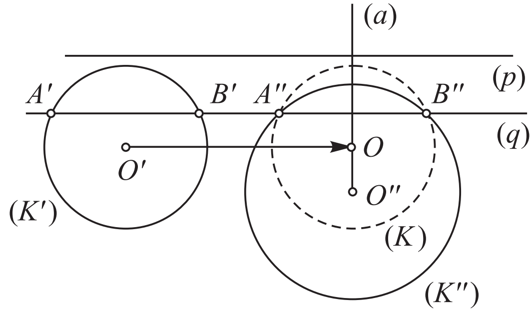

Figure 3

Whether the problem has a solution or not depends on the relationship of the circles (K') and (K). Due to that, the given problem has a solution, and furthermore:

- if $|R'' - R'| \leq |o'' - o| = \overline{OO''} \leq R' + R''$, then the given problem has a unique solution,
- if $|R'' - R'| > |o'' - o| = \overline{OO''}$ or $|o'' - o| = \overline{OO''} > R' + R''$, then the given problem has no solution. ■

4. Let be given two points A, B and let $S = u - z$ be a poin reglection. If $S(A) = A'$ and $S(B) = B'$, then $\overrightarrow{AB} = \overrightarrow{B'A'}$. Prove it!

Solution. Let a, b be the affixes of the points A, B, respectively. The affixes of the points A', B' are $a' = S(a) = u - a$ and $b' = S(b) = u - b$, respectively. Thus,

$$b' - a' = u - b - (u - a) = a - b$$

The latter means that $\overrightarrow{AB} = \overrightarrow{B'A'}$. ■

5. If O' and O'' are centers of a symmetry of the figure **F**, prove that $O = S_{O''}(O')$ is also a center of a symmetry of the figure **F**.

Solution. Let $\frac{o'}{2}$ and $\frac{o''}{2}$ be affixes of the points O' and O'' respectively, and let $A(a)$ be any point on the figure **F**. Then, the affix of O is

$$o = S_{O''}\left(\frac{o'}{2}\right) = o'' - \frac{o'}{2}.$$

Thereby the condition, the points A_1, A_2, A_3 whose affixes are

$$a_1 = S_{O''}(a) = o'' - a,$$
$$a_2 = S_{O'}(a_1) = o' - (o'' - a) = o' - o'' + a,$$
$$a_3 = S_{O''}(a_2) = o'' - (o' - o'' + a) = 2o'' - o' - a,$$

belong on the figure **F**. Finally, the arbitrariness of the point A and the equality

$$S_O(a) = 2o'' - o' - a = a_3$$

imply that $O = S_{O''}(O')$ is also a center of a symmetry of the figure **F** (figure 4). ■

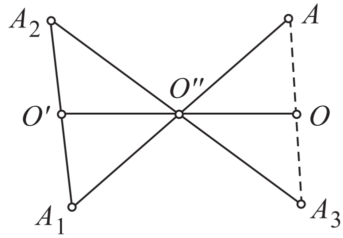

Figure 4

6. Given two lines (p) and (q) and a point A. Through A draw a line (a), so that A is the midpoint of the line segment MN, where $M = (a) \cap (p)$ and $N = (a) \cap (q)$.

Guidelines. Consider a poin reglection centered at A. ■

7. Let the circles $K'(O', R')$ and $K''(O'', R'')$ meet at a point A. Through the point A draw a line (a) on which the circles intersept congruent line segments.

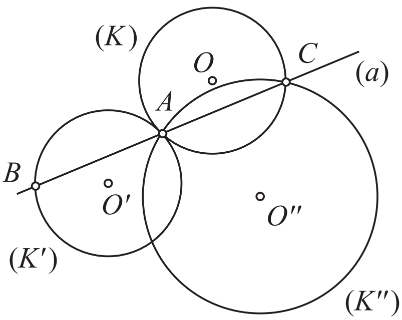

Figure 5

Solution. Let's suppose that the given problem is solved and let (a) be the required line (figure 5). Thereby $\overline{AB} = \overline{AC}$, a point A is the midpoint of a line segment BC. It means that $C = S_A(B)$. A point $B \in (K')$, thus the point $C = S_A(B)$ will be on the circle $(K) = S_A(K')$. So, $C \in (K) \cap (K'')$.

The problem has a unique solution. ■

8. Given four points A, B, C, D, so that $\overline{AB} = \overline{CD}$, but $\overrightarrow{AB} \neq \overrightarrow{CD}$. Prove that it exists a rotation S such that $S(A) = C,\ S(B) = D$!

How many such rotations exist?

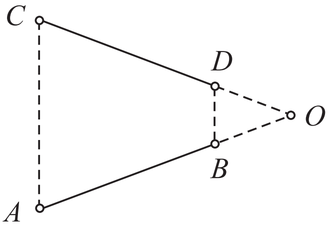

Figure 6

Guidelines. Consider the case when a line AC is parallel to a line BD, (figure 6) and the case when a line AC is not parallel to a line BD, (figure 7), separately. ■

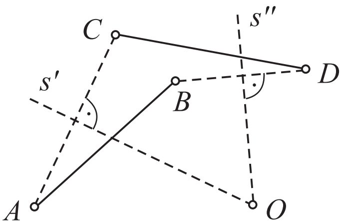

Figure 7

9. Let (K') and (K'') be circles with congruent radii and let they intersect at A and B. Prove that it exists a rotation S around the point A, so that $S(K') = K''$. Moreover, if $X \in (K')$ and $S(X) = X'$, then the line XX' passes through the point B.

Solution. Since $r' = r''$, the line AB is a bisector of the line segment $O'O''$, thus A is a center of the rotation $S_{A,\alpha}$ such that $S_{A,\alpha}(K') = K'$, where $\alpha = \angle O'AO''$. Let $X \in (K')$ be any point and $X' = XB \cap (K'')$, (figure 8). Since $\angle AXB$ and $\angle AX'B$ are inscribed angles of congruent arcs, the angles are conguent, and thus $\overline{AX} = \overline{AX'}$. If $S_{A,\alpha}(X) = X^*$, we get that $\overline{AX} = \overline{AX^*}$ and $X^* \in (K'')$, therefore $X' = X^*$.

Finally, for any point $X \in (K')$, the points X, B and $S_{A,\alpha}(X)$ are collinear. ■

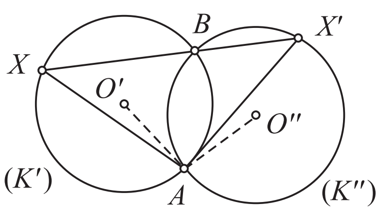

Figure 8

10. Given lines (p) and (q) and a point A. Construct an equilateral triangle ABC, such that $B \in (p)$ and $C \in (q)$.

Solution. Let assume that the given problem is solved and ABC is the required triangle, (figure 9). Since the triangle ABC is an equilateral, we obtain that $\overline{AB} = \overline{AC}$ and $\angle BAC = 60^\circ$. That is, $S_{A,60^\circ}(B) = C$ or $S_{A,-60^\circ}(B) = C$. The point B is on the line (p), and therefore the point $S_{A,60^\circ}(B) = C$ is on the line $S_{A,60^\circ}(p)$. On the other hand C is on the line (q), thus $C = (q) \cap S_{A,60^\circ}(p)$. For the rotation $S_{A,-60^\circ}$ the latter holds true, analogously.

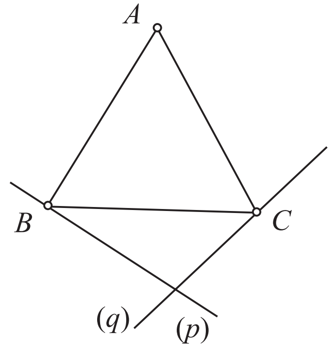

Figure 9

Let
$$S_{A,60^\circ}(p) = (p_1) \text{ and } S_{A,-60^\circ}(p) = (p_2).$$
If
$$C_1 = (q) \cap (p_1),\ C_2 = (q) \cap (p_2),$$
$$S_{A,-60^\circ}(C_1) = B_1 \text{ and } S_{A,60^\circ}(C_2) = B_2,$$
then AB_1C_1 and AB_2C_2 are the required triangles, (figure 10). Further, at least one of the lines $S_{A,60^\circ}(p)$ and $S_{A,-60^\circ}(p)$ meet the line (q), which means that the problem always has at least one possible solution. ■

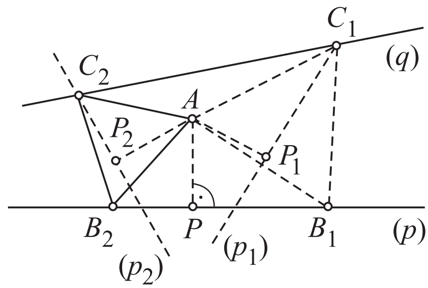

Figure 10

11. Given lines (p) and (q) and a point O. Construct a square $ABCD$ centered at the point O, such that two adjacent vertices are on the lines (p) and (q), respectively.

Solution. Let assume that the given problem is already solved and $ABCD$ is the required square, (figure 11). We have $\angle AOB = -90^\circ$, therefore $S_{O,-90^\circ}(A) = B$. Further, $A \in (p)$ implies that $B \in (q) \cap S_{O,-90^\circ}(p)$.

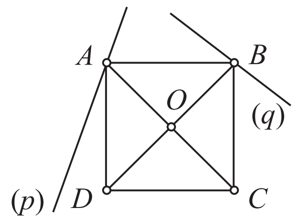

Figure 11

Let's consider the rotations $S_{O,90^\circ}$ and $S_{O,-90^\circ}$.

So,
$$(p_1) = S_{O,90^\circ}(p),\ (p_2) = S_{O,-90^\circ}(p),$$
$$B_1 = (p_1) \cap (q),\ B_2 = (p_2) \cap (q),$$
$$A_1 = S_{O,-90^\circ}(B_1),\ A_2 = S_{O,90^\circ}(B_2).$$
Then the required squares are squares with sides A_1B_1 и A_2B_2, respectively.

Let state that the above problem may have either two solutions or none. Namely, if the lines (p) and (q) are perpendicular to each other, then the lines

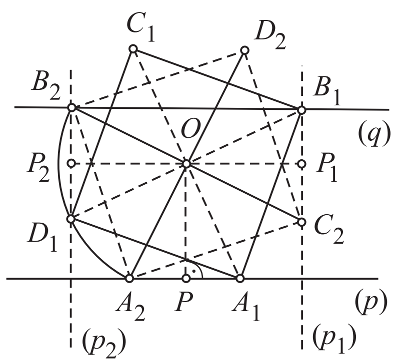

Figure 12

(p_1) and (p_2) are parallel with (q) and the given problem does not have any solution, if lines (p) and (q) are not perpendicular, then the given problem has exactly two solutions. ■

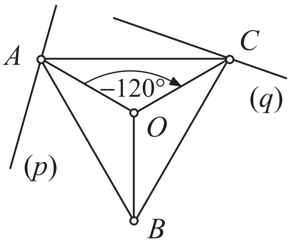

Figure 13

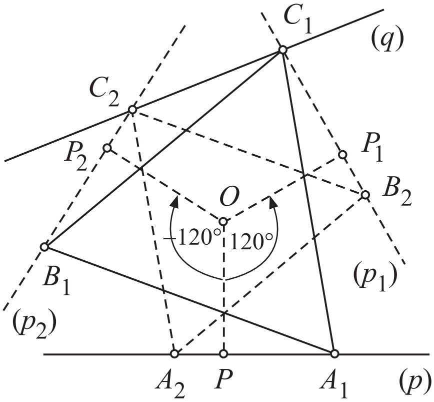

Figure 14

12. Given lines (p) and (q) and a point O. Construct an equilateral triangle ABC centered at the point O, so that its two vertices are placed on the lines (p) and (q), respectively.

Solution. Let assume that the given problem is solved and let the triangle ABC be the required triangle. Thereby $\angle AOC = -120°$, we get that $S_{O,-120°}(A) = C$, therefore $C = S_{O,-120°}(p) \cap (q)$, (figure 13).

Let's consider the rotations $S_{O,120°}$ and $S_{O,-120°}$. Therefore

$$(p_1) = S_{O,120°}(p),\ (p_2) = S_{O,-120°}(p),$$
$$C_1 = (p_1) \cap (q),\ C_2 = (p_2) \cap (q),$$
$$A_1 = S_{O,-120°}(C_1),\ A_2 = S_{O,120°}(C_2).$$

Then the required triangles have sides A_1C_1 and A_2C_2, respectively (figure 14).

The given problem may have two solutions, one solution or none. ■

13. Let $S(z) = az + b$, $a \in \mathbf{R} \setminus \{0,1\}$ be a homothety and M be any point. Prove that the center of a homothety C, the point M and its image N are collinear.

Solution. Let the affix of the point M be z. Then the affixes of the center of a homothety C and a point N are $c = \frac{b}{1-a}$ and $w = az + b$. Then,

$$\frac{\overline{w}-\overline{z}}{\overline{c}-\overline{z}} = \frac{a\overline{z}+\overline{b}-\overline{z}}{\frac{\overline{b}}{1-a}-\overline{z}} = \frac{\overline{b}+(a-1)\overline{z}}{\overline{b}+(a-1)\overline{z}}(1-a) = 1-a = \frac{b+(a-1)z}{b+(a-1)z}(1-a) = \frac{az+b-z}{\frac{b}{1-a}-z} = \frac{w-z}{c-z},$$

and since corollary 1.4 we have that the points C, M and N are collinear. ■

14. Given a circle $K(O,r)$ and a point A on the circle. Determine the locus of midpoints of the chords of the circle (K) at the point A.

Solution. Let AX be an arbitrary chord of a circle $K(O,r)$ and let Y be its midpoint, (figure 15). If the affixes of the points A, X and Y are a, z and w, respectively, then since the condition of the problem it is true that

$$w = \tfrac{1}{2}(z+a) = \tfrac{1}{2}z + \tfrac{1}{2}a,$$

The latter means that the required locus is the image of the circle (K) under the similarity

$$S(z)=\frac{1}{2}z+\frac{1}{2}a\,,$$

with coefficient $\frac{1}{2}$ and center

$$c=\frac{\frac{1}{2}a}{1-\frac{1}{2}}=a\,.$$

Finally, the required locus is the image of a circle (K) under the homothety with center at A and coefficient $\frac{1}{2}$, i.e. that is the circle with diameter OA. ■

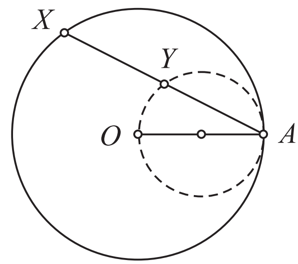

Figure 15

15. On the bases AB and DC of the trapezoid $ABCD$ on the same side of them, are constructed equilateral triangles ABM and DCN. Prove that a line MN passes through the intersection point O of the extensions of the legs.

Guideline. Prove that the homothety with center at O and coefficient $\frac{\overline{AB}}{\overline{DC}}$ maps the point N at M, (figure 16), and further apply the Example 13. ■

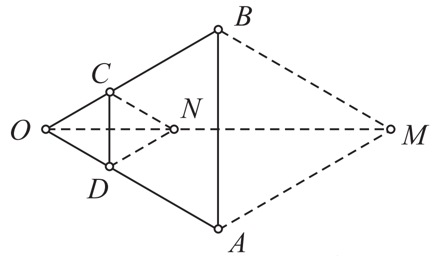

Figure 16

16. Let $ABCD$ be trapezoid with bases AB and CD and let M be the midpoint of AB, N be the midpoint of CD, P be the intersection of diagonals and Q the intersection of extensions of the legs. Prove that the points M, N, P and Q are collinear.

Guideline. Prove that it exists a homothety H with center at Q and coefficient $\frac{\overline{DC}}{\overline{AB}}$ so that $\mathrm{H}(A)=D$ and $\mathrm{H}(B)=C$, further, conclude that $\mathrm{H}(M)=N$, (figure 17), and then apply the example 13. Provethat there exists a homothety H_1 with center P and coefficient $-\frac{\overline{DC}}{\overline{AB}}$ so that $\mathrm{H}_1(A)=C$ and $\mathrm{H}_1(B)=D$, and further conclude that $\mathrm{H}(M)=N$, (figure 17), and then apply the example 13. ■

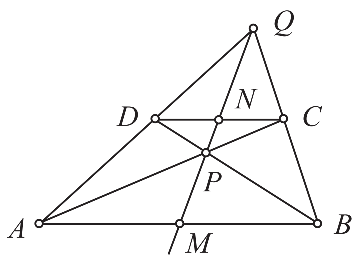

Figure 17

17. Given two concentric circles $K(O,R)$ and $K'(O',R')$, $R>R'$. Draw a line (p) which consecutively meets the circles at A, B, C and D, so that $\overline{AB}=\overline{BC}=\overline{CD}$.

Solution. Let assume that the given problem is solved and let (p) be the required line, (figure 18). Then $\frac{\overline{AB}}{\overline{AD}}=\frac{1}{3}$, thus under homothety H with center at A and coefficient $\frac{1}{3}$ the point D maps at B. This means that B is on the circle H(K). That is, $B\in \mathrm{H}(K)\cap(K')$.

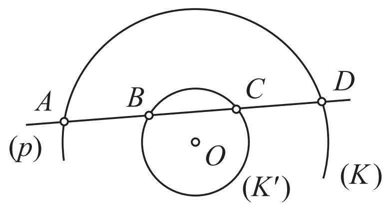

Figure 18

Finally, if we fix a point A on a circle (K) and let $B \in \mathrm{H}(K) \cap (K')$, then the required line $(p) = AB$, (make a figure). ■

18. Construct a triangle ABC if given α, β and h_c.

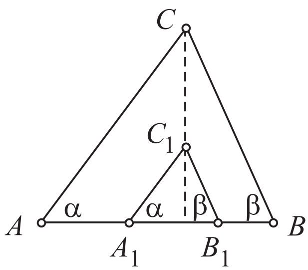

Figure 19

Solution. Let assume that the given problem is solved and ABC is a required triangle (figure 19). If $A_1B_1C_1$ is any triangle with angles α and β, then the triangles ABC and $A_1B_1C_1$ are homothetic with center of homothety at D and coefficient $\frac{h_c}{\overline{C_1D}}$, thus the required triangle can be constructed if we take an arbitrary triangle with angles α and β and map it under homothety H with center D and coefficient $A_1B_1C_1$, (make a figure). ■

19. In a triangle ABC, inscribe a triangle PQR whose sides are perpendicular to the sides of ABC.

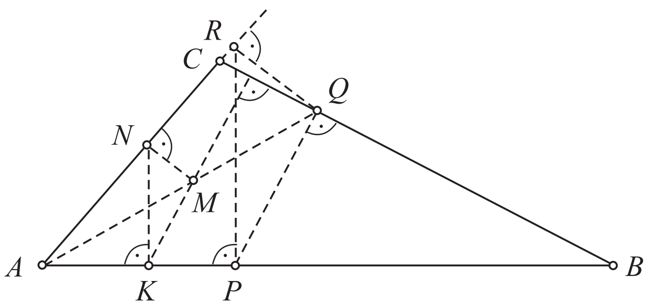

Figure 20

Guideline. Take any point K on the side AB, and construct the triangle KMN so that $NK \perp AB$, $MN \perp AC$. Further, draw the line AM and find the point $Q = AM \cap BC$. Now, the required triangle PQR is an image of the triangle KMN under homothety H with center A and coefficient $\frac{\overline{AQ}}{\overline{AM}}$ (figure 20). ■

20. Prove that if two circles touch each other, their centers and the point of touch are collinear.

Guideline. Determine the affix c of a point of touch of the circles with equations $|z| = 1$ and $|z - a| = R$ and prove that the points with affixes 0, a and c are collinear. ■

21. Given points A and B. Let A' be a point on the line OB, B' be a point on the line OA and Z be a point on AB. Construct a point Z' which divides the line segment $A'B'$ in a same ratio as a point Z divides the line segment AB.

Solution. Let a, a', b, b', z be the affixes of points A, B, A', B', Z, respectively. If $\frac{\overline{BZ}}{\overline{ZA}} = \lambda$, then

$$b - z = \lambda(z - a) \tag{1}$$

Hence, we should obtain the affix z' of a point Z so that

$$b'-z'=\lambda(z'-a'). \qquad (2)$$

Let Z_1 be the point where the OA meets the line which passes through the point Z and is parallel with BB', and Z_2 be the point where the line OB meets the line which passes through the point Z and is parallel with AA'. Then, the similarity of triangles AZZ_1 and ABB', i.e. the similarity of BZZ_2 and BBA' implies that

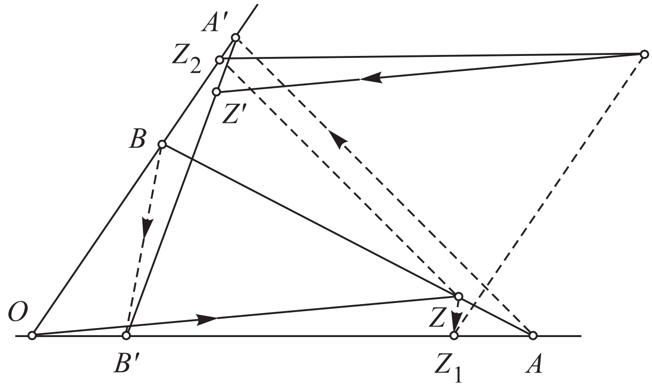

Figure 21

$$b'-z_1=\lambda(z_1-a) \text{ and } b-z_2=\lambda(z_2-a'). \qquad (3)$$

By reducing a and b, in (1) and (3), we obtain that

$$b'-(z_1+z_2-z)=\lambda[(z_1+z_2-z)-a'],$$

Since (2),we get that

$$z'=z_1+z_2-z\pi.$$

Hence, the point Z' is obtained by construction, (show in figure 21). ■

22. Prove that the sum of interior angles of any triangle is π.

Solution. Let a, b, c be affixes of vertices A, B, C of a triangle. Then

$$\angle CBA=\arg\tfrac{a-b}{c-b},\ \angle BAC=\arg\tfrac{c-a}{b-a},\ \angle ACB=\arg\tfrac{b-c}{a-c}.$$

Since each of the interior angles of a triangle is strictly smaller then π, we get that their sum is strictly smaller than 3π, and thus

$$\angle CBA+\angle BAC+\angle ACB=\arg\tfrac{a-b}{c-b}+\arg\tfrac{c-a}{b-a}+\arg\tfrac{b-c}{a-c}$$

$$=\arg\tfrac{a-b}{c-b}\tfrac{c-a}{b-a}\tfrac{b-c}{a-c}=\arg(-1)=\pi,$$

which was supposed to be proven. ■

23. Let $ABCD$ be strictly convex quadrilateral and let the points T_a, T_b, T_c, T_d be centroids of the triangles BCD, ACD, BAD, ABC, respectively. Prove that the medians of the quadrilaterals $ABCD$ and $T_aT_bT_cT_d$ are concurrent.

Solution. According to Example I 4.2. B) the medians MP and NQ of the quadrilateral $ABCD$ intersect at a point T with affix $t=\frac{a+b+c+d}{4}$, which according to Example 15.10 A) is the centroid of the quadrilateral $ABCD$. Analogously, the medians of the quadrilateral $T_aT_bT_cT_d$ intersect at its centroid, whose affix is

$$t'=\frac{t_a+t_b+t_c+t_d}{4}=\frac{\frac{b+c+d}{3}+\frac{a+c+d}{3}+\frac{a+b+d}{3}+\frac{a+b+c}{3}}{4}=\frac{a+b+c+d}{4}.$$

Finally since, $t=t'$ the statement of the given problem is proven. ■

24. a) Given two vertices of an equilateral trianglein a complex plane. Determine its third vertex.

b) Obtain a point z_3 so that the points $z_1 = 2+2i$, $z_2 = 3+i$ and z_3 create an equilateral triangle.

Solution. a) Let be given points *A* and *B* with affixes *a* and *b*. The given problem has two possible solutions *C* and C': $\triangle ABC$ is positively oriented and $\triangle ABC'$ is negatively oriented. Thereby, a point *C* is obtained if the vector $\overrightarrow{AB}$ is rotated around a point *A* at $\frac{\pi}{3}$, and C' is obtained when aif the $\overrightarrow{AB}$ is rotated around a point *A* at $-\frac{\pi}{3}$. Therefore

$$c = a + (b-a)e^{i\frac{\pi}{3}} \text{ and } c' = a + (b-a)e^{-i\frac{\pi}{3}}.$$

b) Since solution a),

$$z_3' = z_1 + (z_2 - z_1)e^{i\frac{\pi}{3}} = 2+2i+(1-i)e^{i\frac{\pi}{3}} = 2+2i+(1-i)\frac{1+i\sqrt{3}}{2} = \frac{5+\sqrt{3}}{2} + i\frac{3+\sqrt{3}}{2}$$

$$z_3'' = z_1 + (z_2 - z_1)e^{-i\frac{\pi}{3}} = 2+2i+(1-i)e^{-i\frac{\pi}{3}} = 2+2i+(1-i)\frac{1-i\sqrt{3}}{2} = \frac{5-\sqrt{3}}{2} + i\frac{3-\sqrt{3}}{2}. \ \blacksquare$$

25. Given an equilateral triangle $\triangle ABC$ let *a* be an affix of the vertex *A*. Determine the affix of the vertex *B* if the origin coincides with:

a) the vertex *C*,

b) the centroid *T* of the $\triangle ABC$,

c) A_1, the foot of the altitude at the vertex *A* to the line segment *BC*.

Solution. a) Since *C* coincides with the origin, $c = 0$. If $\triangle ABC$ is positively oriented, then *B* is obtained by rotation of the point *A* at $\frac{\pi}{3}$ around the point *C* and therefore

$$b = ae^{i\frac{\pi}{3}} = a\frac{1+i\sqrt{3}}{2}.$$

If $\triangle ABC$ is negatively oriented, then B' is obtained by rotation of the point *A* at $-\frac{\pi}{3}$ around the point *C* and therefore

$$b' = ae^{-i\frac{\pi}{3}} = a\frac{1-i\sqrt{3}}{2}.$$

b) Since T coincides with the origin, $t = 0$. If $\triangle ABC$ is positively oriented, thenthe point *B* is obtained by rotation of the point *A* at $\frac{2\pi}{3}$ around *T* and therefore $b = ae^{i\frac{2\pi}{3}} = a\frac{-1+i\sqrt{3}}{2}$. If $\triangle ABC$ is negatively oriented, then the point B' is obtained by rotation of the point *A* at $-\frac{2\pi}{3}$ around *C* and therefore $b' = ae^{-i\frac{2\pi}{3}} = a\frac{-1-i\sqrt{3}}{2}$.

c) Since A_1 coincides with the origin, $a_1 = 0$. Due to $\overline{A_1A} = \overline{A_1B}\sqrt{3}$, we obtain the point *B* by rotation of the point *A* around A_1 at $\frac{\pi}{2}$ if $\triangle ABC$ is positively oriented, that is $-\frac{\pi}{2}$ if $\triangle ABC$ is negatively oriented and the both cases, the obtained results are divided by $\sqrt{3}$. Thus, the given problem has two solutions: $b = \frac{ai}{\sqrt{3}}$ and $b' = \frac{-ai}{\sqrt{3}}$. $\blacksquare$

26. If a, b, c are the affixes of vertices of an equilateral triangle, then $a^2+b^2+c^2=ab+bc+ca$. Prove it!

Solution. Let t be affix of the centroid of an equilateral triangle, (with vertices whose affixes are a, b, c) such that a, b, c are the affixes, of its vertices,and let $u=e^{i\frac{2\pi}{3}}$. Then $b=t+(a-t)u$ and $c=t+(a-t)u^2$. The equality

$$a^2+b^2+c^2=ab+bc+ca$$

is equivalent to the equality

$$(a-b)^2+(b-c)^2+(c-a)^2=0. \tag{1}$$

We will prove the equality (1). Thus

$$\begin{aligned}(a-b)^2+(b-c)^2+(c-a)^2 &= [(1-u)^2+(u-u^2)^2+(u^2-1)^2](a-t)^2\\ &= [1+u^2+(u+1)^2](1-u)^2(a-t)^2\\ &= 2(1+u+u^2)(1-u)^2(a-t)^2\\ &= 2(1-u^3)(1-u)(a-t)^2\\ &= 2(1-e^{2i\pi})(1-u)(a-t)^2\\ &= 2(1-1)(1-u)(a-t)^2=0. \ \blacksquare\end{aligned}$$

27. If a, b and c are complex numbers so that they satisfy $a^2+b^2+c^2=ab+bc+ca$, then either $a=b=c$ or a, b and c are affixes of the vertices of an equilateral triangle. Prove it!

Solution. The equality $a^2+b^2+c^2=ab+bc+ca$ is equivalent to the equality $(b-c)^2=(c-a)(a-b)$, and thus

$$|b-c|^2=|c-a|\cdot|a-b|.$$

Analogously, it can be proven that

$$|c-a|^2=|a-b|\cdot|b-c| \text{ and } |a-b|^2=|b-c|\cdot|c-a|.$$

If the last three equalities are multiplied by $|b-c|$, $|c-a|$ and $|a-b|$, respectively, then, we obtain the following equalities

$$|a-b|^3=|b-c|^3=|c-a|^3=|a-b|\cdot|b-c|\cdot|c-a|,$$

The latter explaines the following $|a-b|=|b-c|=|c-a|$. The above means that either $a=b=c$ or a, b and c are affixes of the vertices of an equilateral triangle. ■

Comment. The statements in the Examples 26 and 27 can be transformed as following

A triangle ABC, where a, b, c are affixes of its vertices is an equilateral triangle if and only if $a^2+b^2+c^2=ab+bc+ca$ *holds true.*

Indeed, a triangle ABC is equilateral if it and only if it is directly similar with the triangle BCA, that is if and only if $\alpha=\beta,\ \beta=\gamma,\ \gamma=\alpha \Leftrightarrow \alpha=\beta=\gamma=\frac{\pi}{3}$, which actually means that if and only if

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b & c & a \end{vmatrix}=0 \Leftrightarrow a^2+b^2+c^2=ab+bc+ca. \ \blacksquare$$

28. On the sides of a triangle ABC three equilateral triangles are constructed, such that the triangles $A'BC$ and $B'AC$ are constructed on an interior side and $C'AB$ is constructed on an exterior side. If M is the center of a triangle $C'AB$, then prove that the triangle $A'B'M$ is an isosceles triangle and furthermore $\angle A'MB'=\frac{2\pi}{3}$ holds true.

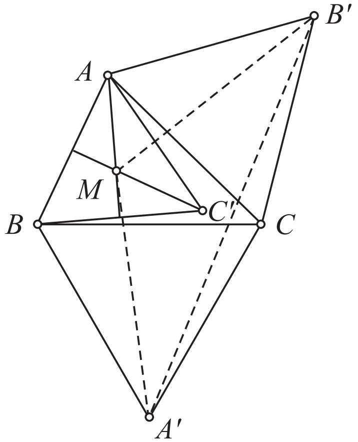

Figure 22

Solution. Let a, b, c be the affixes of vertices A, B, C respectively. Then the condition of the given problem implies that

$$a'=\frac{b-ce^{i\frac{\pi}{3}}}{1-e^{i\frac{\pi}{3}}},\ b'=\frac{c-ae^{i\frac{\pi}{3}}}{1-e^{i\frac{\pi}{3}}} \text{ and}$$

$$c'=\frac{b-ae^{i\frac{\pi}{3}}}{1-e^{i\frac{\pi}{3}}}=be^{i\frac{\pi}{3}}-ae^{i\frac{2\pi}{3}}.$$

Hence,

$$m=\frac{a+b+c'}{3}=\frac{1}{3}a\left(1-e^{i\frac{2\pi}{3}}\right)+\frac{1}{3}b\left(1+e^{i\frac{\pi}{3}}\right)=\frac{a+b}{2}+i\frac{b-a}{6}\sqrt{3}.$$

Finally,

$$(a'-m)e^{i\frac{2\pi}{3}}+m=a'e^{i\frac{2\pi}{3}}+m\left(1-e^{i\frac{2\pi}{3}}\right)$$

$$=\frac{b-ce^{i\frac{\pi}{3}}}{e^{-i\frac{\pi}{3}}}e^{i\frac{2\pi}{3}}+\left(\frac{a+b}{2}+i\frac{b-a}{6}\sqrt{3}\right)\left(\frac{3}{2}-i\frac{\sqrt{3}}{2}\right)$$

$$=-b+ce^{i\frac{\pi}{3}}+\frac{3a+3b+b-a}{4}+i\frac{b-a-a-b}{4}\sqrt{3}$$

$$=ce^{i\frac{\pi}{3}}-a\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=ce^{i\frac{\pi}{3}}-ae^{i\frac{2\pi}{3}}=\frac{c-ae^{i\frac{\pi}{3}}}{e^{-i\frac{\pi}{3}}}=b',$$

the latter implies the statement of the given problem. $\blacksquare$

29. Given a triangle $A_1A_2A_3$ and a point P_0 on its plane. Let $A_s=A_{s-3},\ s\geq 4$. We construct consecutive points $P_1,P_2,P_3,\ldots$ such that the point P_k, under rotation around the point A_{k+1} at $-\frac{2\pi}{3}$, maps at P_{k+1}. If $P_{2013}=P_0$, then the triangle $A_1A_2A_3$ is an equilateral triangle. Prove it!

Solution. By applying the given rotation, we obtain the following

$$S_{A_1,-\frac{2\pi}{3}}(p_0)=p_1,\ S_{A_2,-\frac{2\pi}{3}}(p_1)=p_2,\dots,\ S_{A_{2013},-\frac{2\pi}{3}}(p_{2012})=p_0\,,$$

thus

$$\left(S_{A_{2013},-\frac{2\pi}{3}}\circ S_{A_{2012},-\frac{2\pi}{3}}\circ\dots\circ S_{A_2,-\frac{2\pi}{3}}\circ S_{A_1,-\frac{2\pi}{3}}\right)(p_0)=p_0\,.$$

Further, $3\left(-\frac{2\pi}{3}\right)=-2\pi$, so the theorem 5.13 implies that

$$S_{A_3,-\frac{2\pi}{3}}\circ S_{A_2,-\frac{2\pi}{3}}\circ S_{A_1,-\frac{2\pi}{3}}=S_v\,, \tag{1}$$

where S_w is translation for a vector v. But,

$$S_{A_{2013},-\frac{2\pi}{3}}\circ S_{A_{2012},-\frac{2\pi}{3}}\circ\dots\circ S_{A_2,-\frac{2\pi}{3}}\circ S_{A_1,-\frac{2\pi}{3}}=(S_v)^{671}=S_{671v}=S_w\,,$$

i.e. $S_w(p_0)=p_0$. This means that S_w is a translation which has a fixed point. Now the theorem 5.5. implies that S_w is identity i.e. $w=0$, thus $v=0$, which according to (1) means that $S_{A_2,-\frac{2\pi}{3}}\circ S_{A_1,-\frac{2\pi}{3}}=S_{A_3,\frac{2\pi}{3}}$. The points A_1 and A_2 are fixed points for the rotations $S_{A_1,-\frac{2\pi}{3}}$ and $S_{A_2,-\frac{2\pi}{3}}$. Thus

$$S_{A_1,-\frac{2\pi}{3}}=e^{-i\frac{2\pi}{3}}z+\left(1-e^{-i\frac{2\pi}{3}}\right)a_1 \text{ and } S_{A_2,-\frac{2\pi}{3}}=e^{-i\frac{2\pi}{3}}z+\left(1-e^{-i\frac{2\pi}{3}}\right)a_2\,,$$

which according to the theorem 5.13 means that an affix a_3 of the center of rotation $S_{A_3,\frac{2\pi}{3}}$ satisfies the following equality

$$a_3=\frac{e^{-i\frac{2\pi}{3}}\left(1-e^{-i\frac{2\pi}{3}}\right)a_2+\left(1-e^{-i\frac{2\pi}{3}}\right)a_1}{1-e^{-i\frac{2\pi}{3}}e^{-i\frac{2\pi}{3}}}\,,\text{ i.e. } a_3\left(1+e^{-i\frac{2\pi}{3}}\right)=e^{-i\frac{2\pi}{3}}a_2+a_1\,.$$

The last equality is equivalent to the following equality

$$a_1-a_3=(a_2-a_3)e^{i\frac{\pi}{3}}\,,$$

The latter implies that the triangle $A_1A_2A_3$ is an equilateral triangle. ■

30. Determine the points c and d, which together with the points $a=1+i$ and $b=2+3i$ form such a square onto the Oxy plane, that a and b are its two adjacent vertices and one of the other two vertices is placed on the second quadrant.

Solution. $c=4i$ and $d=-1+2i$. ■

31. The points $a=1+i$ and $c=-1+i\sqrt{3}$ are opposite vertices of a square. Determine the other two vertices of that square.

Solution. The affix of the center of that square is $o=i\frac{1+\sqrt{3}}{2}$, and hence

$$b=\frac{\sqrt{3}-1}{2}+i\frac{3+\sqrt{3}}{2} \text{ and } d=\frac{1-\sqrt{3}}{2}+i\frac{-1+\sqrt{3}}{2}\,.\ ■$$

32. The complex numbers $a=1+i$ and $b=2+2i$ are adjacent vertices of a square. Determine the other two vertices c and $d$of that square.

Solution. The given problem has two possible solutions:

$$c'=1+3i,\ d'=2i \text{ and } c''=3+i,\ d''=2\,. \blacksquare$$

33. Given a square $ABCD$ and a as the affix of A. Determine the affixes b, c, d of B, C, D if the origin coincides with:

a) the vertex B,

b) the vertex C,

c) the center of the square.

Solution. a) The given problem has two possible solutions and in both of them $b=0$.

If the square is positively oriented, then the point C is obtained by rotation of the point A at $-\frac{\pi}{2}$ around B, thus $c'=-ia$. Now, thereby $\overrightarrow{AD}=\overrightarrow{BC}$ we get that $d'-a=c'-b$, thus $d'=a+c'=(1-i)a$.

If the square is negatively oriented, then the point C is obtained by rotation of a point A at $\frac{\pi}{2}$ around B, thus $c''=ia$ and $d''=a+c''=(1+i)a$.

b) The given problem has two possible solutions, and in both cases $c=0$, since the center O of the square is the midpoint of its diagonal, we get that the affix of the center is $o=\frac{a}{2}$. If the square is positively oriented then the point B is obtained by rotation of A around O at $\frac{\pi}{2}$, and D by rotation at $-\frac{\pi}{2}$, thus $b=\frac{a}{2}+i\frac{a}{2}$ and $d=\frac{a}{2}-i\frac{a}{2}$. If the square $ABCD$ is negatively oriented, then b and d will only change their positions.

c) To determine the points B, C, D we only rotate the point A around O at $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ respectively, and thus $b=ia,\ c=-a,\ d=-ia$. $\blacksquare$

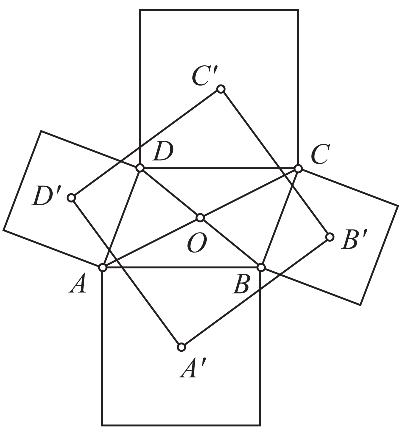

Figure 23

34. Squares are constructed to the outside of the parallelogram $ABCD$ on each side. Prove that their centers form a square.

Solution. Let the intersection of diagonals of a square $ABCD$ coincide with the origin and let a, b, c, d be the affixes of A, B, C, D, respectively, and a', b', c', d', be the affixes of A', B', C', D' (the centers of the constructed squares), respectively (see the figure 23). Then $c=-a,\ d=-b$, thus

$$(a-a')e^{-i\frac{\pi}{2}}+a'=b \text{, i.e. } a'=\frac{b+ai}{1+i}.$$

Similarly,

$$b' = \frac{c+bi}{1+i},\ c' = \frac{d+ci}{1+i} \text{ and } d' = \frac{a+di}{1+i},$$

and hence

$$b' = \frac{-a+bi}{1+i},\ c' = \frac{-b-ai}{1+i},\ d' = \frac{a-bi}{1+i}.$$

Finally, since

$$b' - a' = \frac{-(a+b)+(b-a)i}{1+i} = d' - c'$$

and

$$c' - b' = \frac{a-b-(b+a)i}{1+i} = i\frac{-(a+b)+(b-a)i}{1+i} = i(b' - a'),$$

we get that the quadrilateral is a square. ■

35. Squares are constructed to the outside of the quadrilateral *ABCD* on each side. If A', B', C' and D' are centers of the squares constructed on the sides *AB*, *BC*, *CD* and *DA*, respectively, *E* is the midpoint of $A'C'$, *F* is the midpoint of *BD*, *G* is the midpoint of $B'D'$ and *H* is the midpoint of *AC*, prove that the quadrilateral *EFGH* is a square.

Solution. Let the affixes of points be denoted by the appropriate lower case letters. The properties of rotation and the condition of the given problem, imply the following

$$a' = \frac{a-bi}{1-i},\ b' = \frac{b-ci}{1-i},\ c' = \frac{c-di}{1-i},\ d' = \frac{d-ai}{1-i}.$$

Further,

$$e = \frac{a'+c'}{2} = \frac{a+c-(b+d)i}{2(1-i)},$$

$$g = \frac{b'+d'}{2} = \frac{b+d-(a+c)i}{2(1-i)},$$

$$f = \frac{b+d}{2},\ h = \frac{a+c}{2}.$$

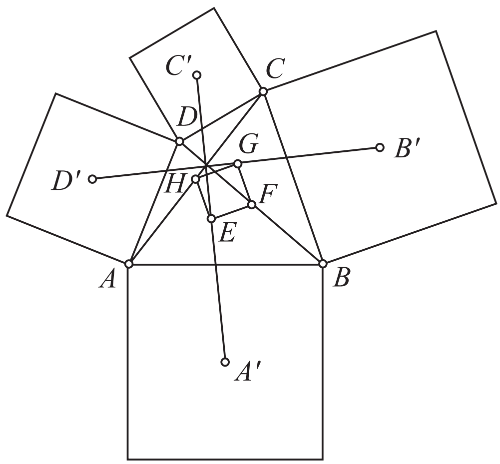

Figure 24

Hence,

$$f - e = \frac{b+d}{2} - \frac{a+c-(b+d)i}{2(1-i)} = \frac{b+d}{2(1-i)} - \frac{a+c}{2(1-i)},$$

$$g - h = \frac{b+d-(a+c)i}{2(1-i)} - \frac{a+c}{2} = \frac{b+d}{2(1-i)} - \frac{a+c}{2(1-i)}, \text{ and}$$

$$g - f = \frac{b+d-(a+c)i}{2(1-i)} - \frac{b+d}{2} = \left(\frac{b+d}{2(1-i)} - \frac{a+c}{2(1-i)}\right) = i(f - e),$$

therefore quadrilateral *EFGH* is a square. ■

36. Squares centered at *P*, *Q*, *R* are constructed to the outside of the triangle *ABC* on each side. Squares centered at A', B', C' are constructed to the inside of the *PQR* on each side. Prove that the points A', B', C' are midpoints of the sides of the triangle *ABC*.

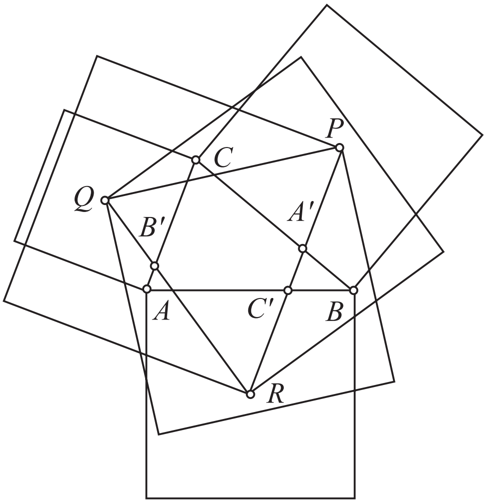

Figure 25

Solution. Let the affixes of points be denoted by appropriate lower case letters and let A', B', C' be centers of the squares constructed on the sides QR, RP, PQ, respectively (see the figure 25). Then, the condition of the given problem and the properties of rotation imply that

$$p = \frac{b-ci}{1-i},\ q = \frac{c-ai}{1-i},\ r = \frac{a-bi}{1-i}, \qquad (1)$$

$$a' = \frac{r-qi}{1-i},\ b' = \frac{p-ri}{1-i},\ c' = \frac{q-pi}{1-i}. \qquad (2)$$

If we substitute the expressions for p, q, r, in (2) we get that

$$a' = \frac{b+c}{2},\ b' = \frac{c+a}{2},\ c' = \frac{a+b}{2},$$

which was supposed to be proven. ■

37. Let a, b be complex,but not real numbers, and are such that they satisfy the following $|a-b|=2$ and $ab=1$. Prove that the quadrilateral $ABCD$ whose vertices have affixes -1, a, 1, b, respectively, is an isosceles trapezoid.

Solution. Since $\overline{AC} = |1-(-1)| = 2 = |b-a| = \overline{BD}$ it is sufficient to prove that the quadrilateral $ABCD$ is a cyclic. But,

$$\frac{[b-(-1)](1-a)}{[1-(-1)](b-a)} = \frac{1-a+b-ab}{2(b-a)} = \frac{b-a}{2(b-a)} = \frac{1}{2} \in \mathbf{R},$$

thus, the remark 3.6 b) implies that the points -1, a, 1, b are placed on a same circle, but they are not on a same line, because in that case it must be the real axis, which is contradictory with the given condition. Hence, the quadrilateral $ABCD$ is an isosceles trapezoid. ■

38. Let $ABCD$ be a cyclic quadrilateral and let H_A, H_B, H_C and H_D be the orthocenters of the triangles BCD, CDA, DAB and ABC, respectively. Prove that the quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent.

Solution. Without loss of generality we assume that the circumcircle of the quadrilateral $ABCD$ is the unit circle. So, we have that

$$h_a = b+c+d,\ h_b = c+d+a,\ h_c = d+a+b,\ h_d = a+b+c.$$

To prove that quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent it is sufficient to prove that for all $x, y \in \{a,b,c,d\}$

$$|x-y| = |h_x - h_y|$$

holds true (why?), which is easy to be proven. Indeed, for example

$$|h_a - h_b| = |b+c+d-(c+d+a)| = |b-a| = |a-b|. \ ■$$

39. Let a, b, c be complex numbers such that they satisfy the following

$$a+b+c=0 \text{ and } |a|=|b|=|c|.$$

Then a, b, c are vertices of an equilateral triangle. Prove it!

Solution.

$$|a-b|^2=(a-b)(\overline{a}-\overline{b})=a\overline{a}-a\overline{b}-b\overline{a}+b\overline{b}=4-a\overline{a}-a\overline{b}-b\overline{a}-b\overline{b}$$

$$=4-(a+b)(\overline{a}+\overline{b})=4-|a+b|^2=4-|-c|^2=4-|c|^2=3.$$

Hence, $|a-b|=\sqrt{3}$. Similarly, it can be proven that $|b-c|=\sqrt{3}$ and $|c-a|=\sqrt{3}$, which actually means that a, b, c are vertices of an equilateral triangle. ■

40. Let the complex numbers a, b, c have equal modules and let a, b, c be affixes of vertices of an equilateral triangle. Prove that the complex numbers ab, bc, ca are also the affixes of vertices of an equilateral triangle.

Solution. Let $|a|=|b|=|c|=r$ and $|a-b|=|b-c|=|c-a|=x$ hold true. Hence,

$$|ab-bc|=|b|\cdot|a-c|=xr,$$

$$|bc-ca|=|c|\cdot|b-a|=xr,$$

$$|ca-ab|=|a|\cdot|c-b|=xr,$$

the above imply that ab, bc, ca are affixes of vertices of an equilateral triangle. ■

41. The squares $BCDE$, $CAFG$ and $ABHI$ are constructed to the outside of the triangle $\triangle ABC$, on each side BC, CA and AB. Let $GCDQ$ and $EBHP$ be parallelograms. Prove that the $\triangle APQ$ is an isosceles right angled triangle.

Solution. The point h is obtained by rotation of a point a around b at $\frac{\pi}{2}$ in a positive direction, which means that

$$h=a+(b-a)e^{i\frac{\pi}{2}}=(1-i)a+ib.$$

Similarly, $d=(1-i)b+ic$ and $g=(1-i)c+ia$. The quadrilateral $BCDE$ is a square, thus the midpoints of sides CE and BD coincide with each other, which implies that $d+b=e+c$, thus $e=(1+i)b-ic$. Analogously, $g=(1+i)c-ia$. Further, the quadrilaterals $BEPH$ and $CGQD$ are parallelograms, thus $p+b=e+h$ and $c+q=g+d$, that is $p=ia+b-ic$ and $q=-ia+ib+c$. Finally, by rotation of the point p around a at $\frac{\pi}{2}$ we get that

$$a+(p-a)e^{i\frac{\pi}{2}}=a+i(ia+b-ic-a)=a-a+ib+c-ia=-ia+ib+c=q.$$

Finally, the point Q is obtained by rotation of the point P around A at $\frac{\pi}{2}$, hence the triangle $\triangle APQ$ is isosceles right angled triangle.■

42. The equilateral triangles BCB_1, CDC_1, DAD_1 are constructed to the outside of a convex quadrilateral $ABCD$, on each side BC, CD, DA. If the points P, Q and R are the midpoints of the sides B_1C_1, C_1D_1 and AB, respectively, prove that the triangle PQR is an equilateral triangle.

Solution. The points B_1, C_1, D_1 are obtained by rotation of the points B, C, D around C, D, A at $\frac{\pi}{3}$ in a positive direction, respectively. Hence, by letting the $\varepsilon = e^{i\frac{\pi}{3}}$ we get that

$$b_1 = c + (b-c)\varepsilon,\ c_1 = d + (c-d)\varepsilon,\ d_1 = a + (d-a)\varepsilon\,.$$

Further, thereby P is the midpoint of B_1C_1 we get that

$$p = \frac{b_1+c_1}{2} = \frac{b\varepsilon+c+(1-\varepsilon)d}{2}\,.$$

Similarly, $q = \frac{c\varepsilon+d+(1-\varepsilon)a}{2}$. Clearly, $r = \frac{a+b}{2}$. Hence,

$$\begin{aligned} r+(p-r)\varepsilon &= \frac{a+b}{2} + \left(\frac{b\varepsilon+c+(1-\varepsilon)d}{2} - \frac{a+b}{2}\right)\varepsilon \\ &= \frac{c\varepsilon+a(1-\varepsilon)+d(\varepsilon-\varepsilon^2)+b(1-\varepsilon+\varepsilon^2)}{2} \\ &= \frac{c\varepsilon+d+(1-\varepsilon)a}{2} = q \end{aligned}$$

thereby $\varepsilon^2 - \varepsilon + 1 = 0$, (why?). Hence, the point Q is obtained by rotation of the point P around R. Therefore, the triangle PQR is equilateral triangle. ■

43. Let $ABCD$ be a convex quadrilateral so that $\overline{AC} = \overline{BD}$. On the exterior side of the quadrilateral on its sides are constructed equilateral triangles. Let O_1, O_2, O_3, O_4 be the centers of triangles constructed on the sides AB, BC, CD, DA, respectively. Prove that the lines O_1O_3 and O_2O_4 are perpendicular to each other.

Solution. Since, a point A is obtained by rotation of B around O_1 at $\frac{2\pi}{3}$ in a positive direction, and by taking that $\varepsilon = e^{i\frac{2\pi}{3}}$ we get that $a = o_1 + (b-o_1)\varepsilon$, i.e. $o_1 = \frac{a-b\varepsilon}{1-\varepsilon}$. Analogously, $o_2 = \frac{b-c\varepsilon}{1-\varepsilon}$, $o_3 = \frac{c-d\varepsilon}{1-\varepsilon}$ and $o_4 = \frac{d-a\varepsilon}{1-\varepsilon}$. Further, to prove that $O_1O_3 \perp O_2O_4$ it is sufficient to prove that $\frac{o_1-o_3}{\overline{o_1}-\overline{o_3}} = -\frac{o_2-o_4}{\overline{o_2}-\overline{o_4}}$, i.e. it is sufficient to prove that

$$\frac{a-c-(b-d)\varepsilon}{\overline{a}-\overline{c}-(\overline{b}-\overline{d})\overline{\varepsilon}} = -\frac{b-d-(c-a)\varepsilon}{\overline{b}-\overline{d}-(\overline{c}-\overline{a})\overline{\varepsilon}}\,.$$

We can be directly assured in the validity of the latter by using the $\varepsilon\overline{\varepsilon} = 1$, i.e. $\overline{\varepsilon} = \frac{1}{\varepsilon}$ and

$$(a-c)\overline{(a-c)} = |a-c|^2 = |b-d|^2 = (b-d)\overline{(b-d)}\,. \ ■$$

44. Let M and N be two distinct points on a plane of a triangle $\triangle ABC$ so that

$$\overline{AM} : \overline{BM} : \overline{CM} = \overline{AN} : \overline{BN} : \overline{CN}\,.$$

Prove that the line MN passes through the circumcenter of the triangle $\triangle ABC$.

Solution. Without loss of generality we consider the circumcircle of the triangle $\triangle ABC$ as the unit circle. Then $o = 0$ and $\overline{a} = \frac{1}{a}$, $\overline{b} = \frac{1}{b}$ and $\overline{c} = \frac{1}{c}$. The proportion $\overline{AM} : \overline{BM} = \overline{AN} : \overline{BN}$ is written as following

$$1 = \frac{|a-m|\cdot|b-n|}{|a-n|\cdot|b-m|}\,,$$

thus,

$$1=\frac{|a-m|^2|b-n|^2}{|a-n|^2|b-m|^2}=\frac{(a-m)(\overline{a}-\overline{m})(b-n)(\overline{b}-\overline{n})}{(a-n)(\overline{a}-\overline{n})(b-m)(\overline{b}-\overline{m})}. \quad (1)$$

Further,

$$(a-m)(\overline{a}-\overline{m})(b-n)(\overline{b}-\overline{n})=\left(1-\tfrac{m}{a}-a\overline{m}+m\overline{m}\right)\left(1-\tfrac{n}{b}-b\overline{n}+n\overline{n}\right),$$

$$(a-n)(\overline{a}-\overline{n})(b-m)(\overline{b}-\overline{m})=\left(1-\tfrac{n}{a}-a\overline{n}+n\overline{n}\right)\left(1-\tfrac{m}{b}-b\overline{m}+m\overline{m}\right)$$

If we substitute in (1) we obtain the following equality

$$\left(1-\tfrac{m}{a}-a\overline{m}+m\overline{m}\right)\left(1-\tfrac{n}{b}-b\overline{n}+n\overline{n}\right)=\left(1-\tfrac{n}{a}-a\overline{n}+n\overline{n}\right)\left(1-\tfrac{m}{b}-b\overline{m}+m\overline{m}\right)$$

After reducing and dividing the latter by $a-b$ we get that

$$\frac{m}{ab}-\overline{m}-\frac{n}{ab}+\frac{(a+b)\overline{m}n}{ab}-\frac{m\overline{m}n}{ab}+\overline{n}-\frac{(a+b)m\overline{n}}{ab}+\overline{m}\overline{n}+\frac{mn\overline{n}}{ab}-\overline{m}n\overline{n}=0. \quad (2)$$

Analogously, since the proportion $\overline{AM}:\overline{CM}=\overline{AN}:\overline{CN}$, whereby in (2) b is substitute by c, and thereby symmetry we get that

$$\frac{m}{ac}-\overline{m}-\frac{n}{ac}+\frac{(a+c)\overline{m}n}{ac}-\frac{m\overline{m}n}{ac}+\overline{n}-\frac{(a+c)m\overline{n}}{ac}+\overline{m}\overline{n}+\frac{mn\overline{n}}{ac}-\overline{m}n\overline{n}=0. \quad (3)$$

If we subtract (3) from (2), after reducing and dividing the so-obtained equality by $b-c$ we get the following

$$-\frac{m}{abc}+\frac{n}{abc}-\frac{\overline{m}n}{bc}+\frac{m\overline{m}n}{abc}+\frac{m\overline{n}}{bc}-\frac{mn\overline{n}}{abc}=0. \quad (4)$$

Further, thereby the symmetry, reapplying the same procedure to the proportions $\overline{AM}:\overline{BM}=\overline{AN}:\overline{BN}$ and $\overline{BM}:\overline{CM}=\overline{BN}:\overline{CN}$ we obtain the following equality

$$-\frac{m}{abc}+\frac{n}{abc}-\frac{\overline{m}n}{ac}+\frac{m\overline{m}n}{abc}+\frac{m\overline{n}}{ac}-\frac{mn\overline{n}}{abc}=0. \quad (5)$$

Finally, if we subtract (5) from (4), and further the so-obtained equality we divide by $\frac{1}{ac}-\frac{1}{bc}$ we get that $m\overline{n}-n\overline{m}=0$, which is equivalent to

$$\frac{m-o}{\overline{m}-\overline{o}}=\frac{n-o}{\overline{n}-\overline{o}},$$

therefore, the points M, N and O are collinear. ■

45. The quadrilateral $ABCD$ is inscribed into a circle, such that AC is its diameter. Lines AB and CD meet at M, and the tangents at B and D meet at N. Prove that $MN\perp AC$.

Solution. Let the quadrilateral $ABCD$ be inscribed into a unit circle. Since AC is a diameter we have that $c=-a$. Further, thereby Remark 3.13 holds true we have that affix of M is

$$m=\frac{(a+b)cd-(c+d)ab}{cd-ab}=\frac{2bd+ad-ab}{d+b}$$

and affix of N is $n=\frac{2bd}{b+d}$. Further, since

$$\overline{a}=\frac{1}{a},\ \overline{b}=\frac{1}{b} \text{ and } \overline{d}=\frac{1}{d}$$

we get that

$$m-n=\frac{a(d-b)}{b+d} \text{ and } \overline{m}-\overline{n}=\frac{\overline{a}(\overline{d}-\overline{b})}{\overline{b}+\overline{d}}=\frac{b-d}{a(b+d)},$$

The latter means that

$$\frac{\overline{m}-\overline{n}}{m-n}=a^2 .$$

But,

$$\frac{\overline{a}-\overline{c}}{a-c}=-\frac{\overline{2a}}{2a}=-a^2 ,$$

Therefore, the complex gradient of *MN* and *AC* holds true

$$\frac{\overline{m}-\overline{n}}{m-n}=a^2=-\frac{\overline{a}-\overline{c}}{a-c} ,$$

which means that $MN \perp AC$. ■

46. Let *H* be the orthocenter of $\triangle ABC$ and let *P* be placed on its circumcircle. Let *E* be the foot of the altitude *BH*, the quadrilaterals *PAQB* and *PARC* be parallelograms and *AD* and *HR* meet at *X*. Prove that *EX* and *AP* are parallel.

Solution. Without loss of generality we get that the circumcircle of $\triangle ABC$ is the unit circle. Since Theorem 15.2 we have that $h=a+b+c$, and since solution of Example 1.9 we get that the affix of *E* is the following $e=\frac{1}{2}\left(a+b+c-\frac{ac}{b}\right)$. Further, the equilateral *PAQB* is parallelogram, and therefore the midpoints of the line segments *PQ* and *AB* coincide, i.e. $q=a+b-p$. Analogously, since the quadrilateral *PARC* is parallelogram we have that $r=a+c-p$. But, *A*, *Q*, *X* are collinear, and therefore

$$\frac{\overline{x}-\overline{a}}{x-a}=\frac{\overline{a}-\overline{q}}{a-q}=\frac{\overline{p}-\overline{b}}{p-b}=-\overline{pb} \text{, i.e. } \overline{x}=\frac{pb+a^2-ax}{abp} .$$

Analogously, the points *H*, *R*, *X* are collinear, and therefore

$$\frac{\overline{x}-\overline{h}}{x-h}=\frac{\overline{h}-\overline{r}}{h-r}=\frac{\overline{p}+\overline{b}}{p+b}=\overline{pb} \text{, i.e. } \overline{x}=\frac{x-a-b-c+p+\frac{bp}{a}+\frac{bp}{c}}{ab} .$$

By equating the obtained equalities for $\overline{x}$ we express *x*

$$x=\frac{1}{2}\left(2a+b+c-p-\frac{bp}{c}\right).$$

Finally, to prove that *EX* and *AP* are parallel, it is sufficient to prove that

$$\frac{\overline{e}-\overline{x}}{e-x}=\frac{\overline{a}-\overline{p}}{a-p}=-\overline{ap}$$

holds true. The latter can be directly checked if we consider that

$$e-x=\frac{1}{2}\left(p+\frac{bp}{c}-a-\frac{ac}{b}\right)=\frac{(b+c)(bp-ac)}{2bc} . \ ■$$

47. Let *ABCD* be a cyclic quadrilateral. The points *P* and *Q* are symmetric to *C* with respect to the lines *AB* and *AD*, respectively. Prove that *PQ* passes through the orthocenter of the triangle *ABD*.

Solution. Without loss of generality we can consider that the quadrilateral *ABCD* is inscribed into the unit circle. Since the solution of Example 1.9, the affixes of *P* and *Q* are

$$p=a+b-\frac{ab}{c},\ \ q=a+d-\frac{ad}{c} . \qquad (1)$$

The Theorem 15.2 implies that the orthocenter of $\triangle ABD$ has affix $h=a+b+d$, and therefore (1) implies that

$$\frac{p-h}{\overline{p}-\overline{h}}=\frac{a+b-\frac{ab}{c}-a-b-d}{\frac{1}{a}+\frac{1}{b}-\frac{c}{ab}-\frac{1}{a}-\frac{1}{b}-\frac{1}{d}}=\frac{abd}{c}=\frac{a+d-\frac{ad}{c}-a-b-d}{\frac{1}{a}+\frac{1}{d}-\frac{c}{ad}-\frac{1}{a}-\frac{1}{b}-\frac{1}{d}}=\frac{q-h}{\overline{q}-\overline{h}},$$

The latter means that the line *PQ* passes through the orthocenter of the triangle *ABD*. ■

48. Let *ABC* be a given triangle, *H* be the orthocenter, *O* be the sircumcenter and *R* be the circumradius of its circumcircle. Let the point *D* be symmetric to *A* with respect to the line *BC*, the point *E* be symmetric to *B* with respect to *CA* and *F* be symmetric to *C* with respect to *AB*. Prove that the points *D*, *E* and *F* are collinear if and only if $\overline{OH}=2R$.

Solution. Without loss of the generality we can consider that the triangle is inscribed into the unit circle. Then $o=0$, $R=1$ and since 15.2 we have $h=a+b+c$. Thereby Example 1.9 the affixes of the points *D*, *E* and *F* are the following

$$d=b+c-\frac{bc}{a},\ e=a+c-\frac{ac}{b},\ f=a+b-\frac{ab}{c}. \qquad (1)$$

Further, the points *D*, *E* and *F* are collinear if and only if

$$\frac{d-e}{\overline{d}-\overline{e}}=\frac{f-e}{\overline{f}-\overline{e}}. \qquad (2)$$

holds true. If (1) is substituted in (2), and after reducing we get that the points *D*, *E* and *F* are collinear if and only if

$$(c-a)(abc-a^2b-ab^2-a^2c-ac^2-b^2c-bc^2)=0,$$

Thereby $c-a\neq 0$, we get that the points *D*, *E* and *F* are collinear if and only if

$$abc-a^2b-ab^2-a^2c-ac^2-b^2c-bc^2=0 \iff$$

$$\frac{a^2b+ab^2+abc+a^2c+ac^2+abc+b^2c+bc^2+abc}{abc}=4 \iff$$

$$\frac{ab(a+b+c)+ac(a+b+c)+bc(a+b+c)}{abc}=4 \iff$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=4 \iff (a+b+c)(\overline{a}+\overline{b}+\overline{c})=4 \iff$$

$$|h-o|^2=h\overline{h}=4R^2 \iff \overline{OH}=2R. \ ■$$

49. Let *ABC* be a triangle so that the tangent of its circumcirle at the vertex *A* meets the midsegment of a triangle (parallel to *BC*) at A_1. The points B_1 and C_1 are defined analogously. Prove that the points A_1, B_1 and C_1 are collinear and moreover the line which passes through these points is perpendicular to the Euler line of $\triangle ABC$.

Solution. Without loss of generality we can consider that the triangle $\triangle ABC$ is inscribed into the unit circle. Then, $h=a+b+c$ and according to 17.4 the equation of the Euler line is the following $z=\frac{h}{\overline{h}}\overline{z}$. Further, if A', B', C' are the midpoints of the line segments *BC*, *CA*, *AB* respectively, then their affixes are

$$a'=\frac{b+c}{2},\ b'=\frac{c+a}{2},\ c'=\frac{a+b}{2}.$$

Therefore, the equation of the midsegment $B'C'$ parallel to BC is

$$z-\frac{c+a}{2}=-bc\left(\overline{z}-\frac{a+c}{2ac}\right), \tag{1}$$

And the equation of the tangent at the vertex A is

$$z+a^2\overline{z}=2a. \tag{2}$$

Since (2) we obtain the following expression $\overline{z}=\frac{2a-z}{a^2}$ and by substituting in (1) we get the following equation

$$z-\frac{c+a}{2}=-bc\left(\frac{2a-z}{a^2}-\frac{a+c}{2ac}\right),$$

whose solution $a_1=\frac{a^2(a+b+c)-3abc}{2(a^2-bc)}$ is the affix of the point A_1. Symmetrically,

$$b_1=\frac{b^2(a+b+c)-3abc}{2(b^2-ca)} \text{ and } c_1=\frac{c^2(a+b+c)-3abc}{2(c^2-ab)}.$$

Further,

$$a_1-b_1=\frac{a^2(a+b+c)-3abc}{2(a^2-bc)}-\frac{b^2(a+b+c)-3abc}{2(b^2-ca)}=-\frac{c(a-b)^3(a+b+c)}{2(a^2-bc)(b^2-ca)}$$

it is easy to check that following holds true

$$\frac{a_1-b_1}{\overline{a_1}-\overline{b_1}}=-\frac{(a+b+c)abc}{ab+bc+ca}=-\frac{h}{\overline{h}},$$

the latter means that the line A_1B_1 is perpendicular to the Euler line. Symmetrically, the line B_1C_1 is perpendicular to the Euler line, and therefore the points A_1, B_1 and C_1 are collinear. ■

50. Let H be the orthocenter of $\triangle ABC$. Prove that the Euler circles of the triangles ABC, ABH, BCH, CAH coincide.

Solution. Without loss of generality we can consider that the triangle $\triangle ABC$ is inscribed into the unit circle. The center of the Euler circle of the triangle $\triangle ABC$ is the point E with affix $e=\frac{a+b+c}{2}$. The solution of Example 17.9 implies that the circumradius of the triangles ABC and ABH are congruent and the point O' with affix $o'=a+b$ is the circumcenter of the triangle ABH. Thereby $CH\perp AB$ and $BC\perp AH$, C is the orthocenter of the triangle ABH. If E' is the center of the Euler circle of the triangle ABH, and $H'(h')$ is its orthocenter, we get that $h'=c$ and E' is the midpoint of the line segment $O'H'$, i.e. $O'C$, and therefore its affix is $e'=\frac{(a+b)+c}{2}=e$. The latter means that the points E and E' coincide. But, the radius of the Euler circle is half of the length of the circumradius. The above stated implies that the Euler circles of the triangles ABC and ABH coincide.

Analogously, it can be proven that the Euler circles of triangles ABC and BCH, i.e. of triangles ABC and CAH coincide.

Finally, the above stated implies that the Euler circles of triangles ABC, ABH, BCH, CAH coincide. ■

51. Let $ABCD$ be a cyclic quadrilateral. Prove that

a) the Euler circles of the triangles ABC, BCD, CDA, DAB meet at a unique point.

b) the centers of the Euler circles of the triangles ABC, BCD, CDA, DAB are vertices of a cyclic quadrilateral.

Solution. a) Without loss of generality we can consider that the quadrilateral $ABCD$ is inscribed into the unit circle. The quadrilateral $ABCD$ is cyclic, and therefore the centers of circumcircles of the triangles ABC, BCD, CDA, DAB coincide. If E_1, E_2, E_3, E_4 are the centers of the Euler circles of the triangles ABC, BCD, CDA, DAB, then their affixes will be

$$e_1 = \tfrac{a+b+c}{2},\ e_2 = \tfrac{b+c+d}{2},\ e_3 = \tfrac{c+d+a}{2},\ e_4 = \tfrac{d+a+b}{2}\ ,$$

respectively. If H_1, H_2, H_3, H_4 are the orthocenters of the triangles ABC, BCD, CDA, DAB, then their affixes will be

$$h_1 = a+b+c,\ h_2 = b+c+d,\ h_3 = c+d+a,\ h_4 = d+a+b\ ,$$

respectively. The point E with affix $e = \tfrac{a+b+c+d}{2}$ is midpoint of line segments DH_1, AH_2, BH_3, CH_4 and the following hold true

$$|e-e_1| = \left|\tfrac{a+b+c+d}{2} - \tfrac{a+b+c}{2}\right| = \left|\tfrac{d}{2}\right| = \tfrac{1}{2},$$

$$|e-e_2| = \left|\tfrac{a+b+c+d}{2} - \tfrac{b+c+d}{2}\right| = \left|\tfrac{a}{2}\right| = \tfrac{1}{2},$$

$$|e-e_3| = \left|\tfrac{a+b+c+d}{2} - \tfrac{c+d+a}{2}\right| = \left|\tfrac{b}{2}\right| = \tfrac{1}{2},$$

$$|e-e_4| = \left|\tfrac{a+b+c+d}{2} - \tfrac{d+a+b}{2}\right| = \left|\tfrac{c}{2}\right| = \tfrac{1}{2}.$$

The above means that it belongs to the Euler circles of the triangles ABC, BCD, CDA, DAB.

b) The proof is directly implied by the following equalities

$$|e-e_1| = |e-e_2| = |e-e_3| = |e-e_4| = \tfrac{1}{2}\ . \ \blacksquare$$

52. Let AA_1, BB_1 and CC_1 be the altitudes of $\triangle ABC$ and let $\overline{AB} \neq \overline{AC}$. Let M be the midpoint of BC, H be the orthocenter of $\triangle ABC$ and D be the point of intersection of BC and B_1C_1. Prove that $DH \perp AM$.

Solution. Let the circumcircle of $\triangle ABC$ be the unit circle. The condition of the given problem implies that

$$b_1 = \tfrac{1}{2}\left(a+b+c-\tfrac{ac}{b}\right) \text{ and } c_1 = \tfrac{1}{2}\left(a+b+c-\tfrac{ab}{c}\right),\ m = \tfrac{b+c}{2} \text{ and } h = a+b+c\ .$$

The equation of the line BC is $z-b = \frac{c-b}{\overline{c}-\overline{b}}(\overline{z}-\overline{b})$, i.e.

$$z-b = -bc\overline{z}+c\ . \tag{1}$$

The equation of the line B_1C_1 is $z-b_1 = \frac{c_1-b_1}{\overline{c_1}-\overline{b_1}}(\overline{z}-\overline{b_1})$, i.e.

$$z-b_1 = -a^2(\overline{z}-\overline{b_1})\ . \tag{2}$$

Since (1), $\overline{z}=\frac{c+b-z}{bc}$. By substituting in (2) and after reducing we obtain the following expression for the affix of D

$$d=\frac{a^2b+a^2c+ab^2+ac^2-b^2c-bc^2-2abc}{2(a^2-bc)}.$$

Finally, to prove that $DH\perp AM$ it is sufficient to check that

$$\frac{d-h}{\overline{d}-\overline{h}}=-\frac{m-a}{\overline{m}-\overline{a}},$$

where $d-h=\frac{(b+c-2a)(ab+bc+ca+a^2)}{2(a^2-bc)}$ and $m-a=\frac{b+c-2a}{2}$. The details are left as an exercise. ■

53. Let ABC be an acute triangle, so that $\overline{BC}>\overline{CA}$ and let O be the circumcenter, H be the orthocenter and F be the foot of the altitude CH. If the line through F, perpendicular to OF, intersects the side CA at P, then prove that $\angle FHP=\angle BAC$.

Solution. Without loss of generality we consider the circumcircle of triangle $\triangle ABC$ as a unit circle. The affix of F is the following $f=\frac{1}{2}\left(a+b+c-\frac{ab}{c}\right)$. The equation of the line CA is the following

$$z-a=\frac{c-a}{\overline{c}-\overline{a}}(\overline{z}-\overline{a}),\text{ i.e. } z+ac\overline{z}=a+c,$$

and the equation of the line which passes through F and is perpendicular to OF is the following

$$z-f=-\frac{f}{\overline{f}}(\overline{z}-\overline{f}).$$

By solving the system of the last two equations we obtain the affix of P as following

$$p=f\frac{2ac\overline{f}-(a+c)}{ac\overline{f}-f}=\frac{\left(a+b+c-\frac{ab}{c}\right)c^2}{b^2+c^2}.$$

Let $\angle PHF=\varphi$ and $\angle BAC=\alpha$. Then

$$\frac{f-h}{\overline{f}-\overline{h}}=\frac{p-h}{\overline{p}-\overline{h}}e^{2i\varphi}\text{ and }\frac{c-a}{\overline{c}-\overline{a}}=\frac{b-a}{\overline{b}-\overline{a}}e^{2i\alpha},$$

i.e.

$$e^{2i\varphi}=\frac{(f-h)(\overline{p}-\overline{h})}{(\overline{f}-\overline{h})p-h}\text{ and }e^{2i\alpha}=\frac{(c-a)(\overline{b}-\overline{a})}{(\overline{c}-\overline{a})(b-a)}=\frac{c}{b}.$$

Then thereby

$$p-h=-b\frac{ab+bc+ca+c^2}{b^2+c^2},\ \overline{p}-\overline{h}=-c\frac{ab+bc+ca+c^2}{ab(b^2+c^2)},$$

$$f-h=\frac{ab+bc+ca+c^2}{2c}\text{ and }\overline{f}-\overline{h}=\frac{ab+bc+ca+c^2}{abc}$$

it is true that $e^{2i\varphi}=\frac{c}{b}=e^{2i\alpha}$. The latter implies that $\varphi=\alpha$ or $\alpha=\varphi+\pi$. But, $\triangle ABC$ is an acute triangle, and therefore $\alpha=\varphi+\pi$ is not possible. So, $\varphi=\alpha$, i.e. $\angle FHP=\angle BAC$. ■

54. If a Symson line $l(P,ABC)$ passes through Q which is diametrically opposite of P, then it passes through the centroid of $\triangle ABC$. Prove that!

Solution. Without loss of generality we consider the circumcircle of triangle $\triangle ABC$ as the unit circle. According to the condition of a problem, the line $l(P,ABC)$ consists of a point Q with affix $q=-p$. Further, according to Example 24.4 the line $l(P,ABC)$ consists of a point O_p with affix $o_p=\frac{a+b+c+p}{2}$, and the centroid T of $\triangle ABC$ has affix $t=\frac{a+b+c}{3}$. Then,

$$\frac{t-q}{o_p-q}=\frac{\frac{a+b+c}{3}+p}{\frac{a+b+c+p}{2}+p}=\frac{2}{3}\frac{a+b+c+3p}{a+b+c+3p}=\frac{2}{3}.$$

Since Corollary 1.4. we get that points Q, T and are collinear, which actually means that T is placed on the line $l(P,ABC)$. ■

55. Prove that the Symson line of any point P (P is a point placed on a circumcircle of $\triangle ABC$) bisects a line segment PH where H is the orthocenter of a $\triangle ABC$.

Solution. Without loss of generality we consider the circumcircle of the triangle $\triangle ABC$ as a unit circle. The orthocenter H of $\triangle ABC$ has affix $h=a+b+c$, and therefore the midpoint Q of the line segment PH has affix $q=\frac{a+b+c+p}{2}$, which obviously satisfies the equation

$$z-\overline{z}\frac{acb}{p}+\frac{abc}{2p}(\overline{a}+\overline{b}+\overline{c}+\overline{p})-\frac{1}{2}(a+b+c+p)=0$$

of the Symson line $l(P,ABC)$. The latter means that $l(P,ABC)$ bisects the line segment PH. ■

56. Let $\triangle ABC$ be a triangle and let D be on the circumcircle of the triangle $\triangle ABC$. Determine the locus of meeting points of the Symson lines $l(A,BCD)$, $l(B,ACD)$, $l(C,ABD)$, $l(D,ABC)$, when D moves on a circumcircle of $\triangle ABC$.

Solution. Without loss of generality we consider the circumcircle of triangle $\triangle ABC$ as a unit circle. If a, b, c, d are the affixes of A, B, C, D, respectively, then according to the Example 23.4 the point of intersection of lines $l(A,BCD)$, $l(B,ACD)$, $l(C,ABD)$, $l(D,ABC)$ has affix $x=\frac{1}{2}(a+b+c+d)$ So, the required locus of points is a set of all points $x=\frac{1}{2}(a+b+c+d)$, when d moves on a circle. That actually is a circle with radius $\frac{1}{2}$ and center $\frac{a+b+c}{2}$, i.e. it is a circle centered at the midpoint of the line segment whose ends are the orthocenter and the circumenter of $\triangle ABC$, and the radius is congruent to half of the circumradius.■

57. Let $\triangle ABC$ be such a triangle that $\overline{AB}\neq\overline{AC}$ and let D be a point of intersection of the tangent to the circumcircle of $\triangle ABC$ at A and the line BC. If E and F are such points of bisectors of line segments AB and AC, respectively, that the lines BE and CF are perpendicular to BC, then the points D, E and F are collinear. Prove it!

Solution. Without loss of generality we consider the circumcircle of triangle $\triangle ABC$ as a unit circle. The equation of the line BC is the following

$$z - b = \frac{c-b}{\overline{c}-\overline{b}}(\overline{z} - \overline{b}), \text{ i.e. } z + bc\overline{z} = b + c,$$

and the equation of the tangent at A is $z + a^2\overline{z} = 2a$. By solving the system of the last two equations we obtain d, the affix of point of intersection between the line BC and the tangent to the circle at A, as following

$$d = \frac{a^2(b+c)-2abc}{a^2-bc}.$$

The point E is placed on the bisector of the line segment AB, and thus $OE \perp AB$, therefore $\frac{e-o}{\overline{e}-\overline{o}} = -\frac{a-b}{\overline{a}-\overline{b}}$, i.e. $\overline{e} = \frac{e}{ab}$. Further, $BE \perp BC$ implies that $\frac{e-b}{\overline{e}-\overline{b}} = -\frac{c-b}{\overline{c}-\overline{b}}$, and therefore $\overline{e} = \frac{c+e-b}{bc}$. So, $\frac{e}{ab} = \frac{c+e-b}{bc}$, thus $e = \frac{a(c-b)}{c-a}$. Analogously, $f = \frac{a(b-c)}{b-a}$.

Finally,

$$d - f = \frac{a^2(b+c)-2abc}{a^2-bc} - \frac{a(b-c)}{b-a} = \frac{ab(a-c)(b+c-2a)}{(a^2-bc)(b-a)} \text{ and}$$

$$d - e = \frac{a^2(b+c)-2abc}{a^2-bc} - \frac{a(c-b)}{c-a} = \frac{ac(a-b)(b+c-2a)}{(a^2-bc)(c-a)}$$

imply

$$\frac{\overline{d}-\overline{f}}{\overline{d}-\overline{e}} = \frac{\overline{b}(\overline{a}-\overline{c})^2}{\overline{c}(\overline{a}-\overline{b})^2} = \frac{a^2b^2c(c-a)^2}{c^2a^2b(b-a)^2} = \frac{b(a-c)^2}{c(a-b)^2} = \frac{d-f}{d-e},$$

the above means that the points D, E and F are collinear. ■

58. (Brokar theorem). Let $ABCD$ be a cyclic quadrilateral. The lines AB and CD intersect at E, the lines AD and BC intersect at F and the lines AC and BD intersect at G. Prove that the circumcenter O of the quadrilateral coincides to the orthocenter of $\triangle EFG$.

Solution. Let assume that $ABCD$ is inscribed into the unit circle. According to the Remark 3.13 c) the affixes of E, F and G are

$$e = \frac{ab(c+d)-cd(a+b)}{ab-cd}, \quad f = \frac{ad(b+c)-bc(a+d)}{ad-bc}, \quad g = \frac{ac(b+d)-bd(a+c)}{ac-bd}. \tag{1}$$

To prove that O is the orthocenter of $\triangle EFG$ it is sufficient to prove that $OF \perp EG$ and $OG \perp EF$. Since (1) it is easy to find that

$$\frac{f-o}{\overline{f}-\overline{o}} = \frac{ad(b+c)-bc(a+d)}{a+d-(b+c)}, \tag{2}$$

$$e - g = \frac{(a-d)(b-c)[(b+c)ad-(a+d)bc]}{(ab-cd)(ac-bd)} \tag{3}$$

$$\overline{e} - \overline{g} = \frac{(a-d)(b-c)(b+c-(a+d))}{(ab-cd)(ac-bd)}. \tag{4}$$

Now, (2), (3) and (4) imply that

$$\frac{e-g}{\overline{e}-\overline{g}} = \frac{\frac{(a-d)(b-c)[(b+c)ad-(a+d)bc]}{(ab-cd)(ac-bd)}}{\frac{(a-d)(b-c)(b+c-(a+d))}{(ab-cd)(ac-bd)}} = \frac{(b+c)ad-(a+d)bc}{b+c-(a+d)} = -\frac{ad(b+c)-bc(a+d)}{a+d-(b+c)} = -\frac{f-o}{\overline{f}-\overline{o}},$$

The latter means that $OF \perp EG$. Since the symmetry we conclude that $OG \perp EF$, i.e. O is the orthocenter of $\triangle EFG$. ■

59. Let ABC be an isosceles triangle, $\overline{AB} = \overline{AC}$. Let P be a point on the extension of the side BC and X and Y be the points on the sides AB and AC, respectively, so that $PX \parallel AC,\ PY \parallel AB$. If T is the midpoint of the arc $\widehat{BC}$, then $PT \perp XY$. Prove it!

Solution. Let the circumcircle of the triangle $\triangle ABC$ be the unit circle and $a=1$. Then $c=\overline{b}$ and $t=-1$. Further, since P is on the line BC, its affix p satisfies the following $\overline{p} = b + \frac{1}{b} - p$. Further, since X is a point on the side AB it is true that $\overline{x} = \frac{1+b-x}{b}$, and since $PX \parallel AC$ we get that $\overline{x} = \overline{p} + bp - bx$. So, $x = \frac{b(p+1)}{b+1}$ thereby the last three equations. Analogously, $\overline{y} = \frac{1+c-y}{c}$ and $\overline{y} = \overline{p} + cp - cy$, thus $y = \frac{p+1}{b+1}$. Finally,

$$\frac{x-y}{\overline{x}-\overline{y}} = \frac{\frac{(p+1)(b-1)}{b+1}}{-\frac{(\overline{p}+1)(b-1)}{b+1}} = -\frac{p+1}{\overline{p}+1} = -\frac{p-t}{\overline{p}-\overline{t}},$$

implies that $PT \perp XY$. ■

60. Let $ABCD$ be a cyclic quadrilateral and let K, L, M, N be the midpoints of the sides AB, BC, CD, DA, respectively. Prove that the orthocenters of the triangles AKN, BKL, CLM, DMN form a parallelogram.

Solution. Let the circumcircle of a quadrilateral be the unit circle. The affixes of the points K, L, M, N are

$$k = \frac{a+b}{2},\ l = \frac{b+c}{2},\ m = \frac{c+d}{2},\ n = \frac{d+a}{2}.$$

We have to determine the affix h_1 of the orthocenter H_1 of the triangle AKN. Since $KH_1 \perp AN,\ NH_1 \perp AK$, the following holds true

$$\frac{k-h_1}{\overline{k}-\overline{h_1}} = -\frac{a-n}{\overline{a}-\overline{n}} = -\frac{a-d}{\overline{a}-\overline{d}} = ad \text{ and } \frac{n-h_1}{\overline{n}-\overline{h_1}} = -\frac{a-k}{\overline{a}-\overline{k}} = -\frac{a-b}{\overline{a}-\overline{b}} = ab,$$

that is

$$\overline{h_1} = \frac{\overline{k}ad-k+h_1}{ad} \text{ and } \overline{h_1} = \frac{\overline{n}ab-n+h_1}{ab},$$

which imply that

$$h_1 = \frac{2a+b+d}{2}.$$

Analogously, the affixes of the orthocenters of the triangles BKL, CLM, DMN are

$$h_2 = \frac{2b+c+a}{2},\ h_3 = \frac{2c+b+d}{2},\ h_4 = \frac{2d+a+c}{2},$$

respectively. Finally, since

$$\frac{h_1+h_3}{2} = a+b+c+d = \frac{h_2+h_4}{2},$$

the midpoints of the diagonals of the quadrilateral coincide. The latter means that the quadrilateral is a parallelogram. ■

61. The incircle of a $\triangle ABC$ centered at O tangents the sides AB, BC, CA at M, K, E. If $P = MK \cap AC$, then $OP \perp BE$. Prove it!

Solution. Let the incircle of $\triangle ABC$ be the unit circle. Then, according to the Remark 3.13 d), it is true that

$$a = \frac{2em}{e+m} \text{ and } b = \frac{2mk}{m+k}.$$

Since P is on the chord MK we get that P, M, K are collinear, and therefore their affixes satisfy the following

$$\overline{p} = \frac{m+k-p}{mk}.$$

Further, P is on the line AC. Thereby this line tangents the circle at E, we get that $PE \perp OE$, and therefore

$$\frac{e-p}{\overline{e}-\overline{p}} = -\frac{e-o}{\overline{e}-\overline{o}} = -e^2, \text{ i.e. } \overline{p} = \frac{2e-p}{e^2}$$

holds true. By equating the last two expressions for $\overline{p}$ and after reducing, we obtain that the affix of P is the following

$$p = \frac{(m+k)e^2-2mke}{e^2-mk}.$$

Finally, it is easy to be checked that the affixes o, p, b, e of the points O, P, B, E satisfy the following

$$\frac{p-o}{\overline{p}-\overline{o}} = -\frac{e-b}{\overline{e}-\overline{b}},$$

(check it!). The latter means that $OP \perp BE$. ■

62. A circle centered at O is incircle of a quadrilateral $ABCD$ and tangents the sides AB, BC, CD, DA at K, L, M, N respectively. The lines KL and MN meet at S. Prove that $OS \perp BD$.

Solution. Let the incircle be the unit circle. Then Remark 13.3 implies that

$$a = \frac{2nk}{n+k},\ b = \frac{2kl}{k+l},\ c = \frac{2lm}{l+m},\ d = \frac{2mn}{m+n},\ s = \frac{kl(m+n)-mn(k+l)}{kl-mn}.$$

Further,

$$\overline{s} = \frac{\overline{kl(m+n)-mn(k+l)}}{\overline{kl-mn}} = \frac{k+l-(m+n)}{kl-mn},$$

$$b-d = \frac{2kl(m+n)-2mn(k+l)}{(k+l)(m+n)} \text{ and}$$

$$\overline{b}-\overline{d} = \frac{2(m+n)-2(k+l)}{(k+l)(m+n)}$$

and therefore

$$\frac{\overline{s}-\overline{o}}{s-o}=\frac{\frac{kl(m+n)-mn(k+l)}{kl-mn}}{\frac{k+l-(m+n)}{kl-mn}}=-\frac{kl(m+n)-mn(k+l)}{(m+n)-(k+l)}=-\frac{\frac{2kl(m+n)-2mn(k+l)}{(k+l)(m+n)}}{\frac{2(m+n)-2(k+l)}{(k+l)(m+n)}}=-\frac{\overline{b}-\overline{d}}{b-d},$$

The latter actually means that $OS\perp BD$. ■

63. Let ABC be an acute triangle, whose incircle tangents the sides AB and AC at Q and R, respectively. Let X and Y be the points of intersection between the bisectors of the angles $\angle ACB$ and $\angle ABC$ with the line QR, respectively and let Z be the midpoint of the line segment BC. Prove that the triangle XYZ is an equilateral triangle if and only if $\angle BAC=\frac{\pi}{3}$.

Solution. Without loss of generality we consider the incircle of the triangle as a unit circle. Let P be the point where the line BC tangents the incircle. Then,

$$a=\frac{2qr}{q+r},\ b=\frac{2pr}{p+r},\ c=\frac{2pq}{p+q}\ \text{ and }\ z=\frac{b+c}{2}=\frac{pr}{p+r}+\frac{pq}{p+q}.$$

Since the bisector of $\angle ACB$ passes through the center of the incircle, we get that the points B, O and X are collinear, and therefore the affix x of the point X satisfies the following $x=\alpha c=\alpha\frac{2pq}{p+q}$, $\alpha\in\mathbf{R}$. Similarly, the affix y of Y is $y=\beta\frac{2pr}{p+r}$, $\beta\in\mathbf{R}$. Further, the constants α and β are determined by the following given conditions that $X,Y\in QR$, i.e. the points X, Q, R are collinear likewise Y, Q, R. So,

$$\frac{\overline{q}-\overline{r}}{q-r}=\frac{\overline{x}-\overline{r}}{x-r}\ \text{ and }\ \frac{\overline{q}-\overline{r}}{q-r}=\frac{\overline{y}-\overline{q}}{y-q},$$

By direct calculation we get that,

$$\alpha=\frac{(p+q)(q+r)}{2q(p+r)}\ \text{ and }\ \beta=\frac{(p+r)(q+r)}{2r(p+q)},$$

thus

$$x=\frac{p(q+r)}{p+r}\ \text{ and }\ y=\frac{p(q+r)}{p+q}.$$

We have to prove that

$$\angle BAC=\frac{\pi}{3}\ \text{ if and only if } \triangle XYZ \text{ is an equilateral triangle.}$$

The first condition is equivalent to $\angle QOR=\frac{2\pi}{3}$, i.e. $q=re^{i\frac{2\pi}{3}}$. The second condition is equivalent to $y-z=(x-z)e^{i\frac{\pi}{3}}$. So,

$$y-z=\frac{p(q+r)}{p+q}-\left(\frac{pr}{p+r}+\frac{pq}{p+q}\right)=\frac{pr(r-q)}{(p+q)(p+r)},$$

$$x-z=\frac{p(q+r)}{p+r}-\left(\frac{pr}{p+r}+\frac{pq}{p+q}\right)=\frac{pq(q-r)}{(p+q)(p+r)}.$$

Therefore,

$$y-z=(x-z)e^{i\frac{\pi}{3}}\ \Leftrightarrow\ \frac{pr(r-q)}{(p+q)(p+r)}=\frac{pq(q-r)}{(p+q)(p+r)}e^{i\frac{\pi}{3}}\ \Leftrightarrow$$

$$r=-qe^{i\frac{\pi}{3}}\ \Leftrightarrow\ q=re^{i\frac{2\pi}{3}},$$

The last equivalence is implied if the equation before the last is multiplied by $e^{i\frac{2\pi}{3}}$ and also having on mind that $e^{i\pi}=-1$. ■

64. (Newton theorem). Let $ABCD$ be a cyclic quadrilateral. Let M and N be the midpoints of the diagonals AC and BD and S be the center of its incircle. Prove that M, N and S are collinear.

Solution. Let the quadrilateral be inscribed into the unit circle and let P, Q, R, S be the points where the circles tangents the sides AB, BC, CD, DA, respectively. Then,

$$a=\frac{2ps}{p+s},\ b=\frac{2pq}{p+q},\ c=\frac{2qr}{q+r},\ d=\frac{2rs}{r+s},$$

and therefore

$$m=\frac{a+c}{2}=\frac{pqs+prs+pqr+qrs}{(p+s)(q+r)},\quad \overline{m}=\frac{p+q+r+s}{(p+s)(q+r)},$$

$$n=\frac{b+d}{2}=\frac{pqr+pqs+prs+qrs}{(p+q)(r+s)},\quad \overline{n}=\frac{p+q+r+s}{(p+q)(r+s)}.$$

Thus,

$$\frac{m-o}{\overline{m}-\overline{o}}=\frac{pqr+pqs+prs+qrs}{p+q+r+s}=\frac{n-o}{\overline{n}-\overline{o}},$$

The latter actually means that M, N and S are collinear. ■

65. Let $ABCD$ be a quadrilateral and let its incircle tangents the sides AB, BC, CD, DA at points M, N, P, Q, respectively. Prove that the lines AC, BD, MP, NQ are *concurrent* (meet at a unique point).

Solution. Let the incircle of the quadrilateral $ABCD$ be the unit circle. Therefore,

$$b=\frac{2mn}{m+n},\ d=\frac{2pq}{p+q}.$$

If $X=MP\cap NQ$, then

$$x=\frac{mp(n+q)-nq(m+p)}{mp-nq}.$$

So,

$$b-d=2\frac{mn(p+q)-pq(m+n)}{(m+n)(p+q)},\qquad \overline{b}-\overline{d}=2\frac{p+q-(m+n)}{(m+n)(p+q)},$$

$$b-x=\frac{(m-n)[mn(p+q)-pq(m+n)]}{(m+n)(mp-nq)},\qquad \overline{b}-\overline{x}=\frac{(m-n)[p+q-(m+n)]}{(m+n)(mp-nq)},$$

Thus,

$$\frac{b-x}{\overline{b}-\overline{x}}=\frac{mn(p+q)-pq(m+n)}{p+q-(m+n)}=\frac{2\frac{mn(p+q)-pq(m+n)}{(m+n)(p+q)}}{2\frac{p+q-(m+n)}{(m+n)(p+q)}}=\frac{b-d}{\overline{b}-\overline{d}},$$

The above means that X is placed on the line BD. Further, by applying the symmetry, we conclude that X is placed on the line AC. The above stated means that the lines AC, BD, MP, NQ are concurrent. ■

66. The incircle of a triangle ABC tangents the sides BC, CA, AB at D, E, F, respectively, and X, Y, Z are the midpoints of the sides EF, FD, DE, respectively. Prove that the center of incircle is placed on the line determined by the circumcenters of the triangles XYZ and ABC.

Solution. Let the incircle of $\triangle ABC$ be the unit circle. According to the Remark 22.13 the affix o of the circumcenter of the triangle ABC is the following

$$o=\frac{2def(d+e+f)}{(d+e)(e+f)(f+d)}.$$

Further, the affixes of points X, Y, Z are

$$x=\frac{e+f}{2},\ y=\frac{d+f}{2},\ z=\frac{d+e}{2},$$

thereby the Remark 3.4 and Example 3.3 b),the affix o' of circumcenter of $\triangle XYZ$ is:

$$o'=\frac{x\overline{x}(z-y)+y\overline{y}(x-z)+z\overline{z}(y-x)}{x\overline{y}+y\overline{z}+z\overline{x}-\overline{x}y-\overline{y}z-\overline{z}x}=\frac{d+e+f}{2}.$$

So,

$$o-i=\frac{2def(d+e+f)}{(d+e)(e+f)(f+d)},\ \overline{o}-\overline{i}=\frac{2(de+ef+fd)}{(d+e)(e+f)(f+d)},$$

$$o'-i=\frac{d+e+f}{2},\ \overline{o'}-\overline{i}=\frac{de+ef+fd}{2def},$$

thus,

$$\frac{o-i}{\overline{o}-\overline{i}}=\frac{def(d+e+f)}{de+ef+fe}=\frac{o'-i}{\overline{o'}-\overline{i}}.$$

The above means that the points I, O, O' are collinear, which actually was supposed to be proven. ■

67. The incircle of a triangle ABC, centered at I, tangents the sides BC, CA, AB at D, E, F, respectively. Let $AI\cap EF=K$, $ED\cap KC=N$ and $DF\cap KB=M$. Prove that $MN\parallel BC$.

Solution. Let the triangle ABC be inscribed into the unit circle. Then,

$$a=\frac{2fe}{f+e},\ b=\frac{2fd}{f+d}\ \text{and}\ c=\frac{2ed}{e+d}.$$

Further, the affix of the midpoint of the line segment EF is $\frac{e+f}{2}$ and since

$$\frac{a-o}{\overline{a}-\overline{o}}=\frac{\frac{2ef}{e+f}}{\frac{2\overline{e}\overline{f}}{\overline{e}+\overline{f}}}=ef=\frac{\frac{f+e}{2}-o}{\frac{\overline{f}+\overline{e}}{2}-\overline{o}}$$

we get that $k=\frac{e+f}{2}$. Further, the equations of the lines DF and KB are

$$z-d=\frac{d-f}{\overline{d}-\overline{f}}(\overline{z}-\overline{d})\ \text{and}\ z-k=\frac{k-b}{\overline{k}-\overline{b}}(\overline{z}-\overline{k}),\qquad (1)$$

respectively, and if we substitute for

$$k=\frac{e+f}{2}\ \text{and}\ b=\frac{2fd}{f+d},$$

by reducing the system (1) we obtain the following expression for the affix of M

$$m=\frac{4ef^2d+efd^2-e^2d^2-e^2f^2-2f^2d^2-f^3e}{6efd-e^2d-ed^2-ef^2-e^2f-d^2f-df^2}.$$

The equations of the lines ED and KC are

$$z-d=\frac{d-e}{\overline{d}-\overline{e}}(\overline{z}-\overline{d}) \text{ and } z-k=\frac{k-c}{\overline{k}-\overline{c}}(\overline{z}-\overline{k}), \qquad (2)$$

respectively, and if we substitute the following expressions for $k=\frac{e+f}{2}$ and $c=\frac{2ed}{e+d}$, and by solving the system (2) we obtain the following expression for the affix of N

$$n=\frac{4e^2fd+efd^2-f^2d^2-e^2f^2-2e^2d^2-e^3f}{6efd-e^2d-ed^2-ef^2-e^2f-d^2f-df^2}.$$

Finally, it is sufficient to prove that $MN\perp ID$, namely to prove that

$$\frac{m-n}{\overline{m}-\overline{n}}=\frac{d-o}{\overline{d}-\overline{o}}=-d^2.$$

Details are left as your exercise. ■

68. Let $\triangle ABC$ be any triangle, with orthocenter H, circumcenter O, incenter I and K the point where the side BC tangents the incircle of $\triangle ABC$. If $IO\parallel BC$, then $AO\parallel HK$. Prove it!

Solution. Let the incircle of $\triangle ABC$ be the unit and let it tangent BC, CA, AB at K, L, M, respectively. According to the Remark 22.13 it is true that

$$o=\frac{2klm(k+l+m)}{(k+l)(l+m)(m+k)} \text{ and } h=\frac{2\left(k^2l^2+l^2m^2+m^2k^2+klm(k+l+m)\right)}{(k+l)(l+m)(m+k)}.$$

Further, $IO\parallel BC$ implies that $IO\perp KL$, and therefore

$$\frac{o-i}{\overline{o}-\overline{i}}=-\frac{k-i}{\overline{k}-\overline{i}}=-k^2$$

By substituting for o and $\overline{o}$, and after reducing we obtain that

$$klm(k+l+m)+k^2(kl+lk+mk)=0. \qquad (1)$$

We will prove that if the condition (1) is satisfied, then $AO\parallel HK$. The affix of A is the following $a=\frac{2ml}{m+l}$, thus

$$a-o=\frac{2ml}{m+l}-\frac{2klm(k+l+m)}{(k+l)(l+m)(m+k)}=\frac{2m^2l^2}{(k+l)(l+m)(m+k)} \text{ and } \overline{a}-\overline{o}=\frac{2k^2}{(k+l)(l+m)(m+k)}.$$

On the other hand, if we use the condition (1) we get that

$$h-k=\frac{(kl+lm+mk)^2[(k+l+m)^2+k^2]}{(k+l+m)^2(k+l)(l+m)(m+k)} \text{ and } \overline{h}-\overline{k}=\frac{(k+l+m)^2+k^2}{(k+l)(l+m)(m+k)}.$$

Thus,

$$\frac{h-k}{\overline{h}-\overline{k}}=\frac{(kl+lm+mk)^2}{(k+l+m)^2}=\text{(acording to (1))}=\frac{m^2l^2}{k^2}=\frac{a-o}{\overline{a}-\overline{o}}.$$

So, $AO\parallel HK$. ■

69. Let AH_1, BH_2, CH_3 be the altitudes of an acute triangle $\triangle ABC$. The incircle of the $\triangle ABC$ tangents the sides BC, CA, AB at points T_1, T_2, T_3, respectively. Let the lines l_1, l_2, l_3 be symmetric to the lines H_2H_3, H_3H_1, H_1H_2 with respect to the lines T_2T_3, T_3T_1, T_1T_2, respectively. Prove that the lines l_1, l_2, l_3 form a triangle whose vertices are on the incircle of the $\triangle ABC$.

Solution. Let the incircle of the $\triangle ABC$ be the unit circle. Thus $c = \frac{2t_1t_2}{t_1+t_2}$. Let's determine the affix h_3 of the point H_3. Since the given conditions $H_3T_3 \perp T_3I$ and $H_3C \parallel T_3I$ it is true that

$$\frac{h_3-t}{\overline{h_3}-\overline{t_3}} = -\frac{t_3-o}{\overline{t_3}-\overline{o}} = -t_3^2 \text{ and } \frac{h_3-c}{\overline{h_3}-\overline{c}} = \frac{t_3-o}{\overline{t_3}-\overline{o}} = t_3^2 .$$

By solving the system of the last two equations we get the following expression for h_3

$$h_3 = \frac{1}{2}(2t_3 + c - \overline{c}t_3^2) = t_3 + \frac{t_1t_2-t_3^2}{t_1+t_2} .$$

Analogously $h_2 = t_2 + \frac{t_1t_3-t_2^2}{t_1+t_3}$. Further in order to determine the line l_1 which is symmetric to H_2H_3 with respect to the line T_2T_3, it is sufficient to determine the points P_2 and P_3 which are symmetric to H_2 and H_3 with respect to the line T_2T_3, respectively. The equation of the line T_2T_3 is

$$z - t_2 = \frac{t_3-t_2}{\overline{t_3}-\overline{t_2}}(\overline{z} - \overline{t_2}) .$$

Since the example 1.9, the affix of P_2 is

$$p_2 = \frac{\overline{h_2}(t_2-t_3)+\overline{t_2}t_3-t_2\overline{t_3}}{\overline{t_2}-\overline{t_3}} = \frac{t_1(t_2^2+t_3^2)}{t_2(t_1+t_3)} .$$

Analogously, the affix of P_3 is

$$p_3 = \frac{\overline{h_3}(t_2-t_3)+\overline{t_2}t_3-t_2\overline{t_3}}{\overline{t_2}-\overline{t_3}} = \frac{t_1(t_2^2+t_3^2)}{t_3(t_1+t_2)} .$$

Further,

$$p_2 - p_3 = \frac{t_1(t_2^2+t_3^2)}{t_2(t_1+t_3)} - \frac{t_1(t_2^2+t_3^2)}{t_3(t_1+t_2)} = \frac{t_1^2(t_2^2+t_3^2)(t_3-t_2)}{t_2t_3(t_1+t_2)(t_1+t_3)} ,$$

thus, the equation of the line l_1 is

$$z - p_2 = \frac{p_2-p_3}{\overline{p_2}-\overline{p_3}}(\overline{z} - \overline{p_2}) ,$$

i.e.

$$z - \frac{t_1(t_2^2+t_3^2)}{t_2(t_1+t_3)} = -t_1^2\left(\overline{z} - \frac{t_2^2+t_3^2}{t_2t_3(t_1+t_3)}\right). \qquad (1)$$

Analogously, the equation of the line l_2 which is symmetric to H_3H_1 with respect to the line T_3T_1 is

$$z - \frac{t_2(t_3^2+t_1^2)}{t_3(t_2+t_1)} = -t_2^2\left(\overline{z} - \frac{t_3^2+t_1^2}{t_3t_1(t_2+t_1)}\right). \qquad (2)$$

By solving the system of the equations (1) and (2) we get that $m_1 = \frac{t_1 t_2}{t_3}$ is affix of the point of intersection of the lines l_1 and l_2. Analogously, $m_2 = \frac{t_2 t_3}{t_1}$ is affix of point of intersection of the lines l_2 and l_3 has affix and $m_3 = \frac{t_3 t_1}{t_2}$ is affix of the point of intersection of the lines l_3 and l_1. Finally, the statement is implied by the fact that $|m_1| = |m_2| = |m_3| = 1$. Prove it ■

70. Let O and R be the circumcenter and the circumradius of $\triangle ABC$, and Z and r be the incenter and the inradius of $\triangle ABC$, respectively. If K is the centroid of the triangle whose vertices are the points where the incircle tangents the sides of $\triangle ABC$ prove that $Z \in OK$ and also that $\overline{OZ} : \overline{ZK} = \frac{3R}{r}$.

Solution. Let the incircle of the triangle $\triangle ABC$ be the unit circle and let d, e, f be the affixes of its tangent points with the sides BC, CA, AB respectively. According to the Remark 22.13 it is true that $o = \frac{2def(d+e+f)}{(d+e)(e+f)(f+d)}$. Since Theorem 15.8, $k = \frac{d+e+f}{3}$. Thus,

$$\frac{\overline{o-z}}{o-z} = \frac{\overline{\frac{2def(d+e+f)}{(d+e)(e+f)(f+d)}} - 0}{\frac{2(d+e+f)}{(d+e)(e+f)(f+d)} - 0} = def = \frac{\overline{\frac{d+e+f}{3}} - 0}{\frac{d+e+f}{3def} - 0} = \frac{\overline{k-z}}{k-z}$$

Therefore the points K, Z and O are collinear. Further,

$$\frac{\overline{OZ}}{\overline{ZK}} = \frac{|o-z|}{|z-k|} = \frac{\left|\frac{2def(d+e+f)}{(d+e)(e+f)(f+d)}\right|}{\left|\frac{d+e+f}{3}\right|} = \frac{3}{\frac{1}{2}|(d+e)(e+f)(f+d)|} = \frac{3R}{r}.$$

The latter was supposed to be proven. ■

71. Let P be the intersection of the diagonals of a convex quadrilateral $ABCD$ so that $\overline{AB} = \overline{AC} = \overline{BD}$ and let O and I be the circumcenter and the incenter of $\triangle ABP$, respectively. Prove that if $O \neq I$, then $OI \perp CD$.

Solution. Let $\triangle ABP$ be inscribed into the unit circle and let u, v, w be complex numbers as given in Theorem 22.14, so that $a = u^2$, $b = v^2$, $p = w^2$, hold true. Then, according to the stated theorem $i = -uv - vw - wu$. But, $\overline{AB} = \overline{AC}$, and therefore for $\alpha = \angle CAB$ it is true that

$$c - a = e^{i\alpha}(b - a), \tag{1}$$

(make a figure). Further, the points A, C and P are collinear, and therefore $\alpha = \angle CAB = \angle PAB$. The latter means

$$\frac{-vw-u^2}{\overline{-vw-u^2}} = e^{2i\frac{\alpha}{2}} \frac{v^2-u^2}{\overline{v^2-u^2}}, \text{ i.e. } e^{i\alpha} = -\frac{w}{v}.$$

By substituting in (1) we get $c - u^2 = -\frac{w}{v}(v^2 - u^2)$, i.e.

$$c = \frac{u^2w+u^2v-v^2w}{v}. \tag{2}$$

Analogously,

$$d = \frac{v^2w+v^2u-u^2w}{u}. \tag{3}$$

Finally, (2) and (3) imply

$$c-d = \frac{(u^2-v^2)(uv+vw+wu)}{uv},$$

therefore

$$\frac{c-d}{\overline{c}-\overline{d}} = -\frac{uv+vw+wu}{u+v+w}uvw = -\frac{i-o}{\overline{i}-\overline{o}}.$$

The latter means that $OI \perp CD$. ■

72. Let I be the circumcenter of $\triangle ABC$, $\overline{AB} \neq \overline{AC}$. The point O_1 is symmetric to O, the circumcenter of $\triangle ABC$, with respect to the line BC. Prove that the points A, I and O_1 are collinear if and only if $\angle BAC = 60°$.

Solution. Let the circumcircle of $\triangle ABC$ be the unit circle. According to the Theorem 22.14 there exist complex numbers u, v, w so that

$$a = u^2,\ b = v^2,\ c = w^2 \text{ and } i = -uv - vw - wu.$$

According to the Example 1.9, the affix of O_1 is the following

$$o_1 = \frac{\overline{0}(b-c)+\overline{b}c-b\overline{c}}{\overline{b}-\overline{c}} = b + c = v^2 + w^2.$$

Further, the points A, I and O_1 are collinear if and only if

$$\frac{a-o_1}{\overline{a}-\overline{o_1}} = \frac{a-i}{\overline{a}-\overline{i}},$$

i.e. if and only if

$$\frac{v^2+w^2-u^2}{\overline{v^2+u^2}-\overline{u^2}} = \frac{u^2+uv+vw+wu}{\overline{u^2}+\overline{uv+vw+wu}} \quad \Leftrightarrow$$

$$\frac{v^2+w^2-u^2}{u^2(v^2+w^2)-v^2w^2}u^2v^2w^2 = \frac{u(u+v+w)+vw}{vw+uw+uv+u^2}u^2vw \quad \Leftrightarrow$$

$$\frac{v^2+w^2-u^2}{u^2(v^2+w^2)-v^2w^2}vw = 1 \quad \Leftrightarrow$$

$$wv^3 + v^2w + vw^3 - vwu^2 - u^2v^2 - u^2w^2 = 0 \quad \Leftrightarrow$$

$$(wv - u^2)(vw + v^2 + w^2) = 0.$$

That is, the points A, I and O_1 are collinear if and only if either $u^2 = vw$ or $vw + v^2 + w^2 = 0$. If $u^2 = vw$, then

$$\frac{u^2-o}{\overline{u^2}-\overline{o}} = u^4 = v^2w^2 = \frac{o-(-vw)}{\overline{o}-\overline{(-vw)}},$$

which means that A, O and O_1 are collinear, thus $\overline{AB} \neq \overline{AC}$, which is contradictory. So, the points A, O and O_1 are collinear if and only if the following holds true

$$vw+v^2+w^2=0,$$

i.e.

$$(v+w)^2=vw,$$

thus $\overline{AB}\neq\overline{AC}$, which is contradictory. So, the points A, O and O_1 are collinear if and only if the following holds true

$$vw+v^2+w^2=0,$$

i.e.

$$(v+w)^2=vw,$$

therefore

$$|v+w|^2=|vw|=1 \iff |vw+w^2|=|vw|=1 \iff |w^2-(-vw)|=|o-(-vw)|,$$

i.e. if and only if the triangle with vertices $w^2, -vw, o$ is an equilateral, that is if and only if $\angle BAC=60°$. ■

73. Let $\triangle ABC$ be any triangle. Let A_1, B_1, C_1 be the midpoints of the sides *BC*, *CA*, *AB*, respectively, *P*, *Q*, *R* be the points where the incircle *k* tangents the sides *BC*, *CA*, *AB*; P_1, Q_1, R_1 be the midpoints of the arcs *QR*, *RP*, *PQ* and P_2, Q_2, R_2 be the midpoints of arcs *QPR*, *RPQ*, *PRQ*, respectively. Prove that both lines A_1P_1, B_1Q_1 and C_1R_1, and lines A_1P_2, B_1Q_2 and C_1R_2 are concurrent.

Guidelines. Let the incircle be a unit circle**.** Since Theorem 22.14 there exist complex numbers *u*, *v*, *w* so that

$$p=u^2,\ q=v^2,\ r=w^2 \text{ and } p_1=-vw,\ q_1=-wu,\ r_1=-uv.$$

The points P_2, Q_2, R_2 are symmetric with respect to the center of *k* with the points P_1, Q_1, R_1, thus

$$p_2=vw,\ q_2=wu,\ r_2=uv.$$

Further,

$$a=\frac{2v^2w^2}{v^2+w^2},\ b=\frac{2w^2u^2}{w^2+u^2},\ c=\frac{2u^2v^2}{u^2+v^2},$$

therefore,

$$a_1=\frac{w^2u^2}{w^2+u^2}+\frac{u^2v^2}{u^2+v^2},\ b=\frac{v^2w^2}{v^2+w^2}+\frac{u^2v^2}{u^2+v^2},\ c=\frac{w^2u^2}{w^2+u^2}+\frac{v^2w^2}{v^2+w^2}.$$

Use that the equations of lines A_1P_1, B_1Q_1 are

$$z-a_1=\frac{a_1-p_1}{\overline{a_1-p_1}}(\overline{z}-\overline{a_1}) \text{ and } z-b_1=\frac{b_1-q_1}{\overline{b_1-q_1}}(\overline{z}-\overline{b_1}),$$

Determine the affix *n* of the point of intersection and verify whether it satisfies the equation of the line C_1R_1.

The second part of the statement should be proved analogously. ■

74. The squares $ABB'B''$, $ACC'C''$ and *BCXY* are constructed on the outside of a triangle $\triangle ABC$. Let *P* be the center of the square *BCXY*. Prove that the lines CB'', BC'' and *AP* are concurrent.

Solution. Let the point A coincide with the origin, i.e. $a=0$. Thus

$$c''-a=e^{i\pi/2}(c-a)\text{, i.e. } c''=ic\,.$$

Similarly,

$$b''=-ib\,,\ x-c=e^{i\pi/2}(b-c)\text{, i.e. } x=(1-i)c+ib$$

And since P is the midpoint of BX we get that

$$p=\tfrac{1+i}{2}b+\tfrac{1-i}{2}c\,.$$

The equation of the lines BC'' and AP are

$$z-b=\frac{b-c''}{\overline{b}-\overline{c''}}(\overline{z}-\overline{b}) \qquad (1)$$

$$z-a=\frac{a-p}{\overline{a}-\overline{p}}(\overline{z}-\overline{a}) \qquad (2)$$

By solving the system of equations (1) and (2), we obtain the affix of Q, point of intersection between the lines BC'' and AP as following

$$q=\frac{(\overline{b}c+b\overline{c})[(1+i)b+(1-i)c]}{(b-ic)(\overline{b}+i\overline{c})}\,.$$

The equation of line $B''C$ is following

$$z-b''=\frac{b''-c}{\overline{b''}-\overline{c}}(\overline{z}-\overline{b''}) \qquad (3)$$

By solving the system of equations (2) and (3), we obtain the affix of Q', the point of intersection of the lines $B''C$ and AP as following

$$q'=\frac{(\overline{b}c+b\overline{c})[(1+i)b+(1-i)c]}{(b-ic)(\overline{b}+i\overline{c})}\,.$$

Finally, the statement of the given problem is implied by the equality $q'=q$. ■

75. Let $ABCD$ be any quadrilateral, O be the intersection of its diagonals, M be the midpoint of the side AB and N be the midpoint of the side CD. Prove that if $OM\perp CD$ and $ON\perp AB$, then $ABCD$ is a cyclic quadrilateral.

Solution. Let the intersection of the diagonals coincide with the origin, i.e. $o=0$. The points A, O and C are collinear and also the points B, O and D are collinear, and thus $a\overline{c}=c\overline{a}$ and $b\overline{d}=d\overline{b}$. Further, $m=\frac{a+b}{2}$ and $n=\frac{c+d}{2}$. Thereby, $OM\perp CD$ and $ON\perp AB$ the following holds true

$$\frac{\frac{c+d}{2}-o}{\frac{\overline{c}+\overline{d}}{2}-\overline{o}}=-\frac{a-b}{\overline{a}-\overline{b}} \text{ and } \frac{\frac{a+b}{2}-o}{\frac{\overline{a}+\overline{b}}{2}-\overline{o}}=-\frac{c-d}{\overline{c}-\overline{d}}\,,$$

i.e.

$$c=\frac{da(\overline{a}b-2\overline{b}b+a\overline{b})}{b(\overline{a}b-2\overline{a}a+a\overline{b})} \text{ and } c=\frac{da(\overline{a}b+2\overline{b}b+a\overline{b})}{b(\overline{a}b+2\overline{a}a+a\overline{b})}\,,$$

therefore

$$(\overline{a}b+a\overline{b})(a\overline{a}-b\overline{b})=0\,. \qquad (1)$$

We have to prove that the condition (1) is sufficient for the points A, B, C, D to be on a same circle, which according to Remark 25.3 means that the condition (1) is sufficient to $\frac{(c-d)(b-a)}{(b-d)(c-a)} \in \mathbf{R}$, that is

$$\frac{(c-d)(b-a)}{(\overline{c}-\overline{d})(\overline{b}-\overline{a})} = \frac{(b-d)(c-a)}{(\overline{b}-\overline{d})(\overline{c}-\overline{a})} \qquad (2)$$

The points B, O and D are collinear, and thus $\frac{b-d}{\overline{b}-\overline{d}} = \frac{b}{\overline{b}}$ and also points A, O and C are collinear and thus $\frac{a-c}{\overline{a}-\overline{c}} = \frac{a}{\overline{a}}$. If $\overline{a}b + a\overline{b} = 0$, then

$$c - d = d\frac{2ab(\overline{a}-\overline{b})}{b(\overline{a}b-2a\overline{a}+a\overline{b})},$$

and if $a\overline{a} - b\overline{b} = 0$, then

$$c - d = \frac{d(a-b)(\overline{a}b+a\overline{b})}{b(\overline{a}b-2a\overline{a}+a\overline{b})}.$$

By direct checking we assure that in both cases the condition (2) is satisfied. The latter means that the points A, B, C, D are placed on the same circle. Details are left as your exercise. ■

76. Let F be the point on the base AB of a trapezoid $ABCD$, such that $\overline{DF} = \overline{CF}$, $E = AC \cap BD$ and O_1 and O_2 be the circumcenters of the triangles ADF and FBC, respectively. Prove that $FE \perp O_1O_2$.

Solution. Let the origin coincide with the point F, i.e. $f = o$ and let $d = \overline{c}$. $CD \parallel AF$ implies that

$$\frac{a-f}{\overline{a}-\overline{f}} = \frac{c-d}{\overline{c}-\overline{d}} = -1,$$

i.e. $\overline{a} = -a$ and similarly, $\overline{b} = -b$. Further, the above statated and the Example 3.3 imply that

$$o_1 = \frac{ad(\overline{d}-\overline{a})}{\overline{a}d-a\overline{d}} = \frac{\overline{c}(a+c)}{c+\overline{c}} \text{ and } o_2 = \frac{cb(\overline{c}-\overline{b})}{b\overline{c}-\overline{b}c} = \frac{c(\overline{c}+b)}{c+\overline{c}}.$$

The equations of the lines AC and BD are

$$z - a = \frac{c-a}{\overline{c}-\overline{a}}(\overline{z} - \overline{a}) \text{ and } z - b = \frac{d-b}{\overline{d}-\overline{b}}(\overline{z} - \overline{b}).$$

The solution of the system consists of the last two equations is the affix of the point E, thus

$$e = \frac{a\overline{c}-bc}{a+\overline{c}-b-c}.$$

Finally,

$$o_1 - o_2 = \frac{\overline{c}a-cb}{c+\overline{c}}$$

By direct checking we get that the following holds true

$$\frac{o_1-o_2}{\overline{o}_1-\overline{o}_2} = -\frac{e-f}{\overline{e}-\overline{f}},$$

therefore $FE \perp O_1O_2$. ■

77. Let the diagonals of a convex quadrilateral $ABCD$ meet at O and let T_1 and T_2 be the centroids of the triangles AOD and BOC, and H_1 and H_2 be the orthocenters of the triangles AOD and BOC, respectively. Prove that $T_1T_2 \perp H_1H_2$.

Solution. Let the point O and the origin coincide. Then the affixes of the orthocenters H_1 and H_2 and centroids T_1 and T_2 are

$$h_1 = \frac{(a-b)(\overline{a}b+a\overline{b})}{a\overline{b}-\overline{a}b},\ h_2 = \frac{(c-d)(\overline{c}d+c\overline{d})}{c\overline{d}-\overline{c}d},\ t_1 = \frac{a+d}{3} \text{ and } t_2 = \frac{b+c}{3}.$$

The points A, C and O are collinear, and also B, D and O are collinear, therefore $\overline{c} = \frac{c\overline{a}}{a}$ and $\overline{d} = \frac{d\overline{b}}{b}$, that is

$$h_2 = \frac{(c-d)(\overline{b}a+b\overline{a})}{a\overline{b}-\overline{a}b}.$$

Further,

$$h_1 - h_2 = \frac{(a+d-b-c)(\overline{a}b+a\overline{b})}{a\overline{b}-\overline{a}b},\ t_1 - t_2 = \frac{a+d-b-c}{3}$$

By direct checking we obtain that

$$\frac{t_1-t_2}{\overline{t_1}-\overline{t_2}} = -\frac{h_1-h_2}{\overline{h_1}-\overline{h_2}},$$

therefore $T_1T_2 \perp H_1H_2$. The details are left as your exercise. ■

78. Let the tangents of a circle Γ at points A and B meet at C. The circle Γ_1 is such a circle that passes through C, tangents the line AB at B and meets Γ at M. Prove that the line AM bisects the line segment BC.

Solution. Let Γ be the unit circle. Then $c = \frac{2ab}{a+b}$. Let O_1 be the center of the circle Γ_1. Then $O_1B \perp AB$, thus

$$\frac{o_1-b}{\overline{o_1}-\overline{b}} = -\frac{a-b}{\overline{a}-\overline{b}} = ab,$$

So, we obtain that $\overline{o_1} = \frac{o_1+a-b}{ab}$. Further, $|o_1 - b| = |o_1 - c|$, and by squaring we get

$$(o_1 - b)(\overline{o_1} - \overline{b}) = (o_1 - c)(\overline{o_1} - \overline{c}), \text{ i.e. } \overline{o_1} = \frac{o_1}{b^2} - \frac{a-b}{b(a+b)}.$$

Thus,

$$\frac{o_1+a-b}{ab} = \frac{o_1}{b^2} - \frac{a-b}{b(a+b)}, \text{ i.e. } o_1 = \frac{ab}{a+b} + b.$$

The point M is placed on the unit circle Γ, therefore $\overline{m} = \frac{1}{m}$ and since it is placed on the circle whose radius is O_1B and is centered at O_1 we get that

$$|o_1 - b| = |o_1 - m|, \text{ i.e. } \overline{o_1}m^2 - \left(\frac{o_1}{b} + \overline{o_1}b\right)m + o_1 = 0$$

holds. The solutions of the last quadratic equation are m and b, and thereby the Viet formulas it is true that

$$b + m = \frac{o_1}{\overline{o_1}b} + b, \text{ i.e. } m = b\frac{2a+b}{a+2b}.$$

Further, the affix of the midpoint of the line segment BC is $\frac{b+c}{2}$. Finally, to prove that the line AM bisects the line segment BC it is sufficient to prove that

$$\frac{a-\frac{b+c}{2}}{\overline{a}-\frac{\overline{b+c}}{2}}=\frac{a-m}{\overline{a}-\overline{m}}=-am\,,$$

The validity of the latter could be easily checked. Details are left as your exercise. ■

79. Let Γ be a given circle, and AB be its diameter. Let P be an arbitrary point on Γ distinct of A and B. The projection of the point P to AB is a point Q. A circle centered at P and radius PQ meets Γ at points C and D. The lines CD and PQ intersect at a point E. Let F be the midpoint of AQ, and G be the foot of the perpendicular at F to CD. Prove that the points A, G and P are collinear and furthermore

$$\overline{EP}=\overline{EQ}=\overline{EG}\,.$$

Solution. Let Γ be the unit circle and let $b=1$. Then $a=-1$ and thereby $P\in\Gamma$ we get that $\overline{p}=\frac{1}{p}$. Further, the affix of the point Q is $q=\frac{1}{2}\left(p+\frac{1}{p}\right)$, and the affix of F is

$$f=\frac{\frac{1}{2}\left(p+\frac{1}{p}\right)-1}{2}=\frac{(p-1)^2}{4p}\,. \tag{1}$$

The point C is placed on the circle centered at P and radius PQ, thus $|p-q|=|p-c|$, The latter implies that

$$(p-q)(\overline{p}-\overline{q})=(p-c)(\overline{p}-\overline{c})\,. \tag{2}$$

But, $C\in\Gamma$, and thus $\overline{c}=\frac{1}{c}$ and thereby

$$p-q=\frac{1}{2}\left(p-\frac{1}{p}\right)$$

by substituting in (2) we obtain that

$$4pc^2-(p^4+6p^2+1)c+4p^3=0\,. \tag{3}$$

The equation (3) is a quadratic equation with variable c and since the point D satisfies the same conditions as the conditions applied when determined the affix of C, we get that d is the second solution of (3). Now, by applying the Viet roles we get the following

$$c+d=\frac{p^4+6p^2+1}{4p^3},\ \ cd=p^2\,.$$

The point G is placed on the chord CD, therefore C, D, G are collinear, thus we get that $\overline{g}=\frac{c+d-g}{cd}$, and thereby $FG\perp CD$ we have that

$$\frac{g-f}{\overline{g}-\overline{f}}=-\frac{c-d}{\overline{c}-\overline{d}}=cd=p^2\,.$$

By solving the system consisting of the last two equations where f is the expression given in (1), we get that

$$g=\frac{p^3+3p^2-p+1}{4p}\,.$$

To prove that the points A, G and P are collinear it is sufficient to prove that

$$\frac{g-a}{\overline{g}-\overline{a}}=-\frac{a-p}{\overline{a}-\overline{p}}=p\,.$$

The last can be easily checked if we consider that

$$g-a=\frac{p^3+3p^2+3p+1}{4p} \text{ and } \overline{g}-\overline{a}=\frac{p^3+3p^2+3p+1}{4p^2}\,.$$

The point E is placed on the chord CD, therefore C, D and E are collinear, i.e. $\overline{e}=\frac{c+d-e}{cd}$. So, $PE\perp AB$ implies that

$$\frac{e-p}{\overline{e}-\overline{p}}=-\frac{a-b}{\overline{a}-\overline{b}}=-1\text{, i.e. } \overline{e}=p+\frac{1}{p}-e\,.$$

That is,

$$p+\frac{1}{p}-e=\overline{e}=\frac{c+d-e}{cd}\,,$$

therefore

$$e=\frac{3p^2+1}{4p}\,.$$

Now,

$$e-p=\frac{3p^2+1}{4p}-p=\frac{1-p^2}{4p}\,,\ e-q=\frac{p^2-1}{4p} \text{ and } e-g=\frac{p-p^3}{4p}$$

Further, since $|p|=1$, we get that

$$|e-p|=|e-q|=|e-g|\text{, i.e. } \overline{EP}=\overline{EQ}=\overline{EG}\,.\ \blacksquare$$

80. Let H be the orthocenter of a $\triangle ABC$. The tangents at A to the circle whose diameter is BC touch the circle at P and Q. Prove that the points P, Q and H are collinear.

Solution. Let the circle over the diameter BC be a unit circle and let $b=-1$. Then, $c=1$ and the origin is the midpoint of the line segment BC. The point P lies on the unit circle, thus $\overline{p}=\frac{1}{p}$ and since $PA\perp PO$ we get that

$$\frac{a-p}{\overline{a}-\overline{p}}=-\frac{p-o}{\overline{p}-\overline{o}}=-p^2\,,$$

The latter implies

$$\overline{a}p^2-2p+a=0\,. \qquad (1)$$

The equation (1) is a quadratic equation with a variable p and thereby the point Q satisfies the same conditions as the ones used when determining the point P, we get that q is the second solution of (1). Now, by applying the Viet formulae we get the following

$$p+q=\frac{2}{\overline{a}},\ \ pq=\frac{a}{\overline{a}}\,.$$

Let H' be the intersection of the line through A perpendicular to the side BC and the line PQ. the points P, Q and H' are collinear, so

$$\overline{h'}=\frac{p+q-h'}{pq}\text{, i.e. } \overline{h'}=\frac{2-\overline{a}h'}{a}\,.$$

But, $AH'\perp BC$, therefore

$$\frac{a-h'}{\overline{a}-\overline{h'}}=-\frac{b-c}{\overline{b}-\overline{c}}=-1\,,$$

The latter means that $\overline{h'}=a+\overline{a}-h'$. Thus,

$$a+\overline{a}-h'=\overline{h'}=\frac{2-\overline{a}h'}{a},$$

So, we find

$$h'=\frac{a\overline{a}+a^2-2}{a-\overline{a}}.$$

We will prove that $h=h'$, therefore the statement in the given problem shall be implied. In order to do that, it is sufficient to prove that $CH'\perp AB$, that is, to prove that

$$\frac{c-h'}{\overline{c}-\overline{h'}}=-\frac{a-b}{\overline{a}-\overline{b}},$$

holds true (why?). We can be assured in validity of the last equality by direct checking if we use that

$$h'-c=h'-1=\frac{a\overline{a}+a^2-2-a+\overline{a}}{a-\overline{a}}=\frac{(a+1)(a+\overline{a}-2)}{a-\overline{a}} \text{ and } a-b=a+1.$$

Details are left as an exercise. ■

81. Let P be a point on the extension of the diagonal AC of a rectangle $ABCD$ through the point C, so that $\angle BPD=\angle CBP$. Determine the ratio $\overline{PB}:\overline{PC}$.

Solution. Let the intersection of diagonals O of the rectangle be the origin and let the line AB be parallel to the real axis. Then $a+c=0$, $b+d=0$, $c=\overline{b}$ and $d=\overline{a}$. Further, the points P, A, O are collinear, and therefore

$$\frac{p}{\overline{p}}=\frac{a}{\overline{a}}, \text{ i.e. } \overline{p}=-\frac{b}{a}p.$$

Let $\angle DPB=\angle PBC=\varphi$. Then,

$$\frac{d-p}{\overline{d}-\overline{p}}=e^{2i\varphi}\frac{b-p}{\overline{b}-\overline{p}} \text{ and } \frac{p-b}{\overline{p}-\overline{b}}=e^{2i\varphi}\frac{c-b}{\overline{c}-\overline{b}}.$$

If we multiply the last two equalities, and express the obtained equality in terms of a and b, we get

$$\frac{p+b}{bp+a^2}=\frac{a(p+b)^2}{(bp-a^2)^2}.$$

Further, if we express the above equality as a polynomial of p we get the following

$$(b-a)[bp^3+(a^2+3ab+b^2)p^2-a(a^2+3ab+b^2)p-a^3b]=0,$$

i.e.

$$bp^3+(a^2+3ab+b^2)p^2-a(a^2+3ab+b^2)p-a^3b=0. \quad (1)$$

But the point A satisfies $\angle DAB=\angle ABC=\frac{\pi}{2}$, thus one of the points which satisfies the given condition in the given problem is the point A. So, a is one of the roots of the polynomial (1). The latter implies that the polynomial can be divided by $p-a$ and p is a root of the such obtained quotient (why?), i.e.

$$bp^2+(a^2+4ab+b^2)p+a^2b=0. \quad (2)$$

holds true. Finally, by applying the condition (2) we get

$$\frac{\overline{PB}^2}{\overline{PC}^2}=\frac{|p-b|^2}{|p-c|^2}=\frac{(p-b)(\overline{p}-\overline{b})}{(p-c)(\overline{p}-\overline{c})}=\frac{(p-b)\left(-\frac{b}{a}p+a\right)}{(p+a)\left(-\frac{b}{a}p-b\right)}$$

$$=\frac{bp^2-(a^2+b^2)p+a^2b}{bp^2+2abp+a^2b}=\frac{-2(a^2+4ab+b^2)p}{-(a^2+2ab+b^2)p}=2$$

The latter implies that $\overline{PB}:\overline{PC}=\sqrt{2}$. ■

82. In a convex quadrilateral $ABCD$ the diagonal BD is not a bisector neither of $\angle ABC$ nor of $\angle CDA$. A point P placed into the $ABCD$ is such that $\angle PBC=\angle DBA$ and $\angle PDC=\angle BDA$. Prove that the quadrilateral $ABCD$ is cyclic if and only if $\overline{AP}=\overline{CP}$.

Solution. Let the quadrilateral $ABCD$ be cyclic and let the circumcircle be the unit circle. If $\angle PBC=\angle ABD=\varphi$ and $\angle PDC=\angle BDA=\theta$, then

$$\frac{d-b}{\overline{d}-\overline{b}}=e^{2i\varphi}\frac{a-b}{\overline{a}-\overline{b}},\ \frac{c-b}{\overline{c}-\overline{b}}=e^{2i\varphi}\frac{p-b}{\overline{p}-\overline{b}},\ \frac{c-d}{\overline{c}-\overline{d}}=e^{2i\theta}\frac{p-d}{\overline{p}-\overline{d}},\ \frac{b-d}{\overline{b}-\overline{d}}=e^{2i\theta}\frac{a-d}{\overline{a}-\overline{d}}$$

and thereby $\overline{a}=\frac{1}{a}, \overline{b}=\frac{1}{b}, \overline{c}=\frac{1}{c}, \overline{d}=\frac{1}{d}$ by using the first equality, we obtain that $e^{2i\varphi}=\frac{a}{d}$, and since the fourth one, $e^{2i\theta}=\frac{b}{a}$. By substituting at the second and the third equality we get that

$$\frac{a}{d}\frac{p-b}{\overline{p}-\overline{b}}=-bc \text{ and } \frac{b}{a}\frac{p-d}{\overline{p}-\overline{d}}=-cd ,$$

since which

$$p=\frac{ac+bd}{b+d} .$$

Further,

$$a-p=\frac{ab+ad-ac-bd}{b+d},\ \overline{a}-\overline{p}=\frac{bc+cd-ac-bd}{ac(b+d)},$$

$$c-p=\frac{bc+bd-ac-bd}{b+d},\ \overline{c}-\overline{p}=\frac{ab+ad-ac-bd}{ac(b+d)},$$

thus,

$$\overline{AP}^2=|a-p|^2=(a-p)(\overline{a}-\overline{p})=\frac{ab+ad-ac-bd}{b+d}\cdot\frac{bc+cd-ac-bd}{ac(b+d)}$$

$$=\frac{bc+bd-ac-bd}{b+d}\cdot\frac{ab+ad-ac-bd}{ac(b+d)}=(c-p)(\overline{c}-\overline{p})=|c-p|^2=\overline{CP}^2,$$

i.e. $\overline{AP}=\overline{CP}$.

Let $\overline{AP}=\overline{CP}$, i.e.

$$|a-p|=|c-p| \qquad (*)$$

and let suppose that the circumcircle of the triangle ABC is the unit circle. This means that $\overline{a}=\frac{1}{a},\ \overline{b}=\frac{1}{b}, \overline{c}=\frac{1}{c}$. The condition (*), after squaring and reducing, implies that

$$a\overline{p}+\frac{p}{a}=c\overline{p}+\frac{p}{c},$$

that is

$$(a-c)\left(\overline{p}-\frac{p}{ac}\right)=0 ,$$

therefore $\overline{p} = \frac{p}{ac}$. Let D' denote the intersection of a side CD and a unit circle. Then

$$\frac{c-d}{\overline{c-d}} = \frac{c-d'}{\overline{c-d'}} = -cd',$$

thus

$$\overline{d} = \frac{c+d'-d}{cd'}.$$

Thereby the condition of the problem, we have that

$$\angle CBP = \angle DBA = \varphi \text{ and } \angle PDC = \angle ADB = \theta,$$

thus

$$\frac{a-b}{\overline{a-b}} = e^{2i\varphi}\frac{d-b}{\overline{d-b}},\ \frac{p-b}{\overline{p-b}} = e^{2i\varphi}\frac{c-b}{\overline{c-b}},\ \frac{c-d}{\overline{c-d}} = e^{2i\theta}\frac{p-d}{\overline{p-d}},\ \frac{b-d}{\overline{b-d}} = e^{2i\theta}\frac{a-d}{\overline{a-d}}. \qquad (1)$$

Thereby the first two equations in (1) we get that

$$\frac{p-b}{\overline{p-b}}\frac{d-b}{\overline{d-b}} = \frac{a-b}{\overline{a-b}}\frac{c-b}{\overline{c-b}} = ab^2c,$$

And by substitution for $\overline{d}$ and $\overline{p}$, and after reducing we obtain following

$$p = c\frac{bdd'+acd'-abd'-abc+abd-b^2d'}{cd'd-b^2d'+b^2d-b^2c}. \qquad (2)$$

Now, the third and the fourth equality in (1) and $\frac{c-d}{\overline{c-d}} = -cd'$ imply

$$-cd'\frac{a-d}{\overline{a-d}} = \frac{p-d}{\overline{p-d}}\frac{b-d}{\overline{b-d}}. \qquad (3)$$

If in the latter we substitute the above expression for p and $\overline{p}$, then after reducing we obtain a polynomial of $P(d)$, which is obviously at most quartic. By comparing the coefficients of d^4 we get that the polynomial $P(d)$ is exactly a cubic polynomial. Clearly, two of its zeros are a and b. We shall prove that its third zero is d', and therefore $d = d'$, i.e. the quadrilateral $ABCD$ is cyclic. Indeed, if $d = d'$, then

$$\frac{a-d}{\overline{a-d}} = -ad',\ \frac{b-d}{\overline{b-d}} = -bd' \text{ and } \frac{p-d}{\overline{p-d}} = \frac{p-d'}{pd'-ac}acd'.$$

Thus the equality (3) is equivalent to the $p = \frac{ac+bd'}{d'+b}$, which is obviously satisfied, and is obtained by letting $d = d'$ in (2). ■

83. Three triangles KPQ, QLP and PQM, so that $\angle QPM = \angle PQL = \alpha$, $\angle PQM = \angle QPK = \beta$ and $\angle PQK = \angle QPL = \gamma$ for $\alpha < \beta < \gamma$ and $\alpha + \beta + \gamma = 180°$, are constructed on a same side of a line segment PQ. Prove that the triangle KLM is similar to, and moreover, is the same oriented with the triangles KPQ, QLP and PQM.

Solution. Let $p = 0$ and $q = 1$. Since $\angle MPQ = \alpha$

$$\frac{m-p}{\overline{m-p}} = e^{2i\alpha}\frac{q-p}{\overline{q-p}},$$

holds thus $\frac{m}{\overline{m}} = e^{2i\alpha}$. Further, $\angle PQM = \beta$ implies

$$e^{2i\beta}\frac{m-q}{\overline{m-q}} = \frac{p-q}{\overline{p-q}},$$

thus $e^{2i\beta}\frac{m-1}{\overline{m}-1}=1$. If we remember that $e^{2i(\alpha+\beta+\gamma)}=1$, then by solving the system

$$\begin{cases} e^{2i\alpha}=\frac{m}{\overline{m}} \\ e^{2i\beta}\frac{m-1}{\overline{m}-1}=1 \end{cases}$$

we get

$$m=\frac{e^{2i(\alpha+\gamma)}-1}{e^{2i\gamma}-1}.$$

Analogously $l=\frac{e^{2i(\beta+\gamma)}-1}{e^{2i\beta}-1}$ and $k=\frac{e^{2i(\alpha+\beta)}-1}{e^{2i\alpha}-1}$. According to the theorem 4.9 in order to prove that the triangle *KLM* is simmilar and same oriented as the triangle *KPQ*, it is sufficient to prove that

$$\frac{k-l}{l-m}=\frac{k-p}{p-q}=-k,$$

in which we can be convinced by immediate validation. Finally, since the triangles *KPQ*, *QLP* and *PQM* are similar and same oriented, we get that each of the four triangles is similar and same oriented as the other ones. ■

84. Prove that the area of the triangle whose vertices are feet of the perpendiculars at any vertex of a cyclic pentagon to its sides does not depend on the choice of the vertex of the pentagon.

Solution. Let the circumcircle of the pentagon *ABCDE* be the unit circle and let *X*, *Y*, *Z* be the feet of the perpendiculars at the vertex *A* to the sides *BC*, *CD*, *DE*, respectively. Then $x=\frac{1}{2}\left(a+b+c-\frac{bc}{a}\right)$, $y=\frac{1}{2}\left(a+c+d-\frac{cd}{a}\right)$ and $z=\frac{1}{2}\left(a+d+e-\frac{ed}{a}\right)$, thus

$$P_{\triangle XYZ}=\pm\frac{i}{4}\begin{vmatrix} x & \overline{x} & 1 \\ y & \overline{y} & 1 \\ z & \overline{z} & 1 \end{vmatrix}=\pm\frac{i}{16}\begin{vmatrix} a+b+c-\frac{ab}{a} & \overline{a}+\overline{b}+\overline{c}-\frac{\overline{bc}}{\overline{a}} & 1 \\ a+c+d-\frac{cd}{a} & \overline{a}+\overline{c}+\overline{d}-\frac{\overline{cd}}{\overline{a}} & 1 \\ a+e+d-\frac{ed}{a} & \overline{a}+\overline{e}+\overline{d}-\frac{\overline{ed}}{\overline{a}} & 1 \end{vmatrix}$$

$$=\pm\frac{i}{16}\begin{vmatrix} a+b+c-\frac{ab}{a} & \overline{a}+\overline{b}+\overline{c}-\frac{\overline{bc}}{\overline{a}} & 1 \\ \frac{(a-c)(d-b)}{a} & \frac{\overline{(a-c)(d-b)}}{\overline{a}} & 0 \\ \frac{(e-c)(a-d)}{a} & \frac{\overline{(e-c)(a-d)}}{\overline{a}} & 0 \end{vmatrix}$$

$$=\pm\frac{i}{16}\begin{vmatrix} a+b+c-\frac{ab}{a} & \overline{a}+\overline{b}+\overline{c}-\frac{\overline{bc}}{\overline{a}} & 1 \\ \frac{(a-c)(d-b)}{a} & \frac{(a-c)(d-b)}{bcd} & 0 \\ \frac{(e-c)(a-d)}{a} & \frac{(e-c)(a-d)}{ced} & 0 \end{vmatrix}$$

$$=\pm\frac{i(a-c)(d-b)(e-c)(a-d)}{16}\begin{vmatrix} a+b+c-\frac{ab}{a} & \overline{a}+\overline{b}+\overline{c}-\frac{\overline{bc}}{\overline{a}} & 1 \\ \frac{1}{a} & \frac{1}{bcd} & 0 \\ \frac{1}{a} & \frac{1}{ced} & 0 \end{vmatrix}$$

$$=\pm\frac{i(a-c)(d-b)(e-c)(a-d)}{16}\left(\frac{1}{aced}-\frac{1}{abcd}\right)$$

$$=\pm\frac{i(a-c)(c-e)(e-b)(b-d)(a-d)}{16abcde}.$$

Therefore, the area is the sixteenth of the product of the pentagon diagonals, so it does not depend on the choice of the pentagon vertex. ■

85. The points A_1, B_1, C_1 are positioned on the altitudes of the $\triangle ABC$ plotted at the vertices A, B, C, respectively, and H is the orthocenter of $\triangle ABC$. If

$$P_{\triangle ABC_1}+P_{\triangle BCA_1}+P_{\triangle CAB_1}=P_{\triangle ABC}, \tag{1}$$

prove that the quadrilateral $A_1B_1C_1H$ is cyclic.

Solution. Let the circumcircle of the $\triangle ABC$ be the unit circle. Let A' be the foot of the perpendicular at the vertex A to the side BC. Then

$$P_{\triangle BCA_1}=\frac{1}{2}\overline{BC}\cdot\overline{A_1A'}=\frac{|b-c|\cdot|a_1-a'|}{2} \text{ and } P_{\triangle ABC}=\frac{1}{2}\overline{BC}\cdot\overline{AA'}=\frac{|b-c|\cdot|a-a'|}{2},$$

thus

$$\frac{P_{\triangle BCA_1}}{P_{\triangle ABC}}=\frac{|a_1-a'|}{|a-a'|}=\frac{|a-a'|-|a-a_1|}{|a-a'|}=1-\frac{|a-a_1|}{|a-a'|}=1-\frac{a-a_1}{a-a'},$$

The latter means that the equality (1) can be transformed and rewritten as the following

$$\frac{a-a_1}{a-a'}+\frac{b-b_1}{b-b'}+\frac{c-c_1}{c-c'}=2. \tag{2}$$

Further,

$$a'=\frac{1}{2}\left(a+b+c-\frac{bc}{a}\right),$$

thus

$$a-a'=\frac{(a-b)(a-c)}{2a}.$$

Analogously,

$$b-b'=\frac{(b-c)(b-a)}{2b},\ c-c'=\frac{(c-a)(c-b)}{2c}.$$

If we substitute in (2), and after equivalent transformations, we get that the above condition is equivalent to

$$aa_1(b-c)+bb_1(c-a)+cc_1(a-b)=0. \tag{3}$$

According to the Remark 25.3, in order to prove that the quadrilateral $A_1B_1C_1H$ is cyclic, it is sufficient to prove that

$$\frac{a_1-c_1}{\overline{a_1}-\overline{c_1}}\,\frac{\overline{b_1}-\overline{h}}{b_1-h}=\frac{b_1-c_1}{\overline{b_1}-\overline{c_1}}\,\frac{\overline{a_1}-\overline{h}}{a_1-h}. \tag{4}$$

holds.

The point H is the orthocenter of the $\triangle ABC$, thus $h=a+b+c$ and since $A_1H\perp BC$ we get that $\frac{a_1-h}{\overline{a_1}-\overline{h}}=-\frac{b-c}{\overline{b}-\overline{c}}=bc$ and similary $\frac{b_1-h}{\overline{b_1}-\overline{h}}=ac$. Further, thereby $A_1A\perp BC$, we get $\frac{a_1-a}{\overline{a_1}-\overline{a}}=bc$, thus $\overline{a_1}=\frac{bc+aa_1-a^2}{bc}$ and similarly $\overline{b_1}=\frac{ac+bb_1-b^2}{ac}$ and $\overline{c_1}=\frac{ab+cc_1-c^2}{ab}$. Finally, if we apply the obtained equalities and the condition (4), we immideatly check the validity of (4), which means that the quadrilateral $A_1B_1C_1H$ is cyclic. The details are left as an exercise for the reader. ■

86. The feet of the altitudes at the vertices A, B and C of a $\triangle ABC$ are D, E and F, respectively. The line through D is parallel to EF and meets the lines AC and AB at Q and R, respectively. The line EF meets the line BC at P. Prove that the circumcircle of $\triangle PQR$ consists of the midpoint of the side BC.

Solution. Let the circumcircle of the $\triangle ABC$ be the unit circle. So,

$$d=\frac{1}{2}\left(a+b+c-\frac{bc}{a}\right),\ e=\frac{1}{2}\left(a+b+c-\frac{ac}{b}\right),\ f=\frac{1}{2}\left(a+b+c-\frac{bc}{a}\right),\ a_1=\frac{b+c}{2},$$

where A_1 is the midpoint of BC. Since Q is placed on AC we get that $\overline{q}=\frac{a+c-q}{ac}$. But, $QD\parallel EF$, thus

$$\frac{q-d}{\overline{q}-\overline{d}}=\frac{e-f}{\overline{e}-\overline{f}}=-a^2.$$

By solving the system consisting of the last two equations, we get that

$$q=\frac{a^3+a^2b+abc-b^2c}{2ab}.$$

Similarly,

$$r=\frac{a^3+a^2c+abc-bc^2}{2ac}.$$

Moreover, $P\in BC$, thus

$$\overline{p}=\frac{b+c-p}{bc}$$

And since $P\in EF$, we get that

$$\frac{p-e}{\overline{p}-\overline{e}}=\frac{e-f}{\overline{e}-\overline{f}}=-a^2.$$

By solving the system of the last two equations, we obtain

$$p=\frac{b+c}{2}+\frac{a(b-c)^2}{2(a^2-bc)}.$$

We have to prove that the points P, Q, R and A_1 are concyclic, which according to Remark 25.3 means that we have to prove the equality

$$\frac{p-a_1}{\overline{p}-\overline{a_1}}\frac{q-r}{\overline{q}-\overline{r}}=\frac{q-a_1}{\overline{q}-\overline{a_1}}\frac{p-r}{\overline{p}-\overline{r}},$$

in order to do this it is sufficient to apply that

$$q-r=\frac{a(c-b)(a^2+bc)}{2abc},\ p-r=\frac{(a^2-c^2)(b^2c+abc-a^3-a^2c)}{2ac(a^2-bc)},$$

$$q-a_1=\frac{a^3+a^2b-b^2c-ab^2}{2ab} \text{ and } p-a_1=\frac{a(b-c)^2}{2(a^2-bc)}.$$
The details are left to the reader as an exercise. ■

87. Given two circles Γ_1 and Γ_2 on the plane. Let *A* be their common point. On the circles Γ_1 and Γ_2, with constant velocities the points M_1 and M_2 move, respectively. They pass through *A* at the same moment of time. Prove that it exists a fixed point *P* which at every moment in time is on a same distance of the points M_1 and M_2.

Solution. Let *B* and *C* be the centers of the circles Γ_1 and Γ_2 and let *BC* be the real axis. If the points M_1 and M_2 move in the same direction, then
$$m_1-b=(a-b)e^{i\varphi} \text{ and } m_2-c=(a-c)e^{i\varphi}.$$
The existance of a point *P* with the desirable property is consecutively equivalent to the following conditions
$$|p-m_1|=|p-m_2|,\ (p-m_1)(\overline{p}-\overline{m_1})=(p-m_2)(\overline{p}-\overline{m_2}),$$
$$\overline{p}=\frac{m_1\overline{m_1}-m_2\overline{m_2}-p(\overline{m_1}-\overline{m_2})}{m_1-m_2}. \tag{1}$$
Let $e^{i\varphi}=z$. If we apply that $\overline{b}=b$, $\overline{c}=c$ and $\overline{z}=\frac{1}{z}$, we get that the condition (1) is equivalent to
$$(b+c-a-\overline{p})z^2-[2(b+c)-a-\overline{a}-p-\overline{p}]z+b+c-\overline{a}-p=0, \tag{2}$$
The latter means that the right side of the polynomial, in (2), with a variable z must be identically equal to null. Therefore all of its coefficients must be nulls. Since the free term, we find that $p=b+c-\overline{a}$ and clearly the coefficients of the linear and the quadratic terms are null.(Check it!)

The completely identically procedure is applied in case when the points M_1 and M_2 move in opposite directions. The details are left for reader as an exercise. ■

88. Given a square *ABCD* and a circle Γ with diameter *AB*. Let *P* be any point on the side *CD*, *M* and *N* be the points where the line segments *AP* and *BP* meet Γ which differs from *A* and *B*, and *Q* be the intersection of the lines *DM* and *CN*. Prove that $Q\in\Gamma$ and further that $\overline{AQ}:\overline{BQ}=\overline{DP}:\overline{CP}$.

Solution. Let Γ be the unit circle and let $a=-1$. Then $b=1$, $c=1+2i$ and $d=-1+2i$. Further, the points *A*, *P*, *M* are collinear, therefore
$$\frac{a-p}{\overline{a}-\overline{p}}=\frac{a-m}{\overline{a}-\overline{m}}=-am=m,$$
the latter implies that
$$\overline{p}=\frac{p+1-m}{m}.$$
But the points *C*, *D*, *P* are collinear, therefore
$$\frac{c-p}{\overline{c}-\overline{p}}=\frac{c-d}{\overline{c}-\overline{d}}=1,$$

therefore $\overline{p}=p-4i$. That is,

$$\frac{p+1-m}{m}=\overline{p}=p-4i\text{, i.e. } p=\frac{4im}{m-1}-1.$$

Similarly, the points *B*, *N*, *P* are collinear, and therefore

$$\frac{c-p}{\overline{c}-\overline{p}}=\frac{c-d}{\overline{c}-\overline{d}}=1,$$

that is

$$n=-\frac{b-p}{\overline{b}-\overline{p}}=\frac{m(1-2i)-1}{1+2i-m}.$$

Let $Q'=\Gamma\cap DM$. So,

$$q'\overline{q'}=1 \text{ and } \frac{d-m}{\overline{d}-\overline{m}}=\frac{q'-m}{\overline{q'}-\overline{m}}=-q'm,$$

therefore

$$q'=-\frac{m+1-2i}{m(1+2i)+1}.$$

Further,

$$\frac{q'-c}{\overline{q'}-\overline{c}}=q'\frac{q'-c}{1-q'c}=\frac{m+1-2i}{m(1+2i)+1}\cdot\frac{m(1-2i)-1}{1+2i-m}=-nq'=\frac{q'-n}{\overline{q'}-\overline{n}},$$

The latter means that the points Q', C, N are collinear, which implies that $Q'=CN\cap DM=Q$.

The equality $\overline{AQ}:\overline{BQ}=\overline{DP}:\overline{CP}$ is equivalent to the equality

$$|q-a|\cdot|p-c|=|d-p|\cdot|b-q|,$$

Its validity can be proven by immediate checking and by applying the following

$$|q-a|=\left|\frac{m+1-2i}{m(1+2i)+1}+1\right|=2\left|\frac{m+1}{m(1+2i)+1}\right|,\ |p-c|=\left|\frac{4im}{m-1}-1-1-2i\right|=2\left|\frac{m(i-1)+1+i}{m-1}\right|,$$

$$|d-p|=\left|-1+2i-\frac{4im}{m-1}+1\right|=2\left|\frac{m+1}{m-1}\right|,\ |b-q|=\left|1+\frac{m+1-2i}{m(1+2i)+1}\right|=2\left|\frac{m(1+i)+1-i}{m(1+2i)+1}\right|$$

and apply that

$$i[m(1+i)+1-i]=m(i-1)+i+1.\ \blacksquare$$

89. Given a $\triangle ABC$ and a circle such that it passes through *B* and *C* and remeets the sides *AB* and *AC* at the points C' and B' respectively. Prove that the lines BB', CC' and HH' are concurrent (*H* and H' are the orthcenters of the triangles *ABC* and $A'B'C'$, respectively).

Solution. Let the circumcircle of the quadrilateral $BCB'C'$ be the unit circle. The intersection of lines BB' and CC' is a point *X* with affix

$$x=\frac{bb'(c+c')-cc'(b+b')}{bb'-cc'}.$$

Further, since $BH\perp CB'$ and $CH\perp BC'$ we get

$$\frac{b-h}{\overline{b}-\overline{h}}=-\frac{b'-c}{\overline{b'}-\overline{c}}=-bc' \text{ and } \frac{c-h}{\overline{c}-\overline{h}}=-\frac{b-c'}{\overline{b}-\overline{c'}}=bc',$$

therefore

$$\overline{h}=\frac{bh-b^2+cb'}{bb'c} \text{ and } \overline{h}=\frac{ch-c^2+bc'}{bc'c},$$

So,

$$\frac{bh-b^2+cb'}{bb'c}=\overline{h}=\frac{ch-c^2+bc'}{bc'c}\text{, i.e. } h=\frac{b'c'(b-c)+b^2c'-b'c^2}{bc'-cb'}.$$

Analogously,

$$h'=\frac{bc(b'-c')+b'^2c-bc'^2}{b'c-c'b}.$$

Finally, in the order to prove the statement, it is sufficient to prove that the points H, H', X are collinear, i.e. to prove that

$$\frac{h-h'}{\overline{h}-\overline{h'}}=\frac{h-x}{\overline{h}-\overline{x}},$$

holds true. We can be convinced in the validity by immediate check if we use that

$$h-h'=\frac{(b+b'-c-c')(bc'+cb')}{bc'-cb'} \text{ and } h-x=\frac{b'c'(b^2-c^2)(b'+b-c'-c)}{(bc'-cb')(bb'-cc')}.$$

The details are left to the reader as an exercise. ■

90. Let $ABCDEF$ be a convex hexagon such that

$$\angle B+\angle D+\angle F=360^\circ \text{ and } \overline{AB}\cdot\overline{CD}\cdot\overline{EF}=\overline{BC}\cdot\overline{DE}\cdot\overline{FA}.$$

Then

$$\overline{BC}\cdot\overline{AE}\cdot\overline{FD}=\overline{CA}\cdot\overline{EF}\cdot\overline{DB}.$$

Prove it!

Solution. Let $\angle A=\alpha,\ \angle B=\beta,\ \angle C=\gamma,\ \angle D=\delta,\ \angle E=\varepsilon,\ \angle F=\varphi$. So,

$$\frac{c-b}{|c-b|}=e^{i\beta}\frac{a-b}{|a-b|},\ \frac{e-d}{|e-d|}=e^{i\delta}\frac{c-d}{|c-d|},\ \frac{a-f}{|a-f|}=e^{i\varphi}\frac{e-f}{|e-f|}.$$

If we multiply the last three equalities and consider that

$$\beta+\delta+\varphi=360^\circ \text{ and } |a-b|\cdot|c-d|\cdot|e-f|=|b-c|\cdot|d-e|\cdot|f-a|$$

we get

$$(c-b)(e-d)(a-f)=(a-b)(c-d)(e-f).$$

So, it is easy to conclude that

$$(b-c)(a-e)(f-d)=(c-a)(e-f)(d-b),$$

If we take modulus in the last equality we obtain the required equality. ■

91. Given a triangle $\triangle ABC$ and points X and Y on the sides BC and CA, respectively. Let $R=AX\cap BY$ and $\frac{\overrightarrow{AY}}{\overrightarrow{YP}}=p,\ \frac{\overrightarrow{AR}}{\overrightarrow{RX}}=q$, for $0<p<q$. Determine the ratio $\frac{\overrightarrow{BX}}{\overrightarrow{XC}}$.

Solution. Let consider the $\triangle AXC$. The points B, R and Y are Menelaus' points of the sides CX, AX and AC, respectively. The are collinear thereby the condition $R=AX\cap BY$ (figure 26). According to the Menelaus' theorem it is true that

$$\frac{\overrightarrow{AR}}{\overrightarrow{RX}}\cdot\frac{\overrightarrow{XB}}{\overrightarrow{BC}}\cdot\frac{\overrightarrow{CY}}{\overrightarrow{YA}}=-1.$$

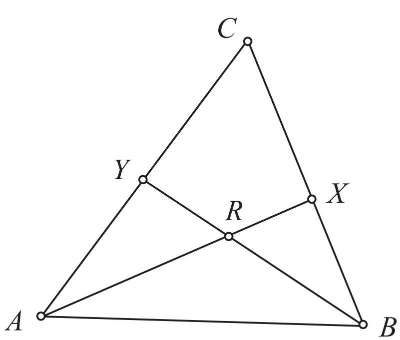

Figure 26

So,

$$\frac{\overrightarrow{BC}}{\overrightarrow{XB}}=\frac{\overrightarrow{AR}}{\overrightarrow{RX}}\cdot\frac{\overrightarrow{CY}}{\overrightarrow{YA}}=-\frac{q}{p}$$

and thereby $\overrightarrow{BC}=\overrightarrow{BX}+\overrightarrow{XC}$ and $\overrightarrow{XB}=-\overrightarrow{BX}$ by substitution in the last equality we get

$$\frac{\overrightarrow{BX}+\overrightarrow{XC}}{\overrightarrow{BX}}=\frac{q}{p}$$

that is

$$\frac{\overrightarrow{BX}}{\overrightarrow{XC}}=\frac{p}{q-p}\,.\ ■$$

92. Given a right angled triangle $\triangle ABC$ whose right angle is at B and sides $\overline{AB}=4$, $\overline{BC}=3$. A point E is the midpoint of the side AB, and the point D is on placed the side AC and moreover $\overline{DA}=1$. Let $F=DE\cap BC$. Determine the length of the line segment BF.

Solution. Let's consider the triangle $\triangle ABC$ (figure 27). The points D, E and F are Menelaus' points of the sides CA, AB and BC, respectively, so they are collinear. The Menelaus' theorem implies

$$\frac{\overline{AE}}{\overline{EB}}\cdot\frac{\overline{FB}}{\overline{FC}}\cdot\frac{\overline{CD}}{\overline{DA}}=1\,. \qquad (1)$$

Since the condition of the given problem,

$$\overline{FC}=\overline{FB}+\overline{CB}=\overline{FB}+3\,,$$
$$\overline{DA}=1 \text{ and } \overline{AE}=\overline{EB}=2\,.$$

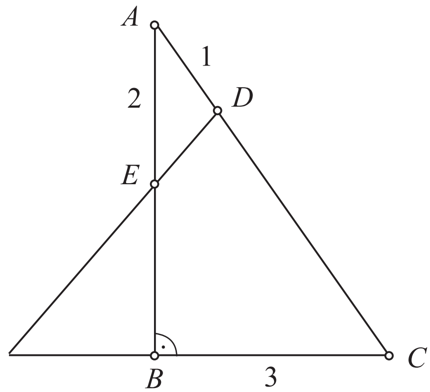

Figure 27

Moreover the Pithagora's theorem implies $\overline{CA}=\sqrt{\overline{BC}^2+\overline{AB}^2}=5$. Due to this, $\overline{CD}=\overline{CA}-\overline{DA}=4$ and if we substitute in (2) and after reducing, we get that $\overline{FB}=1$. ■

93. Let $A_0A_1A_2A_3A_4A_5A_6$ be a regular heptagon. Prove that

$$\frac{1}{\overline{A_0A_1}}=\frac{1}{\overline{A_0A_2}}+\frac{1}{\overline{A_0A_3}}\,. \qquad (1)$$

Solution. Without loss of generality, we consider the case the regular heptagon is inscribed into the unit circle and (1,0) is the affix of the vertex A_0. So, the affixes of the vertices A_k, $k=0,1,2,3,4,5,6$ are $a_k=w^k$, $k=0,1,2,3,4,5,6$, respectively, for $w=e^{i\frac{2\pi}{7}}$. Further, since the properties of the regular heptagon, we get that if the point A_1 is rotated at A_0 around $\frac{2\pi}{7}$ and the point A_2 is rotated at A_0 around $\frac{2\pi}{14}$ we get points which are collinear with the points A_0 and A_3. Let $\varepsilon=e^{i\frac{2\pi}{14}}$, $w=\varepsilon^2$, thus

$$a'_1=1+(a_1-1)w \text{ and } a'_2=1+(a_1-1)\varepsilon\,.$$

To prove the equality (1) it is sufficient to prove that

$$\frac{1}{a'_1-1}=\frac{1}{a'_2-1}+\frac{1}{a_3-1},$$

(why?). The last equality is equivalent to the equality

$$\frac{1}{\varepsilon^2(\varepsilon^2-1)}=\frac{1}{\varepsilon(\varepsilon^4-1)}+\frac{1}{\varepsilon^6-1},$$

Which after reducing, can be rewritten as below

$$\varepsilon^6+\varepsilon^4+\varepsilon^2+1=\varepsilon^5+\varepsilon^3+\varepsilon.$$

But,

$$\varepsilon^5=-\varepsilon^{12},\ \varepsilon^3=-\varepsilon^{10},\ \varepsilon=-\varepsilon^8.$$

Therefore the last equality is equivalent to the equality

$$\varepsilon^{12}+\varepsilon^{10}+\varepsilon^8+\varepsilon^6+\varepsilon^4+\varepsilon^2+1=0,$$

i.e. to the equality

$$w^6+w^5+w^4+w^3+w^2+w+1=0,$$

which is obviously true thereby $w^7=1$. ■

94. Let $A_0A_1...A_{13}A_{14}$ be a regular 15-gon. Prove that

$$\frac{1}{A_0A_1}=\frac{1}{A_0A_2}+\frac{1}{A_0A_4}+\frac{1}{A_0A_7}, \tag{1}$$

holds.

Solution. Without loss of generality we consider $a_k=w^k$, $k=0,1,2,...,14$, for $w=e^{i\frac{2\pi}{15}}$. Further, by rotation of the points A_1, A_2, A_4 at A_0 around $\frac{6\pi}{15}, \frac{5\pi}{15}, \frac{3\pi}{15}$, respectively, we get points with affixes a'_1, a'_2, a'_4 which are collinear with the points A_0 and A_7. Therefore, to prove the equality (1) it is sufficient to prove that

$$\frac{1}{a'_1-1}=\frac{1}{a'_2-1}+\frac{1}{a'_4-1}+\frac{1}{a_7-1}. \tag{2}$$

holds. We set that

$$\varepsilon=e^{i\frac{\pi}{15}},\ w=\varepsilon^2,\ \varepsilon^{30}=1$$

and obtain that

$$a'_1=1+(a_1-1)\varepsilon^6,\ a'_2=1+(a_2-1)\varepsilon^5 \text{ and } a'_4=1+(a_4-1)\varepsilon^3,$$

The latter means that the equality (2) is equivalent to the following equality

$$\frac{1}{\varepsilon^6(\varepsilon^2-1)}=\frac{1}{\varepsilon^5(\varepsilon^4-1)}+\frac{1}{\varepsilon^3(\varepsilon^8-1)}-\frac{\varepsilon^{16}}{\varepsilon^{16}-1}.$$

If the last is multiplied by $\varepsilon^2-1\neq 0$, and after reducing we obtain the following equality

$$\varepsilon^{14}+\varepsilon^{12}+\varepsilon^{10}+\varepsilon^8+\varepsilon^6+\varepsilon^4+\varepsilon^2+1=\varepsilon(\varepsilon^{12}+\varepsilon^8+\varepsilon^4+1)+\varepsilon^3(\varepsilon^8+1)-\varepsilon^{22}. \tag{3}$$

But, $\varepsilon^{15}=e^{i\pi}=-1=-\varepsilon^{30}$, therefore $\varepsilon^{15-k}=-\varepsilon^{30-k}$, which implies

$$\varepsilon^{13}=-\varepsilon^{28},\ \varepsilon^9=-\varepsilon^{24},\ \varepsilon^5=-\varepsilon^{20},\ \varepsilon=-\varepsilon^{16},\ \varepsilon^{11}=-\varepsilon^{26},\ \varepsilon^3=-\varepsilon^{18},$$

so, the equality (3) is equivalent to the equality

$$\varepsilon^{28}+\varepsilon^{26}+\varepsilon^{24}+\varepsilon^{22}+\varepsilon^{20}+\varepsilon^{18}+\varepsilon^{16}+\varepsilon^{14}+\varepsilon^{12}+\varepsilon^{10}+\varepsilon^{8}+\varepsilon^{6}+\varepsilon^{4}+\varepsilon^{2}+1=0,$$

which is obviously true, thereby the left side of the last equality is equal to $\frac{\varepsilon^{30}-1}{\varepsilon^2-1}=0$. ■

95. Given a cyclic quadrilateral *ABCD*. The points A', B', C', D' are the centroids of the triangles *BCD*, *ACD*, *BAD*, *ABC*, respectively. Prove that the quadrilateral $A'B'C'D'$ is also a cyclic quadrilateral.

Solution. The quadrilateral *ABCD* is cyclic, so $\frac{c-b}{a-b}\cdot\frac{a-d}{c-d}\in\mathbf{R}^*$. Further, $a'=\frac{b+c+d}{3}$, $b'=\frac{a+c+d}{3}$, $c'=\frac{a+b+d}{3}$, $d'=\frac{a+b+c}{3}$, so

$$\frac{c'-b'}{a'-b'}\cdot\frac{a'-d'}{c'-d'}=\frac{\frac{b-c}{3}}{\frac{b-a}{3}}\cdot\frac{\frac{d-a}{3}}{\frac{d-c}{3}}=\frac{c-b}{a-b}\cdot\frac{a-d}{c-d}\in\mathbf{R}^*,$$

the latter implies that the quadrilateral $A'B'C'D'$ is cyclic. ■

96. Given a triangle *ABC* and points *P*, *N*, *M* positioned on the sides *AB*, *BC*, *CA*, respectively. Prove that the circumcircles of the triangles *APN*, *BMP*, *CNM* meet at a unique point.

Solution. Let *Q* be the other point of intersection of the circumcircles of the triangles *APN* and *BMP* (see the figure 28). The points *A*, *P*, *Q*, *N* are on a same circle, thus $\frac{q-p}{a-p}\cdot\frac{a-n}{q-n}\in\mathbf{R}^*$, similary $\frac{q-m}{b-m}\cdot\frac{b-p}{q-p}\in\mathbf{R}^*$. So, $\frac{q-m}{q-n}\cdot\frac{a-n}{a-p}\cdot\frac{b-p}{b-m}=\frac{q-p}{a-p}\cdot\frac{a-n}{q-n}\cdot\frac{q-m}{b-m}\cdot\frac{b-p}{q-p}\in\mathbf{R}^*$, which implies $\frac{q-m}{q-n}\cdot\frac{n-c}{m-c}\cdot\frac{m-c}{n-c}\cdot\frac{n-a}{p-a}\cdot\frac{p-b}{m-b}\in\mathbf{R}^*$. But,

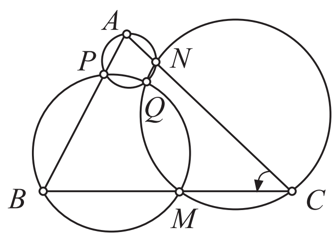

Figure 28

$$\begin{aligned}\arg\left(\frac{m-c}{n-c}\cdot\frac{n-a}{p-a}\cdot\frac{p-b}{m-b}\right)&=\arg\frac{m-c}{n-c}+\arg\frac{n-a}{p-a}+\arg\frac{p-b}{m-b}\\&=\angle MCN+\angle NAP+\angle PBM\\&=\angle ABC+\angle BCA+\angle CAB=\pi,\end{aligned}$$

thus $\frac{m-c}{n-c}\cdot\frac{n-a}{p-a}\cdot\frac{p-b}{m-b}\in\mathbf{R}^*$, which means that $\frac{q-m}{q-n}\cdot\frac{n-c}{m-c}\in\mathbf{R}^*$, i.e. the points *Q*, *M*, *N*, *C* lie on a same circle. Finally, the circumcircles of the *APN*, *BMP*, *CNM* meet at *Q*. ■

97. Four distinct lines intersect each other, so that they form four triangles. Prove that the four circumcircles of these triangles have a common point.

Solution. Since the condition of the given problem, three of the given lines are not concurrent. Let *A*, *B*, *C*, *D*, *E*, *F*, be the point of intersection of the lines, see the figure. Let the circumcircles of the triangles *ABC* and *EFC* intersect at the point *P*. We will prove that the points *E*, *P*, *A*, *D* are concyclic. It is true that,

$$\frac{p-a}{b-a}\cdot\frac{b-c}{p-c},\frac{c-f}{e-f}\cdot\frac{e-p}{c-p}\in\mathbf{R}^*. \quad (1)$$

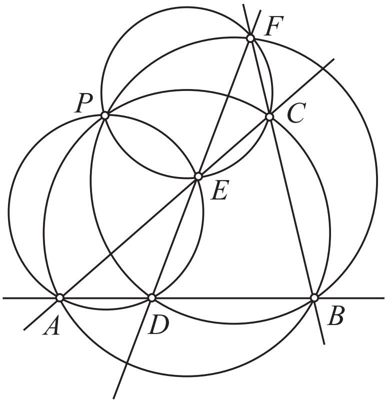

Figure 29

By dividing these two numbers, we get $\frac{p-a}{b-a}\cdot\frac{b-c}{e-p}\cdot\frac{e-f}{c-f}\in\mathbf{R}^*$. Further, the points E, F, D are collinear, and so are the points B, A, D therefore $\frac{e-f}{e-d}=t\in\mathbf{R}^*$ and $\frac{b-a}{d-a}=t'\in\mathbf{R}^*$. If the last two equalities we substitute in (1) we obtain $\frac{a-p}{e-p}\cdot\frac{e-d}{a-d}\cdot\frac{b-c}{f-c}\cdot\frac{t}{t'}\in\mathbf{R}^*$. Since the points B, C, F are collinear, $\frac{b-c}{f-c}\in\mathbf{R}^*$ holds, thus $\frac{a-p}{e-p}\cdot\frac{e-d}{a-d}\in\mathbf{R}^*$, which means that the points A, D, E, P are concyclic. Analogously, it can be proven that the points B, D, F, P are concyclic. So, the four circles consist of the point P. ■

98. In a convex quadrilateral $ABCD$ the sides AB and CD are congruent.

a) The lines AB and CD with the line which connect the midpoints of the sides AD and BC form congruent angles. Prove it!

b) The lines AB and CD with the line which connect the midpoints of the diagonals AC and BD form congruent angles. Prove it!

Solution. a) Let 0, r, c, d where $r\in\mathbf{R}^+$, $c,d\in\mathbf{C}$, be the affixes of A, B, C, D. The points N and M are the midpoints of the line segments AD and BC, respectively, thus $n=\frac{d}{2}$ and $m=\frac{r+c}{2}$. So,

$$(b-a)\bullet(m-n)=r\bullet\frac{r+c-d}{2}=\frac{r^2}{2}+\frac{r\bullet(c-d)}{2}$$

therefore $|c-d|=r$ implies

$$(c-d)\bullet(m-n)=(c-d)\bullet\frac{r+c-d}{2}=\frac{r\bullet(c-d)}{2}+\frac{|c-d|^2}{2}$$

$$=\frac{r\bullet(c-d)}{2}+\frac{r^2}{2}=(b-a)\bullet(m-n)$$

and since $|c-d|=|b-a|=r$ and the previously stated implies

$$\angle(AB,NM)=\angle(NM,DC).$$

b) Let L and K be the midpoints of AC and BD, respectively. Then $l=\frac{c}{2}$ and $k=\frac{r+d}{2}$. Thus,

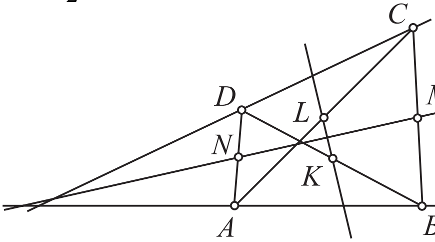

Figure 30

$$(k-l)\bullet(m-n)=\frac{r+d-c}{2}\bullet\frac{r+c-d}{2}$$

$$=\frac{r^2}{4}+\frac{(c-d)\bullet(d-c)}{4}$$

$$=\frac{r^2}{4}-\frac{|c-d|^2}{4}=0,$$

so, $KL\perp MN$. The latter and the statement a) imply

$$\angle(AB,KL)=\angle(KL,DC).\ ■$$

99. In an acute triangle *ABC*, the orthocentar *H* satisfies the following $\overline{HC} = \overline{AB}$. Determine the angle at *C*.

Solution. Let the triangle be inscribed into the unit circle. Then, $h = a + b + c$ therefore $\overline{HC} = |a+b|$ and $\overline{AB} = |a-b|$. Hence,

$$|a+b|^2 = |a-b|^2 \text{, i.e. } (a+b)\bullet(a+b) = (a-b)\bullet(a-b),$$

therefore, $a \bullet b = 0$, that is that $OA \perp OB$, i.e. $\angle AOB = \frac{\pi}{2}$. But, the triangle *ABC* is an acute triangle, and thereby the measure of inscribed angle is half of the measure of its corresponding central angle, we get that $\angle ACB = \frac{\pi}{4}$. ■

100. In a convex quadrilateral *ABCD* the points *P* and *Q* are the midpoints of the diagonals *AC* and *BD*, respectively. Prove that

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\overline{PQ}^2.$$

Solution. The points *P* and *Q* are midpoints of the diagonals *AC* and *BD*, thus $p = \frac{a+c}{2}$ and $q = \frac{b+d}{2}$. Further,

$$\overline{AB}^2 = |b-a|^2 = (b-a)\bullet(b-a) = |a|^2 - 2a\bullet b + |b|^2, \quad \overline{BC}^2 = |b|^2 - 2b\bullet c + |c|^2,$$

$$\overline{CD}^2 = |c|^2 - 2c\bullet d + |d|^2, \quad \overline{DA}^2 = |d|^2 - 2d\bullet a + |a|^2,$$

$$\overline{AC}^2 = |a|^2 - 2a\bullet c + |c|^2, \quad \overline{BD}^2 = |b|^2 - 2b\bullet d + |d|^2,$$

$$4\overline{PQ}^2 = 4(q-p)\bullet(q-p) = (b+d-a-c)\bullet(b+d-a-c)$$

$$= |a|^2 + |b|^2 + |c|^2 + |d|^2 + 2a\bullet c + 2b\bullet d - 2a\bullet b - 2b\bullet c - 2c\bullet d - 2a\bullet d,$$

thus

$$\begin{aligned}\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 &= 2|a|^2 + 2|b|^2 + 2|c|^2 + 2|d|^2 - \\ &\quad - 2a\bullet b - 2b\bullet c - 2c\bullet d - 2a\bullet d \\ &= \overline{AC}^2 + \overline{BD}^2 + 4\overline{PQ}^2,\end{aligned}$$

which was supposed to be proven. ■

101. Let *H* be the orthocenter of an acute triangle *ABC*. A circle through *H* and centered at the midpoint of the line segment *BC*, meets the line *BC* at points A_1 and A_2. Analogously, a circle through *H* and centetred at the midpoint of the line segment *CA*, meets the line *CA* at points B_1 and B_2, a circle through *H* and centetred at the midpoint of the line segment *AB* meets the line *AB* at points C_1 and C_2. Prove that the points $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a same circle.

Solution. Let the triangle be inscribed into the unit circle, A_0 be the midpoint of the line segment *BC* and let $a, b, c, a_0, a_1, a_2, h$ be the affixes of points *A*, *B*, *C*, A_0,

A_1, A_2, H, respectively. Then, $h = a+b+c$, $a\bar{a} = b\bar{b} = c\bar{c} = 1$ and $a_0 = \frac{b+c}{2}$. Since the triangles OA_1A_0 and A_2OA_0 are right angled triangles,

$$\overline{OA_1}^2 = \overline{OA_2}^2 = \overline{OA_0}^2 + \overline{A_0A_1}^2 = \overline{OA_0}^2 + \overline{A_0H}^2$$

$$= \frac{b+c}{2} \cdot \overline{\frac{b+c}{2}} + \left(a+b+c-\frac{b+c}{2}\right)\overline{\left(a+b+c-\frac{b+c}{2}\right)}$$

$$= a\bar{a} + \frac{b\bar{b}}{2} + \frac{c\bar{c}}{2} + \frac{a\bar{b}+\bar{a}b+b\bar{c}+\bar{c}b+c\bar{a}+\bar{c}a}{2}$$

$$= 2 + \frac{a\bar{b}+\bar{a}b+b\bar{c}+\bar{c}b+c\bar{a}+\bar{c}a}{2}.$$

The last expression is symmetric by a, b, c, and by cyclic substitution of the variables we obtain the following $\overline{OA_1}^2 = \overline{OA_2}^2 = \overline{OB_1}^2 = \overline{OB_2}^2 = \overline{OC_1}^2 = \overline{OC_2}^2$, which means that the points A_1, A_2, B_1, B_2, C_1, C_2 are on a same circle. ■

102. Let I be the incenter, and Γ be the circumcircle of a triangle $\triangle ABC$. Let the line AI meets Γ at points A and D. Let E be a point on the arc $\widehat{BDC}$, and F be a point on the line segment BC such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$ holds true. Let G be the midpoint of the line segment IF. Prove that the intersection of DG and EI belongs to Γ.

Solution. Let $\triangle ABC$ be inscribed into the unit circle. Since Theorem 13.3 there exist complex numbers a, b, c such that the points A, B, C have affixes a^2, b^2, c^2, respectively, and the midpoints of the arcs $\widehat{BC}, \widehat{CA}, \widehat{AB}$ which do not consist of the points A, B, C have affixes $-bc, -ca, -ab$, respectively and the incenter I has affix $s = -ab-bc-ca$. Therefore I is the orthocenter of the triangle whose vertices are the midpoints of the arcs $\widehat{BC}, \widehat{CA}, \widehat{AB}$ which do not consist of the points A, B, C.

Without loss of generality we get that the points B and C are symmetric with respect to the real axis. Let F' be the point of intersection of AF and Γ, which differs from A and let the points D, E, F, F', I have affixes d, e, f, f', s, respectively. Hence, $|a| = |b| = |c| = 1$, $c = \bar{b}$, $d = -1$, $f' = \frac{b^2c^2}{e} = \frac{1}{e}$ and $s = -1 - a(b+\bar{b})$. Thereby F is a point of intersection of the lines AF and BC we get that

$$\frac{f-b^2}{\bar{f}-\bar{b}^2} = \frac{b^2-\bar{b}^2}{\bar{b}^2-b^2} = -1 \text{ and } \frac{f-a^2}{\bar{f}-\bar{a}^2} = \frac{a^2-\frac{1}{e}}{\bar{b}^2-\frac{1}{e}} = -\frac{a^2}{e},$$

which imply the following $f + \bar{f} = b^2 + \bar{b}^2$ and $f + \frac{a^2}{e}\bar{f} = a^2 + \frac{1}{e}$, thus

$$f(a^2-e) = a^2(b^2+\bar{b}^2) - a^2e - 1.$$

Let y be the affix of the point of intersection of IE and Γ (which differs from E), and x be the affix of the point of intersection of DG and Γ (which differs from D). Then $\bar{x} = \frac{1}{x}$, $\bar{y} = \frac{1}{y}$, thus

$$-ye = \frac{\overline{y-e}}{y-e} = \frac{\overline{s-e}}{s-e}, \text{ i.e. } y = \frac{s-e}{1-\overline{e}s}.$$

Further,

$$-a(b+\overline{b}) = 1+s,\ a^2\overline{s} = -a^2 - a^2\frac{b+\overline{b}}{a} = 1-a^2+s,\ \overline{a^2-e} = \frac{1}{a^2} - \frac{1}{e} = -\frac{a^2-e}{a^2 e},$$

$$(f+s+2)(a^2-e) = a^2(b^2+\overline{b}^2) - a^2e - 1 + 2a^2 + sa^2 - es - 2e$$

$$= a^2(b+\overline{b})^2 - 1 + a^2(s-e) - 2e - es$$

$$= (1+s)^2 - 1 - 2e - es + a^2(s-e) = (s-e)(2+s+a^2),$$

$$\overline{(f+s+2)(a^2-e)} = \overline{(s-e)(2+s+a^2)} = -\frac{1-e\overline{s}}{e} \cdot \frac{2a^2+a^2\overline{s}+1}{a^2}$$

$$= -\frac{1-e\overline{s}}{e} \cdot \frac{2a^2+1-a^2+s+1}{a^2} = -\frac{(1-e\overline{s})(2+s+a^2)}{a^2 e},$$

imply the following

$$x = \frac{\overline{1+x}}{1+x} = \frac{\overline{g+1}}{g+1} = \frac{\overline{\frac{f+s}{2}+1}}{\frac{f+g}{2}+1} = \frac{\overline{f+s+2}}{f+s+2} = \frac{(f+s+2)\overline{(a^2-e)}}{(f+s+2)(a^2-e)} = -\frac{1}{a^2 e} \cdot \frac{\overline{(f+s+2)(a^2-e)}}{(f+s+2)(a^2-e)}$$

$$= -\frac{1}{a^2 e} \cdot \frac{(s-e)(2+s+a^2)}{-\frac{(1-e\overline{s})(2+s+a^2)}{a^2 e}} = \frac{s-e}{1-e\overline{s}} = y,$$

which actually was supposed to be proven. ■

103. Let P be a point in the inner part of a triangle ABC and let the lines AP, BP, CP remeet the circumcircle Γ of the triangle ABC at points K, L, M, respectively. The tangent of the circle Γ at C meets the line AB at S. Let $\overline{SC} = \overline{SP}$. Prove that $\overline{MK} = \overline{ML}$.

Solution. Without loss of generality we consider that the triangle ABC is inscribed into the unit circle and let 1 be the affix of ABC. If to the points correspond the affixes denoted by the appropriate lower case letters then $|a| = |b| = |k| = |l| = |m| = 1 = c$, $\frac{\overline{a-p}}{a-p} = \frac{\overline{a-k}}{a-k} = -\overline{ak}$, thus $k = \frac{p-a}{1-a\overline{p}}$ and symmetrically $l = \frac{p-b}{1-b\overline{p}}$, $m = \frac{p-1}{1-\overline{p}}$. The point S is the point of intersection of the line AB and the tangent to Γ at C, thus $s + \overline{s} = 2$ and $s + ab\overline{s} = a + b$, therefore we obtain that $s = \frac{a+b-2ab}{1-ab}$. Let T be the midpoint of PC, i.e. $t = \frac{p+1}{2}$. Since the condition of the problem $\overline{SC} = \overline{SP}$, that is T is the foot of the perpendicular at S to PC, i.e. to MC, which implies that $\frac{\overline{t-1}}{t-1} = \frac{\overline{m-1}}{m-1} = -\overline{m}$ and $\frac{\overline{t-s}}{t-s} = -\frac{\overline{m-1}}{m-1} = m$, i.e.

$$p + 1 = 2t = m + 1 + s - m\overline{s}, \text{ i.e. } p = m + s - m\overline{s} = m + s - m(2-s) = s - m + ms.$$

Thereby

$$\frac{p-1}{1-\overline{p}} = m = \frac{p-s}{s-1}$$

it is true that

$$p-s=sp-p-s+1+(p-s)\overline{p}\text{, i.e. }\overline{p}=\frac{2p-sp-1}{p-s}.$$

Thus,

$$m=\frac{p-s}{s-1}=\frac{p-\frac{a+b-2ab}{1-ab}}{\frac{a+b-2ab}{1-ab}-1}=\frac{(1-ab)p-a-b+2ab}{a+b-2ab-1+ab}=-\frac{(1-ab)p-a-b+2ab}{(1-a)(1-b)},$$

$$k=\frac{p-a}{1-a\cdot\frac{2p-sp-1}{p-s}}=\frac{(p-a)(p-s)}{p-2ap+a+(ap-1)s}=\frac{(p-a)\left(p-\frac{a+b-2ab}{1-ab}\right)}{p-2ap+a+(ap-1)\cdot\frac{a+b-2ab}{1-ab}}$$

$$=\frac{(p-a)[(1-ab)p-a-b+2ab]}{p-2ap+a-abp+2a^2bp-a^2b+a^2p+abp-2a^2bp-a-b+2ab}$$

$$=\frac{(p-a)[(1-ab)p-a-b+2ab]}{p-2ap+a^2p-a^2b-b+2ab}=\frac{(p-a)[(1-ab)p-a-b+2ab]}{p(1-a)^2-b(1-a)^2}$$

$$=\frac{(p-a)[(1-ab)p-a-b+2ab]}{(p-b)(1-a)^2}$$

and symmetrically,

$$l=\frac{(p-b)[(1-ab)p-a-b+2ab]}{(p-a)(1-b)^2},$$

So, $m^2=kl$, which implies the statement of the problem.■

104. Let ABC be an acute scalene triangle so that $\overline{AC}>\overline{BC}$, O be the circumcenter H be the orthocenter and F be the foot of the altitude at the vertex C. Let P be a point on the line AB, which differs from A, so that $\overline{AF}=\overline{PF}$, and M be the midpoint of the line segment AC. Let X be the point of intersection of the lines PH and BC, Y be the point of intersection of the lines OM and FX, and Z be the point of intersection of the lines OF and AC. Prove that the points F, M, Y and Z lie on a same circle.

Solution. Without loss of generality we consider the case where the triangle ABC is inscribed into the unit circle. Let the affixes of the points A, B, C, H, F, P, X be a, b, c, h, f, p, x, respectively. So, $h=a+b+c$ and $|a|=|b|=|c|=1$, holds true.

Thereby F is placed on AB, and CF is perpendicular to AB we get that $\frac{f-a}{\overline{f}-\overline{a}}=\frac{b-a}{\overline{b}-\overline{a}}=-ab=-\frac{f-c}{\overline{f}-\overline{c}}$, i.e. $f+ab\overline{f}=a+b$ and $f-ab\overline{f}=c-ab\overline{c}$, thus $f=\frac{a+b+c-ab\overline{c}}{2}$. $p=2f-a=b+c-ab\overline{c}$, thereby $\overrightarrow{AF}=\overrightarrow{FP}$. The point X is placed on BC, hence $\frac{x-b}{\overline{x}-\overline{b}}=\frac{b-c}{\overline{b}-\overline{c}}=-bc$, i.e. $\overline{x}=\frac{b+c-x}{bc}$. But, X is placed on PH, hence

$$\frac{p-x}{\overline{p}-\overline{x}}=\frac{p-h}{\overline{p}-\overline{h}}=\frac{b+c-ab\overline{c}-a-b-c}{\overline{b}-\overline{c}-\overline{a}\overline{b}c-\overline{a}-\overline{b}-\overline{c}}=-\frac{a\overline{c}(b+c)}{-\overline{a}\overline{b}(c+b)}=\frac{a^2b}{c},$$

therefore

$$p-x=\frac{a^2b}{c}\left(\overline{b}+\overline{c}-\overline{a}\overline{b}c-\frac{b+c-x}{bc}\right)=\frac{-ac^2+a^2x}{c^2}\text{, i.e. }x=\frac{c^2p+ac^2}{a^2+c^2}=\frac{2fc^2}{a^2+c^2}.$$

It is sufficient to prove that $OF \perp FX$, which is equivalent to $\frac{f-0}{\overline{f}-0} = -\frac{f-x}{\overline{f}-\overline{x}}$, i.e. $x\overline{f} + \overline{x}f = 2|f|^2$, which is obviously true, thereby

$$x\overline{f} + \overline{x}f = \frac{2fc^2}{a^2+c^2}\overline{f} + \frac{2\overline{f}\,\overline{c}^2}{\overline{a}^2+\overline{c}^2}f = 2|f|^2\left(\frac{c^2}{a^2+c^2} + \frac{a^2}{a^2+c^2}\right) = 2|f|^2. \blacksquare$$

105. Let $\triangle ABC$ not be an isosceles triangle and let AD, BF and CF be the bisectors of its angles ($D \in BC$, $E \in AC$, $F \in AB$). Let K_a, K_b, K_c be points of incircle of the triangle $\triangle ABC$ such that DK_a, EK_b, FK_c are tangents of the incircle and $K_a \notin BC$, $K_b \notin AC$, $K_c \notin AB$. Let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Prove that lines A_1K_a, B_1K_b, C_1K_c concure at the incircle of the triangle $\triangle ABC$.

Solution. Without loss of generality we consider the incircle of the triangle $\triangle ABC$ as the unit circle and let the circle tangents the sides BC, CA, AB at A', B', C', respectively. Let S be the incenter whose affix is 0. If the affixes of the points are denoted by corresponding lowercase letters, we get that $|a'| = |b'| = |c'| = 1$ and $a = \frac{2b'c'}{b'+c'}$, $b = \frac{2a'c'}{a'+c'}$, $c = \frac{2a'b'}{a'+b'}$, thus $a_1 = \frac{b+c}{2} = \frac{a'^2b'+a'^2c'+2a'b'c'}{(a'+b')(a'+c')}$. Since DK_a is a tangent of an incircle we get that $\angle ASK_a = \angle A'SA$ and thereby $|k_a| = 1$, it implies that $\frac{k_a}{a} = \overline{\left(\frac{a'}{a}\right)}$, so, $k_a = \frac{1}{a'} \cdot \frac{a}{\overline{a}} = \frac{b'c'}{a'}$. The point of intersection X, of the incircle and the line A_1K_a satisfies the following $|x| = 1$ and $\frac{x-k_a}{a_1-k_a} = \overline{\left(\frac{x-k_a}{a_1-k_a}\right)}$, that is $\overline{(a_1-k_a)}(x-k_a) = \left(\frac{1}{x} - \frac{1}{k_a}\right)(a_1-k_a)$, and thereby $x \neq k_a$, it is true that $\overline{a_1-k_a} = -\frac{1}{xk_a}(a_1-k_a)$. Since

$$a_1 - k_a = \frac{a'^2b'+a'^2c'+2a'b'c'}{(a'+b')(a'+c')} - \frac{b'c'}{a'} = \frac{(a'^2-b'c')(a'b'+a'c'+b'c')}{a'(a'+b')(a'+c')} \text{ and}$$

$$\overline{a_1 - k_a} = \frac{\frac{b'c'-a'^2}{a'^2b'c'} \cdot \frac{a'+b'+c'}{a'b'c'}}{\frac{(a'+b')(a'+c')}{a'^3b'c'}} = \frac{(b'c'-a'^2)(a'+b'+c')}{b'c'(a'+b')(a'+c')},$$

and $a'^2 \neq b'c'$, since $\triangle ABC$ is not an isosceles triangle,

$$x = -\frac{1}{k_a} \cdot \frac{a_1-k_a}{\overline{a_1-k_a}} = \frac{a'b'+b'c'+c'a'}{a'+b'+c'}.$$

Thereby the obtained expression is symmetric at a', b', c', the lines B_1K_b and C_1K_c meet the incircle at X. $\blacksquare$

4. EXERCISES (CHAPTER 2 AND 3)

1. Dermine the relationship between the points *A* and *B* with affixes *a* and *b*, respectively, if given that

a) $\operatorname{Re} ab = 0$, b) $\operatorname{Im} ab = 0$,

c) $\operatorname{Re} a\bar{b} = 0$, d) $\operatorname{Im} a\bar{b} = 0$.

2. Let A' be the projection of a point A on a real axis. Determine the point A'' such that it is symmetric to A' with respect to the line *OA* (figure 31).

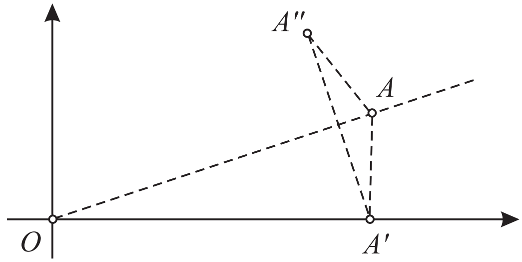

Figure 31

3. Let $\overline{OA} = 1$ be the diameter of a semicircle (figure 32). At *B* and *C* such that $\overline{OB} = \frac{1}{4}$ and $\overline{OC} = \frac{3}{4}$ are drawn perpendiculars to *x*-axis and *E* and *D* are the points of intersection of these perpendiculars and the semicircle. Find the complex number such that it is the affix of *M*, the point of intersection of the lines *OE* and *BD*.

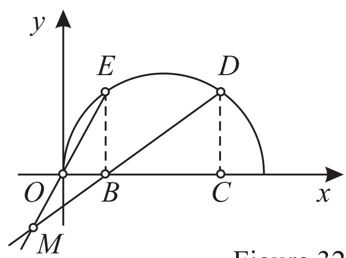

Figure 32

4. Let be given a point *C* with affix $c = 2e^{\frac{\pi}{6}}$. Determine the affixes *a* and *b* of the points *A* and *B* such that they are symmetric with respect to the line *OC*, the distance between each of them and the point *C* is 1 unit and satisfy the following:

a) $|a-b| = 2$, b) $|a-b| = \sqrt{2}$.

5. Given the points *A*, *B*, *C* and *Z* with affixes a, $b = a + e^{i\alpha}$, $c = a + e^{i\beta}$ and *z*, respectively. Determine the distance between the symmetric points Z' and Z'' to the point *Z* with respect to the lines *AB* and *AC*.

6. We shall say that α is a viewing angle for a line segment *AB* of a point *M* a line segment *AB* is viewed at an angle α of a point *M* if $\angle AMB = \alpha$. Let be given points *A* and *B* with affixes *a* and *b*, respectively. Let *W* be the point on the bisector of *AB* and furthermore the viewing angle for the line segment *AB* be α. Prove that $w = \frac{ae^{i\alpha} - b}{e^{i\alpha} - 1}$ is the affix of *W*.

7. The square of the diagonal is equal to the sum of the square of the leg and the product of the length of the bases in an isosceles trapezoid. Prove it!

8. Let a quadrilateral *ABCD* be parallelogram and let *N* be the point of intersection of the semiline *AD* and the circumcircle of the triangle *ABC*. Prove that $\overline{AD} \cdot \overline{AM} = \overline{AC}^2 - \overline{AB}^2$.

9. A circle (*K*) is a circumcircle of a regular pentagon *ABCDE*. Let *M* be a point on the arc $\overset{\frown}{AE}$. Prove that

$$\overline{MA} + \overline{MC} + \overline{ME} = \overline{MB} + \overline{MD} \cdot$$

10. Construct a trapezoid if given all its sides.

11. Given circles $K'(o',R')$ and $K''(o'',R'')$ and a line segment *AB*. Construct a line segment *CD* parallel and congruent to *AB*, such that $C \in (K')$ and $D \in (K'')$.

12. Given a line (*p*), a circle (*K*) and a line segment *AB*. Construct a line segment *CD* parallel and congruent to *AB*, such that $C \in (K)$ and $D \in (p)$.

13. Given lines (*p*) and (*q*) and a line segment *AB*. Construct a line segment *CD* parallel and congruent to *AB*, such that $C \in (p)$ and $D \in (q)$.

14. Let *A*, *B*, *C*, *D* be four given points and let

$$S_A(D) = D_1, \quad S_B(D_1) = D_2, \; S_C(D_2) = D_3,$$
$$S_A(D_3) = D_4, \; S_B(D_4) = D_5, \; S_C(D_5) = D_6.$$

Prove that $D = D_6$!

15. Given points O_i, $i = 1,2,3,4$ and a line segment A_0B_0. Let S_i, $i = 1,2,3,4$ be a point reflection centered at O_i, $i = 1,2,3,4$ and let

$$A_iB_i = S_i(A_{i-1}B_{i-1}), \; i = 1,2,3,4.$$

Prove that $\overline{A_0A_4} = \overline{B_0B_4}$.

16. Does the figure $\mathbf{F} = \{A,B,C\}$ have a center of reflection?

17. In which case a figure consisting of two semilines is a point reflective figure?

18. Given circles $K'(O',R')$ and $K''(O'',R'')$. In which case a figure consisting of the circles (K') and (K'') is a point reflective figure?

19. If a figure **F** is point reflective, then it has either a unique or, infinitely many centers of reflection. Prove it!

20. Given circles $K'(O',R')$ and $K''(O'',R'')$ and point A. Draw a line (a) through A so that A is the midpoint of the line segment MN, for $M \in (a) \cap (K')$ and $N \in (a) \cap (K'')$.

21. Given four distinct points A, B, C, D on a circle (K) and a point M on the chord CD. Determine a point X on the circle (K), so that the lines AX and BX on the chord CD intercept a line segment ST for which M is the middle point.

22. Given a rotation $S_{C,\alpha}$, $\alpha \neq 0, \pi$. Are there any lines which are fixed lines under this rotation?

23. Given a rotation $S_{C,\alpha}$. Prove that a circle $K(O,R)$ is fixed if and only if $O \equiv C$.

24. Given lines (p) and (q). In which case there exists a rotation $S_{C,\alpha}$, so that $S_{C,\alpha}(p) = q$?

25. Given circles $K(O,R)$ and $K'(O',R')$. In which case there exists a rotation $S_{C,\alpha}$, so that $S_{C,\alpha}(K) = K'$?

26. Given two circles (K') and (K'') and a point A. Construct an equilateral triangle ABC, so that $B \in (K')$ and $C \in (K'')$.

27. Given three parallel lines (p), (q) and (r). Construct an equilateral triangle ABC, so that $A \in (p)$, $B \in (q)$, $C \in (r)$.

28. Given three concentric (K'), (K'') and (K'''). Construct an equilateral triangle ABC, so that $A \in (K')$, $B \in (K'')$, $C \in (K''')$.

29. Given a line (p), a circle (K) and a point O. Construct an equilateral triangle ABC centered at O, so that two of its vertices are on (p) and (K), respectively.

30. Given two circles (K), (K') and a point O. Construct an equilateral triangle ABC centered at O, so that two of its vertices are on (K) and (K'), respectively.

31. In a triangle ABC inscribe a rhombus with acute angle $\alpha = 60°$, so that its two adjacent vertices are on the side AB, and the other two on the sides BC and AC, respectively.

32. In a circle $K(O,R)$ inscribe a triangle ABC, which is similar to a given triangle PQR.

33. Given intersecting lines (p), (q) and a circle (K). Construct a circle so that it tangents both lines (p) and (q) and a circle (K).

34. Let H and H_1 be homotheties with a common (mutual) center O and coefficients a and a_1, respectively. Prove that $\mathrm{H} \circ \mathrm{H}_1$ is also a homothety and $\mathrm{H} \circ \mathrm{H}_1 = \mathrm{H}_1 \circ \mathrm{H}$ holds true.

35. a) Prove that the composition of a point reflection and a homothety with coefficient $a \neq -1$ is homothety.
b) Prove that the composition of a homothety with coefficient $a \neq -1$ and a point reflection is homothety.

36. Prove that a composition of rotation around $\alpha \neq 0°, 180°$ and a homothety is a similarity such that it is not a homothety.

37. Prove that a composition of two reflections is either translation or rotation.

38. Prove that each translation can be expressed as a composition of two reflections

39. Prove that each rotation can be expressed as a composition of two reflections.

40. Let (a), (b) and (c) be three parallel lines. Prove that the composition of the reflections σ_a, σ_b and σ_c is a reflection.

41. Prove that a composition of a reflection and a homothety is similarity such that it is not homothety.

42. Prove that there does not exist any similarity such that it is not: movement, homothety, composition of rotation and homothety, composition of reflection and homothety.

43. If z_1, z_2, z_3 and z_4 are four distinct points on a circle, then their двоpазмер is a real number. Prove it!

44. Determine the set of points z such that under the Möbius transformation $w = \frac{z+2i}{2iz-1}$ map to the following set $\{w \,||\, w| = 1\}$.

45. Detrmine such a Möbius transformation that a semiplane $\{z \,|\, \operatorname{Im} z > 0\}$ maps to the circle $\{z \,||\, z| < 1\}$ and furthermore the points $z = i$ and $z = \infty$ under that transformation will be mapped to $w = 0$ and $w = -1$, respectively.

46. Find a condition which has to be satisfied, so that under the Möbius transformation $w = \frac{az+b}{cz+d}$, the circle circle $\{z \,||\, z| < 1\}$ will be mapped to the semiplane $\{z \,|\, \operatorname{Im} z > 0\}$.

47. Determine the Möbius transformation so that the points 0, $-i$, -1 map at i, 1, 0, respectively.

48. Determine the Möbius transformation so that the points i, $-i$, 1 map at 0, 1, ∞, respectively.

49. Prove that any three distinct points on a circle are not collinear.

50. Given the chords AB and CD of a circle so that $\overline{AC} = \overline{BD}$. Prove that either or $AB \,||\, CD$ or $AD \,||\, BC$.

51. In an acute triangle ABC, B' and C' are feet t of the altitudes at the vertices B and C, respectively. The circle with diameter AB meets the line CC' at M and N, and the circle with diameter AC meets the line BB' at P and Q. Prove that the quadrilateral $MNPQ$ is cyclic.

52. Let $ABCD$ be a quadrilateral such that the inner angles at the vertices A, B and C are congruent. Prove that the point D, the circumcentar and the orthocenter of the triangle ABC are collinear.

53. In a circle k is inscribed a hexagon $ABCDEF$, so that the sides AB, CD and EF are congruent with the radius of k. Prove that the midpoints of the other three sides are vertices of an equilateral triangle.

54. Isoscaled triangles *BCD*, *CAE* and *ABF*, whose bases are *BC*, *CA* and *AB*, respectively, are constructed on the outer part of a triangle $\triangle ABC$. Prove that the perpendiculars drawn at the vertices *A*, *B* and *C* to the lines *EF*, *FD* and *DE*, respectively, are concurent.

55. Let the quadrilateral *ABCD* be a cyclic and let *E* and *F* be the feet of the perpendiculars plot at the intersection of diagonals to the sides *AB* and *CD*, respectively. Prove that the line *EF* is perpendicular to the line which passes through the midpoints of the sides *AD* and *BC*.

56. Prove that the midpoints of the altitudes of a triangle are collinear if and only if the triangle is right angled triangle.

57. The feet of the perpendiculars in an acute triangle $\triangle ABC$ are A', B' and C'. If A'', B'' and C'' are the touching points of the incircle of the triangle $\triangle A'B'C'$, then prove that the Euler lines of $\triangle ABC$ and $\triangle A''B''C''$ coincide.

58. Let *ABCD* be a convex quadrilateral such that its diagonals *AC* and *BD* are perpendicular to each other and let $E = AC \cap BD$. Prove that the points symmetric to *E* with respect to the lines *AB*, *BC*, *CD* and *DA* form a cyclic quadrilateral.

59. Let *AK*, *BL*, *CM* be the altitudes ofa triangle *ABC*, *H* be its orthocenter and *P* the midpoint of the line segment *AH*. If $BH \cap MK = S$ and $LP \cap AM = S$, then $TS \perp BC$. Prove it!

60. Let *AD*, *BE*, *CF* be the altitudes of a triangle *ABC*. Let A', B', C' be such that $\overline{AA'} = k\overline{AD}$, $\overline{BB'} = k\overline{BE}$, $\overline{CC'} = k\overline{CF}$, for each $k \in \mathbf{R}$, $k \neq 0$. Determine all *k* such that for any non-isoscale triangle *ABC* the triangles *ABC* and $A'B'C'$ are simillar.

61. Given a triangle $\triangle ABC$ and points *D*, *E*, *F* on its altitudes *BC*, *CA*, *AB*, respectively, so that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = \frac{1-k}{k}, \ k \in \mathbf{R}.$$

Determine the locus of points of the circumcenters of *DEF* for $k \in \mathbf{R}$.

62. Let H' and H'' be the feet of the perpendiculars at the orthocenter *H* of $\triangle ABC$ to the bisector of the outer and the inner angle at *C*. Prove that the line $H'H''$ consists of the midpoint of the side *AB*.

63. Given an acute triangle ABC and a point D on its inner part, so that $\angle ADB = \angle ACB + 90°$ and $\overline{AB} \cdot \overline{CD} = \overline{AD} \cdot \overline{BC}$. Determine

$$\frac{\overline{AB} \cdot \overline{CD}}{\overline{AC} \cdot \overline{BD}}.$$

64. Tangents AM and AN and a line which crosses the circle at k and L are constructed on a circle k at the point A (which is positioned out of the circle k). Let l be any line which is parallel to AM and let KM and LM meet l at the points P and Q, respectively. Prove that the line MN bisects the line segment PQ.

65. On the sides BC, CA and AB of a triangle ABC are given points D, E and F, respectively, so that $\overline{BD} = \overline{CE} = \overline{AF}$. Prove that the triangles ABC and DEF have common circumcenter if and only if the triangle ABC is an equilateral triangle.

66. Let be given a cyclic quadrelateral $ABCD$. Prove that the incenters of the triangles ABC, BCD, CDA, DAB are vertices of a rectangle.

67. Let I be the incenter of atriangle ABC and let D and E be the midpoints of the sides AC and AB, respectively. Let $AB \cap DI = S$ and $AC \cap EI = Q$. Prove that $\overline{AP} \cdot \overline{AQ} = \overline{AB} \cdot \overline{AC}$ if and only if $\angle CAB = 60°$.

68. Let M be a point of the interior part of the square $ABCD$ and A', B', C', D' be the intersections of the lines AM, BM, CM, DM and the circumcircle of the square $ABCD$, respectively. Prove that

$$\overline{A'B'} \cdot \overline{C'D'} = \overline{A'D'} \cdot \overline{B'C'}.$$

69. Let $ABCD$ be a cyclic quadrilateral and let $F = AC \cap BD$ and $E = AD \cap BC$. If M and N are the midpoints of the sides AB and CD, respectively, then prove that $\frac{\overline{MN}}{\overline{EF}} = \frac{1}{2}\left|\frac{\overline{AB}}{\overline{CD}} - \frac{\overline{CD}}{\overline{AB}}\right|$.

70. The points A', B', C' are symmetric to the points A, B, C with respect to the sides BC, CA, AB, respectively. Determine the type of the triangle ABC, so that the triangle $A'B'C'$ is an equilateral triangle?

71. Let O be the circumcenter and R be the circumradius of a triangle ABC. The incircle of the triangle ABC, with radius r, touches the sides BC, CA, AB at points A', B', C', respectively. Let the lines determined by the midpoints of the line segments AB' and AC', BA' and BC', CA' and CB' intersect at C'', A'' and

B''. Prove that the circumcenter of the triangle $A''B''C''$ is O and the circumradius is $R+\frac{r}{2}$.

72. Let the trapezoid *ABCD*, $AB \parallel CD$, $\overline{AB} > \overline{CD}$, not be isoscaled and let it be circumscribed about a circle centered at *I*. The incircle tangents the side *CD* at *E*. Let *M* be the midpoint of the side *AB* and moreover *MI* and *CD* intersect at *F*. Prove that $\overline{DE} = \overline{FC}$ if and only if $\overline{AB} = 2\overline{CD}$.

73. Given a cyclic hexagon *ABCDEF* so that $\overline{AB} = \overline{CD} = \overline{EF}$ and the diagonals *AD*, *BE* and *CF* are concurrent. If $p = AD \cap CE$, then $\frac{\overline{CP}}{\overline{PE}} = \left(\frac{\overline{AC}}{\overline{CE}}\right)^2$. Prove it!

74. Given a triangle *ABC*. A', B', C' are the midpoints of the arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, $\overset{\frown}{AB}$, such that each of them does not consist of the point *A*, *B*, *C*, respectively. The lines $A'B'$, $B'C'$, $C'A'$ divide the sides of the triangle in six parts. Prove that "the middle" parts are congruent if and only if the triangle *ABC* is an equilateral triangle.

75. Let $\triangle ABC$ be such a triangle that $\angle ABC = 60°$. Let the line *IF* be parallel to *AC* (*I* is the incenter, and *F* lies on the side *AB*). The point *P* is on the side *BC* and $3\overline{BP} = \overline{BC}$. Prove that $\angle BFP = \frac{1}{2}\angle ABC$.

76. The angle at *A* is the smallest angle in a $\triangle ABC$. The points *B* and *C* divide the circumcircle of the triangle in two arcs. Let *U* be the interior point of the arc between *B* and *C* which does not consist of *A*. The bisectors of the line segments *AB* and *AC* meet the line *AU* at points *V* and *W*, respectively. The lines *BV* and *BW* meet at *T*. Prove that $\overline{AU} = \overline{TB} + \overline{TC}$.

77. Let *ABCD* be a convex quadrilateral so that *AB* is not parallel to *CD* and *AD* is not parallel to *BC*. The points *P*, *Q*, *R*, *S* are such chosen on the sides *AB*, *BC*, *CD*, *DA*, respectively, that the quadrilateral *PQRS* is parallelogram. Find the locus of the intersections of all such quadrilaterals *PQRS*.

78. The incircle of a triangle *ABC* tangents the sides *BC*, *CA*, *AB* at the points *E*, *F*, *G*, respectively. Let AA', BB', CC' be the intercepts of the bisectors of the inner angles of the triangle *ABC*. Let K_A, K_B, K_C be the points where the second tangents to the incircle drawn at A', B', C', respectively. Let *P*, *Q*, *R* be the midpoints of the sides *BC*, *CA*, *AB*, respectively. Prove that the lines PK_A, QK_B, RK_C concure on the incircle of the triangle *ABC*.

79. Let AD, BE, CF be the altitudes of the triangle ABC, and A', B', C' are points on them respectively, so that $\frac{\overline{AA'}}{\overline{AD}} = \frac{\overline{BB'}}{\overline{BE}} = \frac{\overline{CC'}}{\overline{CF}} = k$ holds. Determine each values of k so that the triangles ABC and $A'B'C'$ are similar.

80. **(Gauss's theorem).** If the line l meets the lines which consist of the sides BC, CA, AB of the triangle ABC at A', B', C', respectively, then prove that the midpoints of the line segments AA', BB', CC' are collinear.

81. Given a triangle ABC and a point T. Let P and Q be the feet of the perpendiculars at T to the lines AB and AC, respectively, and let R and S be the feet of the perpendiculars at A to the lines TC and TB, respectively. Prove that the intersection of the lines PR and QS lies on the line BC.

82. Let $PQRS$ be a cyclic quadrilateral, such that the lines PQ and RS are not parallel. Consider the set of all circles through P and Q and the set of all circles through R and S. Determine the set of all touching points between the circles which belong to these two sets.

83. Given a circle k and a point P positioned in the outer part of the circle. A variable line s, such that it consists of the point P, meets the circle at the points A and B. Let M and N be the midpoints of the arcs determined by the points A and B and let C be a point positioned on the line segment AB so that $\overline{PC}^2 = \overline{PA} \cdot \overline{PB}$ holds. Prove that the angle $\angle MCN$ does not depend on the choise of the line s.

84. Two circles k_1 and k_2 touch at a point M. The radius of k_1 is greater than the radius of k_2. Let A be any point on the circle k_2 such that it is not placed on the line which connects the centers of the circles, B and C be points on k_1 so that AB and AC are its tangents. The lines BM and CM remeet k_2 at E and F, respectively, and D is the intersection of the tangent to k_2 at A and the line EF. Prove that the locus of the point D, when A moves on k_2, is a line.

85. On a plane are given two circles k_1 and k_2 such that they meet at points A and B. The tangents to k_1 at A and B intersect at K. Let M be any point of the circle k_1 and let

$$MA \cap k_2 = \{A, P\},\ MK \cap k_1 = \{M, C\} \text{ and } CA \cap k_2 = \{A, Q\}.$$

Prove that the midpoints of the line segment PQ is placed on the line MC and PQ passes through a fixed point when M moves round the circle k_1.

86. Let ABC be a triangle so that $\angle ACB = 2\angle ABC$ and let D be a point on the line segment BC so that $\overline{CD} = 2\overline{BD}$ holds. The line segment AD is extended through D to the point E so that $\overline{AD} = \overline{DE}$ holds. Prove that the following is satisfied

$$\angle ECB + 180° = 2\angle EBC.$$

87. Given a triangle $A_1A_2A_3$ and a line p which passes through a point P and meets the sides A_2A_3, A_3A_1, A_1A_2 at X_1, X_2, X_3, respectively. Let A_iP meetthe circumcircle of the triangle $A_1A_2A_3$ at a point R_i, for $i = 1,2,3$. Prove that the lines X_1R_1, X_2R_2, X_3R_3 concur at a point which belongs on the circumcircle of the triangle $A_1A_2A_3$.

88. Two circle with defferent radii meet at points A and B. Their mutual tangents are MN and ST. Prove that the orthocenters of the triangles AMN, BMN, AST, BST are vertices of a rectangle.

89. Given a cyclic quadrilateral $ABCD$. The lines AD and BC meet at a point E, so that C is between B and E. The diagonals AC and BD meet at F. Let M be the midpoint of CD and let $N \not\equiv M$ be the point on the circumcircle of the triangle ABM such that $\frac{\overline{AN}}{\overline{BN}} = \frac{\overline{AM}}{\overline{BM}}$. Prove that points E, F and N are collinear.

90. The diameter of a circle k is placed on a line l. Let C and D be points on k. The tangents to k at C and D consecutively meet the line l at B and A, so that the center of the circle is between B and A. Let $E = AC \cap BD$ and F be the foot of the perpendicular at E to l. Prove that EF is the bisector of $\angle CFD$.

91. Let $ABCD$ be a convex quadrilateral whose sides BC and AD are congruent, but not parallel. Let E and F be interior points of the sides BC and AD, respectively, so that $\overline{BE} = \overline{DF}$. The lines AC and BD intersect at P, the lines BD and EF intersect at Q and the lines EF and AC intersect at R. Let's consider the triangles PQR which are get for all points E and F. Prove that the circumcircles of these triangles have a common point, such that it differs from P.

92. Let O be an interior point for the acute triangle $\triangle ABC$. The circles centered at the midpoints of the sides of the triangle $\triangle ABC$, such that each of them passes through O, concur at K, L, M (K, L, M differ from O). Prove that O is the incenter of the triangle $\triangle KLM$ if and only if O is the circumcenter of the triangle $\triangle ABC$.

93. Let M and N be points of the interior of a triangle $\triangle ABC$ such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$ hold. Prove that

$$\frac{\overline{AM}\cdot\overline{AN}}{\overline{AB}\cdot\overline{AC}}+\frac{\overline{BM}\cdot\overline{BN}}{\overline{BA}\cdot\overline{BC}}+\frac{\overline{CM}\cdot\overline{CN}}{\overline{CA}\cdot\overline{CB}}=1.$$

94. Let $\triangle ABC$ be such a triangle that $\angle A = 90°$ and $\angle B < \angle C$ hold. The tangent at A of its circumcircle Γ meets the line BC at D. Let E be the image of A under line symmetry with respect to the line BC, X be the foot of the perpendicular at A to BE and Y be the midpoint of the line segment AX. Let the line BY remeets Γ at a point Z. Prove that the line BD is a tangent to the circumcircle of the $\triangle ADZ$.

95. Given $\triangle ABC$ and points $A_1 \in BC,\ B_1 \in AC,\ C_1 \in AB$ such that $\triangle ABC$ and $\triangle A_1B_1C_1$ are simmilar. If the orthocenters or the incenters of $\triangle ABC$ and $\triangle A_1B_1C_1$ coincide, then $\triangle ABC$ is an equilateral triangle. Prove it!

96. Let be given points A, B and C. Determine the locus of a point D so that

$$\overline{DA}\cdot\overline{DB}\cdot\overline{AB}+\overline{DB}\cdot\overline{DC}\cdot\overline{BC}+\overline{DC}\cdot\overline{DA}\cdot\overline{CA}=\overline{AB}\cdot\overline{BC}\cdot\overline{CA}$$

holds.

97. Prove that the length of the side of a regular nonagon is equal to the difference of the lengths of its longest and shortest diagonal.

98. Prove that for any regular n-gon inscribed into a circle with radius r the, product of all sides and diagonals is equal to $n^{\frac{n}{2}} r^{\frac{n(n-1)}{2}}$.

99. On the circumcircle of a regular $2n$-gon $A_1A_2...A_{2n}$ is chosen an arbitrary point P. Prove that the sum of the squared distances between the point P and the vertices whose indexis are even numbers is equal to the sum of the squared distances between the point P and the vertices whose indexis are odd numbers.

100. Let $A_0A_1A_2...A_{2n}$ be a regular polygon, P be an arbitrary point of the smaller arc $\overset{\frown}{A_0A_{2n}}$ of its circumcircle and m be a positive integer, $0 \le m < n$. Prove that

$$\sum_{k=0}^{n}\overline{PA}_{2k}^{\,2m+1}=\sum_{k=1}^{n}\overline{PA}_{2k-1}^{\,2m+1}.$$

101. Let $A_0A_1...A_{n-1}$ be a regular n-gon inscribed in a circle whose radius is r. Prove that for any point P of the circumcircle and a positive integer $m < n$,

$$\sum_{k=0}^{n-1} \overline{PA_k}^{2m} = \binom{2m}{m} nr^{2m}$$

holds true.

102. Let $h_1, h_2, ..., h_n$ be the distances between an arbitrary point P of the smaller arc A_0A_{n-1} of the circumcircle of regular n-gon $A_0A_1...A_{n-1}$ and the lines A_0A_1, $A_1A_2, ..., A_{n-1}A_0$. Prove that

$$\frac{1}{h_1} + \frac{1}{h_2} + ... + \frac{1}{h_{n-1}} = \frac{1}{h_n}.$$

103. Given a regular n-gon $A_1A_2...A_n$ and a point P of the smaller arc $\widehat{A_1A_n}$. Let d_k be the distance between the points P and A_k. Prove that

$$\frac{1}{d_1d_2} + \frac{1}{d_2d_3} + ... + \frac{1}{d_{n-1}d_n} = \frac{1}{d_1d_n}.$$

104. Let P be any point on the circumcircle of a regular $2n$-gon $A_1A_2...A_{2n}$. If $p_1, p_2, ..., p_{2n}$ are the distances between the point P and the lines which consists of the sides A_1A_2, $A_2A_3, ..., A_{2n}A_1$, respectively then $p_1p_3...p_{2n-1} = p_2p_4...p_{2n}$. Prove it!

105. Let n be a prime number and let H_1 be a convex n-gon. The polygons $H_2, H_3, ..., H_n$ are constructed consecutive: the vertices of the polygon H_{k+1} are obtained by applying the the symmetry through the k-th adjacent vertex to the vertices of the polygon H_k in a positive direction. Prove that the polygons H_1 and H_n are similar.

106. Let $A_0, A_1, ..., A_{2k}$ be cosequtive points on a circle, such that they divide the circle in $2k+1$ congruent arcs. The point A_0 is plot by chords with each other points. These $2k$ chords divide the circle in $2k+1$ parts. These parts are alternately colored with white and black color, such that the number of the white parts is greater for one than the number of the black ones. Prove that the black area is greater than the white one.

107. The vertices of a regular n-gon are coloured with a few colors (each vertex with only one colour) so that the vertices coloured with the same colour form a regular polygon. Prove that two of these polygons are similar.

108. Let the points A, B, C, D and E be such that $ABCD$ is a parallelogram, and $BCED$ is a cyclic quadrilateral. Let l be a line which consists of the point A and intersects the line segment DC at an inner point F, and the line BC at C. If $\overline{EF} = \overline{EG} = \overline{EC}$, then l is the bisector of the angle DAB. Prove it!

109. Let H be the orthocenter of acute triangle ABC. The circle centered at the midpoint of the line segment BC, such that it consists of the point H, meets the line BC at A_1 and A_2. Analogously, the circle centered at the midpoint of the line segment CA, such that it consists of the point H, meets the line CA at B_1 and B_2, and the circle centered at the midpoint of the line segment AB, such that it consists of the point H, meets the line AB at C_1 and C_2. Prove that the points A_1, A_2, B_1, B_2, C_1 and C_2 belong on a same circle.

110. Let $ABCD$ be convex quadrilateral so that $\overline{BA} \neq \overline{BC}$ and k_1 and k_2 be the incircles of the triangles ABC and ADC, respectively. Let it exists a circle k such that it touches the extension of the side BA at A and the extension of the side BC at C, and it likewise touches the lines AD and CD. Prove that the common outer tangents to the circles k_1 and k_2 intersect at a point on the circle k.

111. Let O be the circumcenter of triangle ABC, P and Q be inner point for the line segments CA and AB, respectively, K, L and M be the midpoints of the line segments BP, CQ and PQ, respectively and Γ be a circle which consists of the points K, L and M. If the line PQ is a tangent to the circle Γ, then $\overline{OP} = \overline{OQ}$. Prove it!

112. Let I be the incenter, and Γ be the circumcircle of a triangle $\triangle ABC$. Let the line AI intersect Γ at A and D, and let E be a point on the arc $\widehat{BDC}$, and F be a point on the line segment BC so that

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC .$$

Let G be the midpoint of the line segment IF. Prove that the lines DG and EI intersect at a point of the circle Γ.

113. Let P be a inside point of triangle $\triangle ABC$ and the lines AP, BP and CP remeet the circumcircle Γ of the $\triangle ABC$ at K, L and M, respectively. The tangent to the circle Γ at the point C meets the line AB at S. Let $\overline{SC} = \overline{SP}$. Prove that $\overline{MK} = \overline{ML}$.

114. Let ABC be an acute triangle and let Γ be its circumcircle. Let l be any tangent to the circle Γ and let l_a, l_b and l_c be lines symmetric to l with respect to BC, CA and AB, respectively. Prove that the circumcircle of the triangle determined by the lines l_a, l_b and l_c touches the circle Γ.

115. Let ABC be a scalene acut triangle such that $\overline{AC} > \overline{BC}$ satisfies. Let O be the circumcenter, H be the orthocenter, and F the foot of the altitude at the vertex C.

Let P be a point on the line AB, such that it differes from A, and $\overline{AF} = \overline{PF}$ holds, and M be the midpoint of the line segment AC. Let X be the intersection of PH and BC, Y be the intersection of OM and FX, and Z be the intersection of OF and AC. Prove that the points F, M, Y and Z are on a same circle.

116. In $\triangle ABC$, M and N are points on the sides AB and AC, respectively, so that the line MN is parallel to the side BC. Let P be the intersection of the lines BN and CM. The circumcircles of the triangles $\triangle BMP$ and $\triangle CNP$ meet at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$.

117. Let $\triangle ABC$ be not isoscaled triangle. Let AD, BE, CF be the bisector of the angles of this triangle ($D \in BC$, $E \in AC$, $F \in AB$). Let K_a, K_b, K_c be points on the incircle of the $\triangle ABC$ so that DK_a, EK_b, FK_c are tengents to the incircle and $K_a \notin BC, K_b \notin AC$, $K_c \notin AB$. Let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB. Prove that the lines A_1K_a, B_1K_b, C_1K_c are concurrent on the incircle of $\triangle ABC$.

118. Let $\triangle ABC$ not be isoscaled triangle and k be its incircle centered at S. The circle k tangents the sides BC, CA, AB at points P, Q, R, respectively. The line QR meets BC at M. Let a circle which consists of the points B and C touches k at N. The circumcircle of the triangle MNP meets the line AP at L which differes from P. Prove that the points S, L and M are collinear.

119. In an acute triangle $\triangle ABC$ a point M is the midpoint of the side BC, and points D, E, F are feet of the altitudes at vertices A, B, C, respectively. Let H be the orthocenter of the triangle $\triangle ABC$, S be the midpoint of the line segment AH, and G be the intersection of the line segments FE and AH. If N is point of intersection of the median AH and the circumcircle of the $\triangle BCH$, prove that $\angle HMA = \angle GNS$.

120. In $\triangle ABC$, M and N are points on the sides AB and AC, respectively, so that the line MN is parallel to BC. Let P be the point of intersection of BN and CM. The circumcircles of the triangles $\triangle BMP$ and $\triangle CNP$ meet at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$.

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