38th Balkan Mathematical Olympiad

Shortlisted Problems with Solutions

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Note of Confidentiality

The shortlisted problems should be kept strictly confidential until BMO 2022

Contributing countries

The Organising Committee and the Problem Selection Committee of the BMO 2021 wish to thank the following countries for contributing problem proposals:

- Azerbaijan
- Bulgaria
- Greece
- North Macedonia
- Romania
- Serbia
- United Kingdom
- Uzbekistan

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PROBLEMS

ALGEBRA

A1. Find all functions $f:\mathbb{R}^+\to\mathbb{R}$ and $g:\mathbb{R}^+\to\mathbb{R}$ such that

$$f(x^2+y^2)=g(xy)$$

holds for all $x,y\in\mathbb{R}^+$.

A2. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x^2+y)\geqslant\left(\frac{1}{x}+1\right)f(y)$$

holds for all $x\in\mathbb{R}\setminus\{0\}$ and all $y\in\mathbb{R}$.

A3. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$f(x+f(x)+f(y))=2f(x)+y$$

holds for all $x,y\in\mathbb{R}^+$.

A4. Let f,g be functions from the positive integers to the integers. Vlad the impala is jumping around the integer grid. His initial position is $\mathbf{x}_0=(0,0)$, and for every $n\geqslant 1$, his jump is

$$\mathbf{x}_n-\mathbf{x}_{n-1}=\Big(\pm f(n),\,\pm g(n)\Big)\text{ or }\Big(\pm g(n),\,\pm f(n)\Big),$$

with eight possibilities in total. Is it always possible that Vlad can choose his jumps to return to his initial location $(0,0)$ infinitely many times when

(a) f,g are polynomials with integer coefficients?

(b) f,g are any pair of functions from the positive integers to the integers?

A5. Find all functions $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ such that

$$f(xf(x+y))=yf(x)+1$$

holds for all $x,y\in\mathbb{R}^+$.

A6. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(xy)=f(x)f(y)+f(f(x+y))$$

holds for all $x,y\in\mathbb{R}$.

COMBINATORICS

C1. Let $\mathcal{A}_n$ be the set of n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \{0, 1, 2\}$. A triple x, y, z of distinct elements of $\mathcal{A}_n$ is called *good* if there is some i such that $\{x_i, y_i, z_i\} = \{0, 1, 2\}$. A subset A of $\mathcal{A}_n$ is called *good* if every three distinct elements of A form a good triple.

Prove that every good subset of $\mathcal{A}_n$ has at most $2\left(\frac{3}{2}\right)^n$ elements.

C2. Let K and $N > K$ be fixed positive integers. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be distinct integers. Suppose that whenever $m_1, m_2, \ldots, m_n$ are integers, not all equal to 0, such that $|m_i| \leqslant K$ for each i, then the sum

$$\sum_{i=1}^{n} m_i a_i$$

is not divisible by N. What is the largest possible value of n?

C3. In an exotic country, the National Bank issues coins that can take any value in the interval $[0, 1]$. Find the smallest constant $c > 0$ such that the following holds, no matter the situation in that country:

Any citizen of the exotic country that has a finite number of coins, with a total value of no more than 1000*, can split those coins into* 100 *boxes, such that the total value inside each box is at most* c.

C4. A sequence of $2n + 1$ non-negative integers a_1, a_2,…, a_{2n+1} is given. There's also a sequence of $2n + 1$ consecutive cells enumerated from 1 to $2n + 1$ from left to right, such that initially the number a_i is written on the i-th cell, for i = 1,2, …$2n + 1$. Starting from this initial position, we repeat the following sequence of steps, as long as it's possible:

Step 1: Add up the numbers written on all the cells, denote the sum as s.

Step 2: If s is equal to 0 or if it is larger than the current number of cells, the process terminates. Otherwise, remove the s-th cell, and shift all cells that are to the right of it one position to the left. Then go to Step 1.

Example: $(1, 0, 1, \underline{2}, 0) \to (1, \underline{0}, 1, 0) \to (1, \underline{1}, 0) \to (\underline{1}, 0) \to (0)$.

A sequence a_1, a_2,…, a_{2n+1} of non-negative integers is called *balanced*, if at the end of this process there's exactly one cell left, and it's the cell that was initially enumerated by $(n + 1)$, i.e. the cell that was initially in the middle.

Find the total number of balanced sequences as a function of n.

C5. Angel has a warehouse, which initially contains 100 piles of 100 pieces of rubbish each. Each morning, Angel either clears every piece of rubbish from a single pile, or one piece of rubbish from each pile. However, every evening, a demon sneaks into the warehouse and adds one piece of rubbish to each non-empty pile, or creates a new pile with one piece. What is the first morning when Angel can guarantee to have cleared all the rubbish from the warehouse?

C6. There is a population P of 10000 bacteria, some of which are friends (friendship is mutual), so that each bacterion has at least one friend and if we wish to assign to each bacterion a coloured membrane so that no two friends have the same colour, then there is a way to do it with 2021 colours, but not with 2020 or less.

Two friends A and B can decide to *merge* in which case they become a single bacterion whose friends are precisely the union of friends of A and B. (Merging is not allowed if A and B are not friends.) It turns out that no matter how we perform one merge or two consecutive merges, in the resulting population it would be possible to assign 2020 colours or less so that no two friends have the same colour. Is it true that in any such population P every bacterium has at least 2021 friends?

GEOMETRY

G1. Let ABC be a triangle with $AB < AC < BC$. On the side BC we consider points D and E such that $BA = BD$ and $CE = CA$. Let K be the circumcenter of triangle ADE and let F, G be the points of intersection of the lines AD, KC and AE, KB respectively. Let ω_1 be the circumcircle of triangle KDE, ω_2 the circle with center F and radius FE, and c_3 the circle with center G and radius GD.

Prove that ω_1, ω_2 and ω_3 pass through the same point and that this point of intersection lies on the line AK.

G2. Let I and O be the incenter and the circumcenter of a triangle ABC, respectively, and let s_a be the exterior bisector of angle $\angle BAC$. The line through I perpendicular to IO meets the lines BC and s_a at points P and Q, respectively. Prove that $IQ = 2IP$.

G3. Let ABC be a triangle with $AB < AC$. Let ω be a circle passing through B, C and assume that A is inside ω. Suppose X, Y lie on ω such that $\angle BXA = \angle AYC$ and X lies on the opposite side of AB to C while Y lies on the opposite side of AC to B.

Show that, as X, Y vary on ω, the line XY passes through a fixed point.

G4. Let ABC be a right-angled triangle with $\angle BAC = 90°$. Let the height from A cut its side BC at D. Let I, I_B, I_C be the incenters of triangles ABC, ABD, ACD respectively. Let also E_B, E_C be the excenters of ABC with respect to vertices B and C respectively. If K is the point of intersection of the circumcircles of E_CIB_I and E_BIC_I, show that KI passes through the midpoint M of side BC.

G5. Let ABC be an acute triangle with $AC > AB$ and circumcircle Γ. The tangent from A to Γ intersects BC at T. Let M be the midpoint of BC and let R be the reflection of A in B. Let S be a point so that $SABT$ is a parallelogram and finally let P be a point on line SB such that MP is parallel to AB.

Given that P lies on Γ, prove that the circumcircle of $\triangle STR$ is tangent to line AC.

G6. Let ABC be an acute triangle such that $AB < AC$. Let ω be the circumcircle of ABC and assume that the tangent to ω at A intersects the line BC at D. Let Ω be the circle with center D and radius AD. Denote by E the second intersection point of ω and Ω. Let M be the midpoint of BC. If the line BE meets Ω again at X, and the line CX meets Ω for the second time at Y, show that A, Y and M are collinear.

G7. Let ABC be an acute scalene triangle. Its C-excircle tangent to the segment AB meets AB at point M and the extension of BC beyond B at point N. Analogously, its B-excircle tangent to the segment AC meets AC at point P and the extension of BC beyond C at point Q. Denote by A_1 the intersection point of the lines MN and PQ, and let A_2 be defined as the point, symmetric to A with respect to A_1. Define the points B_2 and C_2, analogously. Prove that $\triangle ABC$ is similar to $\triangle A_2B_2C_2$.

G8. Let ABC be a scalene triangle and let I be its incenter. The projections of I on BC, CA and AB are D, E and F respectively. Let K be the reflection of D over the line AI, and let L be the second point of intersection of the circumcircles of the triangles BFK and CEK. If $\frac{1}{3}BC = AC - AB$, prove that $DE = 2KL$.

NUMBER THEORY

N1. Let $n \geqslant 2$ be an integer and let

$$M = \left\{ \frac{a_1 + a_2 + \cdots + a_k}{k} : 1 \leqslant k \leqslant n \text{ and } 1 \leqslant a_1 < \cdots < a_k \leqslant n \right\}$$

be the set of the arithmetic means of the elements of all non-empty subsets of $\{1, 2, \ldots, n\}$.

Find $\min\{|a - b| : a, b \in M \text{ with } a \neq b\}$.

N2. Denote by $\ell(n)$ the largest prime divisor of n. Let $a_{n+1} = a_n + \ell(a_n)$ be a recursively defined sequence of integers with $a_1 = 2$. Determine all natural numbers m such that there exists some $i \in \mathbb{N}$ with $a_i = m^2$.

N3. Let n be a positive integer. Determine, in terms of n, the greatest integer which divides every number of the form $p + 1$, where $p \equiv 2 \bmod 3$ is a prime number which does not divide n.

N4. Can every positive rational number q be written as

$$\frac{a^{2021} + b^{2023}}{c^{2022} + d^{2024}},$$

where a, b, c, d are all positive integers?

N5. A natural number n is given. Determine all $(n-1)$-tuples of nonnegative integers $a_1, a_2, \ldots, a_{n-1}$ such that

$$\left[\frac{m}{2^n - 1}\right] + \left[\frac{2m + a_1}{2^n - 1}\right] + \left[\frac{2^2 m + a_2}{2^n - 1}\right] + \left[\frac{2^3 m + a_3}{2^n - 1}\right] + \cdots + \left[\frac{2^{n-1} m + a_{n-1}}{2^n - 1}\right] = m$$

holds for all $m \in \mathbb{Z}$.

N6. Let a, b and c be positive integers satisfying the equation $(a, b) + [a, b] = 2021^c$. If $|a - b|$ is a prime number, prove that the number $(a + b)^2 + 4$ is composite.

N7. A *super-integer* triangle is defined to be a triangle whose lengths of all sides and at least one height are positive integers. We will deem certain positive integer numbers to be *good* with the condition that if the lengths of two sides of a super-integer triangle are two (not necessarily different) good numbers, then the length of the remaining side is also a good number. Let 5 be a good number. Prove that all integers larger than 2 are good numbers.

SOLUTIONS

ALGEBRA

A1. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f(x^2 + y^2) = g(xy)$$

holds for all $x, y \in \mathbb{R}^+$.

Proposed by Greece

Solution. Given any $u \geqslant 2$, take $a, b \in \mathbb{R}^+$ such that $a + b = u$ and $ab = 1$. This is possible as the equation $x^2 - ux + 1$ for $u \geqslant 2$ has two positive real solutions. (Discriminant is $u^2 - 4 \geqslant 0$, sum and product of solutions are positive.) Now taking $x = \sqrt{a}, y = \sqrt{b}$ we get $f(u) = g(1)$.

Now given any $t \in \mathbb{R}^+$, taking $x = t/2, y = 2$ we have

$$g(t) = f\left(\frac{t^2}{4} + 4\right) = g(1)$$

as $\frac{t^2}{4} + 4 \geqslant 2$. So g is constant. But since any real number can be written as a sum of two squares, then f is constant as well. So there is a $c \in \mathbb{R}$ such that $f(x) = c$ and $g(x) = c$ for every $x \in \mathbb{R}^+$. Obviosuly any such pair of functions satisfies the equation.

A2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y) \geqslant \left(\frac{1}{x} + 1\right) f(y)$$

holds for all $x \in \mathbb{R} \setminus \{0\}$ and all $y \in \mathbb{R}$.

Proposed by Uzbekistan

Solution. We will show that $f(x) = 0$ for all $x \in \mathbb{R}$ which obviously satisfies the equation.

For $x = -1$ and $y = t + 1$ we get $f(t) \geqslant 0$ for every $t \in \mathbb{R}$.

For $x = \frac{1}{n}$, we get that

$$f\left(y + \frac{1}{n^2}\right) \geqslant (n+1)f(y)\,.$$

Therefore

$$f\left(y + \frac{2}{n^2}\right) \geqslant (n+1)f\left(y + \frac{1}{n^2}\right) \geqslant (n+1)^2 f(y)$$

and inductively we have

$$f\left(y + \frac{k}{n^2}\right) \geqslant (n+1)^k f(y)\,.$$

This holds for each $k, n \in \mathbb{N}$ and each $y \in \mathbb{R}$. In particular, for $k = n^2$ we get

$$f(y+1) \geqslant (n+1)^{n^2} f(y)\,.$$

Now if $f(y) > 0$, then letting n tend to infinity we obtain a contradiction. (E.g. taking $n > f(y+1)/f(y)$ we get $f(y+1) \geqslant (n+1)^{n^2} f(y) \geqslant (n+1)f(y) > f(y+1)$, a contradiction.)

So $f(x) = 0$ for every $x \in \mathbb{R}$.

A3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + f(x) + f(y)) = 2f(x) + y$$

holds for all $x, y \in \mathbb{R}^+$.

Proposed by Greece

Solution 1. We will show that $f(x) = x$ for every $x \in \mathbb{R}^+$. It is easy to check that this function satisfies the equation.

We write $P(x, y)$ for the assertion that $f(x + f(x) + f(y)) = 2f(x) + y$.

We first show that f is injective. So assume $f(a) = f(b)$. Now $P(1, a)$ and $P(1, b)$ show that

$$2f(1) + a = f(1 + f(1) + f(a)) = f(1 + f(1) + f(b)) = 2f(1) + b$$

and therefore $a = b$.

Let $A = \{x \in \mathbb{R}^+ : f(x) = x\}$. It is enough to show that $A = \mathbb{R}^+$.

$P(x, x)$ shows that $x + 2f(x) \in A$ for every $x \in \mathbb{R}^+$. Now $P(x, x + 2f(x))$ gives that

$$f(2x + 3f(x)) = x + 4f(x)$$

for every $x \in \mathbb{R}^+$. Therefore $P(x, 2x + 3f(x))$ gives that $2x + 5f(x) \in A$ for every $x \in \mathbb{R}^+$.

Suppose $x, y \in \mathbb{R}^+$ such that $x, 2x + y \in A$. Then $P(x, y)$ gives that

$$f(2x + f(y)) = f(x + f(x) + f(y)) = 2f(x) + y = 2x + y = f(2x + y)$$

and by the injectivity of f we have that $2x + f(y) = 2x + y$. We conlude that $y \in A$ as well.

Now since $x + 2f(x) \in A$ and $2x + 5f(x) = 2(x + 2f(x)) + f(x) \in A$ we deduce that $f(x) \in A$ for every $x \in \mathbb{R}^+$. I.e. $f(f(x)) = f(x)$ for every $x \in \mathbb{R}^+$.

By injectivity of f we now conclude that $f(x) = x$ for every $x \in \mathbb{R}^+$.

Solution 2. As in Solution 1, f is injective. Furthermore, letting $m = 2f(1)$ we have that the image of f contains (m, ∞). Indeed, if $t > m$, say $t = m + y$ for some $y > 0$, then $P(1, y)$ shows that $f(1 + f(1) + f(y)) = t$.

Let $a, b \in \mathbb{R}$. We will show that $f(a) - a = f(b) - b$. Define $c = 2f(a) - 2f(b)$ and $d = a + f(a) - b - f(b)$. It is enough to show that $c = d$. By interchanging the roles of a and b in necessary, we may assume that $d \geqslant 0$.

From $P(a, y)$ and $P(b, y)$, after subtraction, we get

$$f(a + f(a) + f(y)) - f(b + f(b) + f(y)) = 2f(a) - 2f(b) = c\,. \tag{1}$$

so for any $t > m$ (picking y such that $f(y) = t$ in (1)) we get

$$f(a + f(a) + t) - f(b + f(b) + t) = 2f(a) - 2f(b) = c\,. \tag{2}$$

Now for any $z > m + b + f(b)$, taking $t = z - b - f(b)$ in (2) we get

$$f(z + d) - f(z) = c\,. \tag{3}$$

Now for any $x > m + b + f(b)$ from (3) we get that

$$2f(x+d) + y = 2f(x) + y + 2c\,.$$

Also, for any x large enough, ($x > \max\{m+b+f(b), m+b+f(b)+c-d\}$ will do), by repeated application of (3), we have

$$\begin{aligned} f(x+d+f(x+d)+f(y)) &= f(x+f(x+d)+y)+c \\ &= f(x+f(x)+y+c)+c \\ &= f(x+f(x)+y+c-d)+2c\,. \end{aligned}$$

(In the first equality we applied (3) with $z = x+f(x+d)+y > x > m+b+f(b)$, in the second with $z = x > m+b+f(b)$ and in the third with $z = x+f(x)+y-c+d > x+c-d > m+b+f(b)$.)

In particular, now $P(x+d, y)$ implies that

$$f(x+f(x)+y+c-d) = 2f(x)+y = f(x+f(x)+y)$$

for every large enough x. By injectivity of f we deduce that $x+f(x)+y+c-d = x+f(x)+y$ and therefore $c = d$ as required.

It now follows that $f(x) = x + k$ for every $x \in \mathbb{R}^+$ and some fixed constant k. Substituting in the initial equation we get $k = 0$.

A4. Let f, g be functions from the positive integers to the integers. Vlad the impala is jumping around the integer grid. His initial position is $\mathbf{x}_0 = (0,0)$, and for every $n \geqslant 1$, his jump is

$$\mathbf{x}_n - \mathbf{x}_{n-1} = \Big(\pm f(n),\, \pm g(n) \Big) \text{ or } \Big(\pm g(n),\, \pm f(n) \Big)\,,$$

with eight possibilities in total. Is it always possible that Vlad can choose his jumps to return to his initial location $(0,0)$ infinitely many times when

(a) f, g are polynomials with integer coefficients?

(b) f, g are any pair of functions from the positive integers to the integers?

Proposed by United Kingdom

Solution 1.

(a) Yes it is always possible. The key idea is the following: Let $b(n)$ be the number of 1's in the binary expansion of $n = 0, 1, 2, \ldots$.

Lemma: Given a polynomial f with integer coefficients and degree at most d, then

$$\sum_{k=0}^{2^{d+1}-1} (-1)^{b(k)} f(n+k) = f(n) - f(n+1) - f(n+2) + \cdots \pm f\left(n + (2^{d+1}-1)\right) = 0\,.$$

Proof of Lemma: The result is clear for $d = 0$. For $d \geqslant 1$, we have

$$\sum_{k=0}^{2^{d+1}-1} (-1)^{b(k)} f(n+k) = \sum_{k=0}^{2^d-1} (-1)^{b(k)} \left[f(n+k) - f(n+k+2^d) \right].$$

So set $\tilde{f}(n) = f(n) - f(n+2^d)$, which is a polynomial of degree at most $d-1$. Then

$$\sum_{k=0}^{2^{d+1}-1} (-1)^{b(k)} f(n+k) = \sum_{k=0}^{2^d-1} \tilde{f}(n+k) = 0\,,$$

by induction, completing the proof of the lemma. □

In particular, if we take

$$\mathbf{x}_n - \mathbf{x}_{n-1} = \left((-1)^{b(n)} f(n),\, (-1)^{b(n)} g(n) \right),$$

then $\mathbf{x}_D = \mathbf{0}$ whenever D is a multiple of $2^{1+\max(\deg(f),\deg(g))}$.

(b) No, it is not always possible. Let g be any suitable function. Then, we construct f inductively. There are at most 8^{n-1} possibilities for $\mathbf{x}_{n-1}$, so choose $f(n)$ to be greater than the magnitude of all of them. Consequently $\mathbf{x}_n$ cannot be $\mathbf{0}$.

Solution 2.

(a) Given a polynomial f of degree at most d and integers n, r, we claim that

$$\sum_{k=0}^{2^{d+1}-1} \varepsilon_k f(2^d n + r + k) = 0$$

for some choice of $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2^{d+1}-1} \in \{-1, 1\}$. (Which are allowed to depend on d and f.)

We proceed by induction on d, the case $d = 0$ being immediate. For the inductive step we define the polynomial $g(n) = f(2n + r + 1) - f(2n + r)$ which is a polynomial of degree at most $d - 1$. Then

$$\sum_{k=0}^{2^d-1} \varepsilon_k g(2^{d-1}n + k) = 0$$

for some choice of the ε_k's giving

$$\sum_{k=0}^{2^{d+1}-1} \varepsilon'_k f(2^d n + r + k) = 0$$

where $\varepsilon'_{2k} = -\varepsilon_k$ and $\varepsilon'_{2k+1} = \varepsilon_k$. This completes the proof of the claim.

Now the proof can be completed as in Solution 1.

(b) Apart from magnitude arguments, one could also use modulo arguments. For example, taking $f(0), g(0)$ to be odd and $f(n), g(n)$ to be even for every $n \geqslant 1$ works.

Comments.

(1) We propose to omit part (b) as it is easy and furthermore it suggests that the answer to (a) is most likely affirmative.

(2) Giving a precise self-contained characterisation of $b(n)$ in Solution 1 is not necessary for the lemma. It could instead be phrased as:

There exists a sequence $\beta(k) \in \{-1, +1\}^{\mathbb{N}}$ such that $\sum \beta(k) f(n + k) = 0$.

Then, one constructs $\beta(\cdot)$ inductively as part of the proof via $\beta(k + 2^d) = -\beta(k)$ for $k < 2^d$, which coincides with the original definition, ie $\beta(\cdot) = (-1)^{b(\cdot)}$.

(3) The sequence of signs in both solutions are essentially the same. (Either all signs exactly the same or all signs different.)

A5. Find all functions $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$f(xf(x+y)) = yf(x) + 1$$

holds for all $x, y \in \mathbb{R}^+$.

Proposed by North Macedonia

Solution 1. We will show that that $f(x) = \frac{1}{x}$ for every $x \in \mathbb{R}^+$. It is easy to check that this function satisfies the equation.

We write $P(x, y)$ for the assertion that $f(xf(x+y)) = yf(x) + 1$.

We first show that f is injective. So assume $f(x_1) = f(x_2)$ and take any $x < x_1, x_2$. Then $P(x, x_1 - x)$ and $P(x, x_2 - x)$ give

$$(x_1 - x)f(x) + 1 = f(xf(x_1)) = f(xf(x_2)) = (x_2 - x)f(x) + 1$$

giving $x_1 = x_2$.

It is also immediate that for every $z > 1$ there is an x such that $f(x) = z$. Indeed $P(x, \frac{z-1}{f(x)})$ gives that

$$f\left(xf\left(x + \frac{z-1}{f(x)}\right)\right) = z\,.$$

Now given $z > 1$, take x such that $f(x) = z$. Then $P(x, \frac{z-1}{z})$ gives

$$f\left(xf\left(x + \frac{z-1}{z}\right)\right) = \frac{z-1}{z}f(x) + 1 = z = f(x)\,.$$

Since f is injective, we deduce that $f(x + \frac{z-1}{z}) = 1$.

So there is a $k \in \mathbb{R}^+$ such that $f(k) = 1$. Since f is injective this k is unique. Therefore $x = k + \frac{1}{z} - 1$. I.e. for every $z > 1$ we have

$$f\left(k + \frac{1}{z} - 1\right) = z\,.$$

We must have $k + \frac{1}{z} - 1 \in \mathbb{R}^+$ for each $z > 1$ and taking the limit as z tends to infinity we deduce that $k \geqslant 1$. (Without mentioning limits, assuming for contradiction that $k < 1$, taking $z = \frac{2}{1-k}$ leads to a contradiction.) Set $r = k - 1$.

Now $P(r + \frac{1}{6}, \frac{1}{3})$ gives

$$f\left(\left(r + \frac{1}{6}\right) f\left(r + \frac{1}{6} + \frac{1}{3}\right)\right) = \frac{1}{3}f\left(r + \frac{1}{6}\right) + 1 = \frac{6}{3} + 1 = 3 = f\left(r + \frac{1}{3}\right).$$

But

$$f\left(\left(r + \frac{1}{6}\right) f\left(r + \frac{1}{6} + \frac{1}{3}\right)\right) = f\left(\left(r + \frac{1}{6}\right) f\left(r + \frac{1}{2}\right)\right) = f\left(2r + \frac{1}{3}\right).$$

The injectivity of f now shows that $r = 0$, i.e. that $f(1) = k = 1$.

This shows that $f(\frac{1}{z}) = z$ for every $z > 1$, i.e. $f(x) = \frac{1}{x}$ for every $x < 1$. Now for $x > 1$ consider $P(1, x - 1)$ to get $f(f(x)) = (x - 1)f(1) + 1 = x = f(\frac{1}{x})$. Injectivity of f shows that $f(x) = \frac{1}{x}$.

So for all possible values of x we have shown that $f(x) = \frac{1}{x}$.

Solution 2. $P(1,y)$ shows that $f(f(y+1)) = yf(1)+1$. Now $P\left(f(y+1), \frac{yf(1)}{yf(1)+1}\right)$ shows that

$$f\left(f(y+1)f\left(f(y+1)+\frac{yf(1)}{yf(1)+1}\right)\right) = \frac{yf(1)}{yf(1)+1}f(f(y+1))+1 = yf(1)+1\,.$$

Since f is injective (as in Solution 1) we get that

$$f(y+1)f\left(f(y+1)+\frac{yf(1)}{yf(1)+1}\right) = f(y+1)$$

and therefore there is a unique k such that $f(k)=1$. Furtermore, for every $y>0$ we have

$$f(y+1) = k - \frac{yf(1)}{yf(1)+1}\,. \tag{1}$$

The right hand side of (1) is always positive. But letting y tend to infinity, the right hand side tends to $k-1$ so we must have $k \geqslant 1$.

If $k>1$, then $P(k-1,1)$ gives

$$f(k-1) = f((k-1)f(k)) = f(k-1)+1\,,$$

a contradiction. So $f(1)=k=1$.

For $x<1$, $P(x,1-x)$ gives

$$f(x) = f(xf(x+(1-x))) = (1-x)f(x)+1$$

from which we deduce that $f(x)=\frac{1}{x}$. To show that $f(x)=\frac{1}{x}$ for $x>1$ we can either work as in Solution 1 or take $y=x-1$ in (1) to get that

$$f(x) = 1 - \frac{x-1}{(x-1)+1} = \frac{1}{x}\,.$$

A6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy) = f(x)f(y) + f(f(x + y))$$

holds for all $x, y \in \mathbb{R}$.

Proposed by Romania

Solution 1. We will show that $f(x) = 0$ for every $x \in \mathbb{R}$ or $f(x) = x - 1$ for every $x \in \mathbb{R}$. It is easy to check that both of these functions work.

We write $P(x, y)$ for the assertion that $f(xy) = f(x)f(y) + f(f(x + y))$. For later use we write $Q(x, y)$ for the assertion that $f(xy) = f(x)f(y)$ and $R(x, y)$ for the assertion that $f(xy) = f(x)f(y) + f(x + y - 1)$.

Assume first that $f(0) = 0$.

For each $t \in \mathbb{R}$, $P(0, t)$ gives $f(f(t)) = 0$. Therefore we get that $Q(x, y)$ holds for each $x, y \in \mathbb{R}$. Now $Q(x, 1)$ gives $f(x) = f(x)f(1)$ for each $x \in \mathbb{R}$. But $f(1) \neq 1$ as otherwise we would have $f(f(1)) = f(1) = 1 \neq 0$, a contradiction. Since $f(1) \neq 1$, then $f(x) = f(x)f(1)$ gives $f(x) = 0$. This holds for each $x \in \mathbb{R}$ and gives our first solution.

From now on we assume that $f(0) = a \neq 0$. If $f(1) = 1$, then for $t \in \mathbb{R}$, $P(t - 1, 1)$ gives $f(f(t)) = 0$ so we get that $Q(x, y)$ holds for each $x, y \in \mathbb{R}$. Now $Q(x, 0)$ gives $f(0) = f(x)f(0)$ for each $x \in \mathbb{R}$. Since $f(0) \neq 0$, then $f(x) = 1$ for each $x \in \mathbb{R}$. This however contradicts the fact that $f(f(t)) = 0$ for each $t \in \mathbb{R}$.

So from now on we can further assume that $f(1) = b \neq 1$.

Now $P(x, 0)$ gives

$$f(f(x)) = a - af(x)$$

and $P(x - 1, 1)$ gives

$$f(f(x)) = f(x - 1) - bf(x - 1)\,.$$

Therefore, letting $c = \frac{b-1}{a}$, we get

$$f(x) = cf(x - 1) + 1 \tag{1}$$

for every $x \in \mathbb{R}$.

Claim 1. There is an integer n such that $n^2 \geqslant 4f(n)$.

Proof. If $c = 1$, then inductively from (1) we get that $f(n) = f(0) + n = a + n$ for each $n \in \mathbb{N}$. So for n large enough we have $n^2 \geqslant 4f(n)$.

If $c \neq 1$, then inductively from (1) we get that

$$f(n) = \left(a - \frac{1}{1 - c}\right)c^n + \frac{1}{1 - c}$$

for every $n \in \mathbb{Z}$. (We apply induction once to prove the result for every $n \geqslant 0$ and once to prove the result for every $n < 0$.)

For $|c| < 1$ we have $\lim\limits_{n\to\infty} f(n) = \frac{1}{1-c}$ so we can find n large enough such that $4f(n) \leqslant n^2$.

For $|c| > 1$ we have $\lim\limits_{n\to-\infty} f(n) = \frac{1}{1-c}$ so we can find a negative integer n with $|n|$ large enough such that $4f(n) \leqslant n^2$.

For $|c|=1$, we must have $c=-1$, so $f(n)=\pm(a-\frac{1}{2})+\frac{1}{2}$ and again for n large enough we have $4f(n)\leqslant n^2$. $\square$

Claim 2. $f(1)=0$.

Proof. Let n be as given by Claim 1 and pick $x',y'\in\mathbb{R}$ such that $x'+y'=n$ and $x'y'=f(n)$. This is possible since $n^2\geqslant 4f(n)$. Now $P(x',y')$ gives $f(x')f(y')=0$.

So there is a $d\in\mathbb{R}$ such that $f(d)=0$.

Putting $x=d+1$ in (1) we get $f(d+1)=1$. Now $P(d,1)$ gives $f(f(d+1))=0$ and therefore $b=f(1)=0$. $\square$

Claim 3. $c\neq -1$.

Proof. If $c=-1$, then $f(x)+f(x-1)=1$ for every $x\in\mathbb{R}$. In particular, for every $x\in\mathbb{R}$, we have

$$f(x)+f(x+1)=1=f(x+1)+f(x+2)$$

giving $f(x)=f(x+2)$. So $P(\frac{1}{2},\frac{1}{2})$ and $P(\frac{1}{2},\frac{5}{2})$ give

$$f\left(\frac{5}{4}\right)=f\left(\frac{1}{2}\right)f\left(\frac{5}{2}\right)+f(f(3))=f\left(\frac{1}{2}\right)f\left(\frac{1}{2}\right)+f(f(1))=f\left(\frac{1}{4}\right).$$

But $f\left(\frac{1}{4}\right)+f\left(\frac{5}{4}\right)=1$, therefore $f\left(\frac{1}{4}\right)=f\left(\frac{5}{4}\right)=\frac{1}{2}$. Since $f(1)=0$, then $f(0)=1$ and so

$$\frac{1}{2}=f\left(\frac{1}{4}\right)=f\left(\frac{1}{2}\right)^2+f(f(1))\geqslant f(f(1))=f(0)=1\,,$$

a contradiction. $\square$

Claim 4. $c=1$.

Proof. From (1) we get that $f(2)=1, f(3)=c+1$ and $f(4)=c^2+c+1$. Now $P(3,1)$ and $P(2,2)$ give that

$$f(f(4))=f(3)-f(3)f(1)=c+1\quad\text{and}\quad f(f(4))=f(4)-f(2)^2=c^2+c=c(c+1)\,.$$

Since by Claim 3 $c\neq -1$, then we must have $c=1$. $\square$

Since $f(1)=0$, then $P(x+y-1,1)$ gives $f(x+y-1)=f(f(x+y))$. Thus we have that $R(x,y)$ holds for every $x,y\in\mathbb{R}$.

Now $R(x,y+1)$ gives

$$f(xy+y)=f(x)f(y+1)+f(x+y)$$

and from (1) and the fact that $c=1$ we deduce that

$$\begin{aligned}f(xy+x)&=f(x)f(y)+f(x)+f(x+y)\\&=f(x)f(y)+f(x)+f(x+y-1)+1\\&=f(xy)+f(x)+1\,.\end{aligned}$$

This holds for every $x,y\in\mathbb{R}$. In particular, taking $x\neq 0$ and $y=t/x$, we have

$$f(t+x)=f(t)+f(x)+1 \tag{2}$$

for every $t\in\mathbb{R}, x\in\mathbb{R}\setminus\{0\}$. Note that (2) holds for $x=0$ as well, since $c=1$ implies that $f(0)=-1$.

Defining $g(x) = f(x) + 1$ for each $x \in \mathbb{R}$ then (2) gives that

$$g(t + x) = g(t) + g(x)$$

for every $t, x \in \mathbb{R}$. I.e. g is additive. Furthermore $R(x, y)$ implies that

$$\begin{aligned} g(xy) - 1 &= (g(x) - 1)(g(y) - 1) + g(x + y - 1) - 1 \\ &= g(x)g(y) - g(x) - g(y) + g(x + y - 1) \\ &= g(x)g(y) - 1 \,. \end{aligned}$$

This implies that g is multiplicative.

We know that an additive and multiplicative function is either identically zero or the identity function. [Since g is multiplicative, $g(x^2) = g(x)^2 \geqslant 0$ giving that g takes non-negative values at non-negative arguments. Since also g is additive we get that g is monotone increasing. Since also g is additive it is know that $g(x) = Cx$ for every $x \in \mathbb{R}$ for some contant C. The multiplicativity of g now gives that $C = 0$ or $C = 1$.]

Since g is not identically 0 we get that $g(x) = x$ for every $x \in \mathbb{R}$ giving that $f(x) = x - 1$ for every $x \in \mathbb{R}$.

Solution 2 (Sketch). One can prove directly Claims 3 and 4 without the use of Claims 1 and 2. To prove Claim 3 we can make use of $P(x + 1, y - 1)$ which together with $P(x, y)$ and (1) gives

$$f(xy + y - x) - cf(xy) = f(y) - cf(x) \,. \tag{3}$$

Assuming $c = -1$, then (1) and (3) give that $f(x + 2) = f(x)$ for every $x \in \mathbb{R}$. It follows that $f(x + 2n) = f(x)$ for every $x \in \mathbb{R}$ and every $n \in \mathbb{Z}$. Now with similar ideas as in the proof of Claim 1, it can be shown that for every $u, v \in \mathbb{R}$ there is $n \in \mathbb{N}$ large enough such that $u = xy + x - y + 2n$ and $v = xy + y - x$. Then using (3) we can get

$$f(u) = f(xy + x - y + 2n) = f(xy + x - y) = f(xy + y - x) = f(v) \,.$$

So f is constant and it must be identically equal to $1/2$ which leads to a contradiction.

Now using (3) with $x = y$ and assuming $c \neq 1$ we get $f(x^2) = f(x)$. So f is even. This eventually leads to $f(n) = 1/(1 - c) = a = b$ for every integer n. Now $P(0, 0)$ gives $a = a^2 + f(a)$ and $P(a, -a)$ gives $f(-a^2) = f(a)f(-a) + f(a)$. Since f is even we eventually get $f(a) = 0$ which gives $a = 0$ or $a = 1$ both contraidicting the facts that $a \neq 0$ and $b \neq 1$.

So $c = 1$ and using (1) and (3) one can eventually get $a = -1$. The solution can then finish in the same way as in Solution 1.

COMBINATORICS

C1. Let $\mathcal{A}_n$ be the set of n-tuples $x=(x_1,\ldots,x_n)$ with $x_i\in\{0,1,2\}$. A triple x,y,z of distinct elements of $\mathcal{A}_n$ is called *good* if there is some i such that $\{x_i,y_i,z_i\}=\{0,1,2\}$. A subset A of $\mathcal{A}_n$ is called *good* if every three distinct elements of A form a good triple.

Prove that every good subset of $\mathcal{A}_n$ has at most $2\left(\frac{3}{2}\right)^n$ elements.

Proposed by Greece

Solution 1. We proceed by induction on n, the case $n=1$ being trivial. Let

$$A_0=\{(x_1,\ldots,x_n)\in A : x_n\neq 0\}$$

and define A_1 and A_2 similarly.

Since A is good and A_0 is a subset of A, then A_0 is also good. Therefore, any three of its elements have a coordinate that differs. This coordinate cannot be the last one since 0 cannot appear as a last coordinate. This means that the set A_0' obtained from A_0 by deleting the last coordinate from each of its elements is a good subset of $\mathcal{A}_{n-1}$.

Moreover, if $|A_0|\geqslant 3$ then $|A_0'|=|A_0|$. Indeed, if otherwise, then there is an element $a\in A_0'$ such that $x,y\in A_0$, where x and y are obtained from a by adding to it the digits 1 and 2 respectively as the n-th coordinate. But then if z is any other element of A_0 then x,y,z do not form a good triple, a contradiction. So by the inductive hypothesis

$$|A_0|\leqslant \max\{2,|A_0'|\}\leqslant 2\left(\frac{3}{2}\right)^{n-1}.$$

Similarly,

$$|A_2|,|A_3|\leqslant 2\left(\frac{3}{2}\right)^{n-1}.$$

On the other hand, each element of A appears in exactly two of A_0,A_1,A_2. As a result,

$$|A|=\frac{1}{2}(|A_0|+|A_1|+|A_2|)\leqslant 2\left(\frac{3}{2}\right)^{n}.$$

Solution 2. Let

$$B=\{x=(x_1,\ldots,x_n)\in\mathcal{A}_n : x_i\in\{0,1\}\}$$

Let A be a good subset of $\mathcal{A}_n$ and define $f:A\times B\to\mathcal{A}_n$ by $f(a,b)=a+b=(a_1+b_1,\ldots,a_n+b_n)$ where the addition is done modulo 3.

We claim that if $(a,b),(a',b')$ and (a'',b'') are distinct, then $f(a,b),f(a',b')$ and $f(a'',b'')$ cannot all be equal. Indeed assume $f(a,b)=f(a',b')=f(a'',b'')=(x_1,\ldots,x_n)$. So for each i we have $a_i+b_i=a_i'+b_i'=a_i''+b_i''=x_i$. But then $a_i=x_i-b_i\in\{x_i,x_i-1\}$ and similarly $a_i',a_i''\in\{x_i,x_i-1\}$. So $\{a_i,a_i',a_i''\}\neq\{0,1,2\}$. Since this holds for each i then A cannot be a good set, contradiction.

Therefore $|A||B|\leqslant 2|\mathcal{A}_n|$ which gives $|A|\leqslant 2\left(\frac{3}{2}\right)^n$ as required.

Remark. Writing $f(n)$ for the maximal possible size of a good set, we proved that $f(n)\leqslant 2\left(\frac{3}{2}\right)^n$. We do not know the best possible asymptotic for $f(n)$ but we offer a corresponding lower bound which can increase the difficulty of the proposed problem.

We pick each element of $\mathcal{A}_n$ independently with probability p to form a set A. For each bad triple x, y, z of elements of A we arbitrarily remove one of the elements to end up with a good set B. Note that there are at most 21^n bad triples (x, y, z) since for coordinate i, out of the 27 triples of the form (x_i, y_i, z_i), only 6 of them will make the triple (x, y, z) a good triple. (Actually there are less than 21^n triples since this counts also triples where two or more of the n-tuples are the same.) So we get that

$$\mathbb{E}|B| \geqslant p \cdot 3^n - p^3 \cdot 21^n\,.$$

Taking $p = \dfrac{1}{\sqrt{3 \cdot 7^n}}$ we get

$$\mathbb{E}|B| \geqslant \frac{1}{\sqrt{3}}\left(\frac{9}{7}\right)^{n/2} - \frac{1}{3\sqrt{3}}\left(\frac{9}{7}\right)^{n/2} = \frac{2}{3\sqrt{3}}\left(\frac{9}{7}\right)^{n/2} = C\alpha^n$$

where $\alpha = 1.13389\ldots$ and $C = 0.3849\ldots$. It follows that there is a good set of size at least $C\alpha^n$.

C2. Let K and $N > K$ be fixed positive integers. Let n be a positive integer and let $a_1, a_2, \dots, a_n$ be distinct integers. Suppose that whenever $m_1, m_2, \dots, m_n$ are integers, not all equal to 0, such that $|m_i| \leqslant K$ for each i, then the sum

$$\sum_{i=1}^{n} m_i a_i$$

is not divisible by N. What is the largest possible value of n?

Proposed by North Macedonia

Solution. The answer is $n = \lfloor \log_{K+1} N \rfloor$.

Note first that for $n \leqslant \lfloor \log_{K+1} N \rfloor$, taking $a_i = (K+1)^{i-1}$ works. Indeed let r be maximal such that $m_r \neq 0$. Then on the one hand we have

$$\left| \sum_{i=1}^{n} m_i a_i \right| \leqslant \sum_{i=1}^{n} K(K+1)^{i-1} = (K+1)^n - 1 < N\,.$$

On the other hand we have

$$\left| \sum_{i=1}^{n} m_i a_i \right| \geqslant |m_r a_r| - \left| \sum_{i=1}^{r-1} m_i a_i \right| \geqslant (K+1)^{r-1} - \sum_{i=1}^{r-1} K(K+1)^{i-1} = 1 > 0\,.$$

So the sum is indeed not divisible by n.

Assume now that $n \geqslant \lfloor \log_{K+1} N \rfloor$ and look at all n-tuples of the form $(t_1, \dots, t_n)$ where each t_i is a non-negative integer with $t_i \leqslant K$. There are $(K+1)^n > N$ such tuples so there are two of them, say $(t_1, \dots, t_n)$ and $(t'_1, \dots, t'_n)$ such that

$$\sum_{i=1}^{n} t_i a_i \equiv \sum_{i=1}^{n} t'_i a_i \bmod N\,.$$

Now taking $m_i = t_i - t'_i$ for each i satisfies the requirements on the m_i's but N divides the sum

$$\sum_{i=1}^{n} m_i a_i\,,$$

a contradiction.

C3. In an exotic country, the National Bank issues coins that can take any value in the interval $[0,1]$. Find the smallest constant $c>0$ such that the following holds, no matter the situation in that country:

Any citizen of the exotic country that has a finite number of coins, with a total value of no more than 1000*, can split those coins into* 100 *boxes, such that the total value inside each box is at most* c*.*

Proposed by Romania

Solution 1. The answer is $c=\frac{1000}{91}=11-\frac{11}{1001}$. Clearly, if c' works, so does any $c>c'$. First we prove that $c=11-\frac{11}{1001}$ is good.

We start with 100 empty boxes. First, we consider only the coins that individually value more than $\frac{1000}{1001}$. As their sum cannot overpass 1000, we deduce that there are at most 1000 such coins. Thus we are able to put (at most) 10 such coins in each of the 100 boxes. Everything so far is all right: $10\cdot\frac{1000}{1001}<10<c=11-\frac{11}{1001}$.

Next, step by step, we take one of the remaining coins and prove there is a box where it can be added. Suppose that at some point this algorithm fails. It would mean that at a certain point the total sums in the 100 boxes would be $x_1,x_2,\ldots,x_{100}$ and no matter how we would add the coin x, where $x\leqslant\frac{1000}{1001}$, in any of the boxes, that box would be overflowed, i.e., it would have a total sum of more than $11-\frac{11}{1001}$. Therefore,

$$x_i+x>11-\frac{11}{1001}$$

for all $i=1,2,\ldots,100$. Then

$$x_1+x_2+\cdots+x_{100}+100x>100\cdot\left(11-\frac{11}{1001}\right).$$

But since $1000\geqslant x_1+x_2+\cdots+x_{100}+x$ and $\frac{1000}{1001}\geqslant x$ we obtain the contradiction

$$1000+99\cdot\frac{1000}{1001}>100\cdot\left(11-\frac{11}{1001}\right)\iff 1000\cdot\frac{1100}{1001}>100\cdot 11\cdot\frac{1000}{1001}.$$

Thus the algorithm does not fail and since we have finitely many coins, we will eventually reach to a happy end.

Now we show that $c=11-11\alpha$, with $1>\alpha>\frac{1}{1001}$ does not work.

Take $r\in\left[\frac{1}{1001},\alpha\right)$ and let $n=\left\lfloor\frac{1000}{1-r}\right\rfloor$. Since $r\geqslant\frac{1}{1001}$, then $\frac{1000}{1-r}\geqslant 1001$, therefore $n\geqslant 1001$.

Now take n coins each of value $1-r$. Their sum is $n(1-r)\leqslant\frac{1000}{1-r}\cdot(1-r)=1000$. Now, no matter how we place them in 100 boxes, as $n\geqslant 1001$, there exist 11 coins in the same box. But $11(1-r)=11-11r>11-11\alpha$, so the constant $c=11-11\alpha$ indeed does not work.

Solution 2 (for the upper bound). Amongst all possible arrangements into boxes, pick one where the maximum value inside a box is as small as possible. If there are several arrangements achieving this smallest maximum value, pick one where the number of boxes achieving this value is as small as possible.

Say that the boxes have total values equal to $10+x_1\geqslant 10+x_2\geqslant\cdots\geqslant 10+x_{100}$. respectively. We must have $x_1+\cdots+x_{100}\leqslant 0$. In particular, $0\geqslant x_1+99x_{100}$.

Assume for contradiction that $x_1 > \frac{990}{1001} = \frac{90}{91}$. Remove the coin of smallest denomination from the first box and add it into the 100-th box. Since the total value in the first box is greater than 10, the first box has at least 11 coins and therefore it has a coin of value at most $\frac{10+x_1}{11}$. The total new value in the last box is at most

$$10 + x_{100} + \frac{10 + x_1}{11} \leqslant 10 - \frac{x_1}{99} + \frac{10 + x_1}{11} = 10 + x_1 + \frac{90 - 91x_1}{99} < 10 + x_1 \, .$$

Remark. If we replace $[0,1]$ with $[0,v]$, the total sum with s, and the number of available boxes with n, then the answer to the problem is

$$c = v + \frac{s}{n} - \left(\frac{s}{n} + 1\right) \cdot \frac{1}{s+1} = v + \frac{s^2 - n}{n(s+1)}.$$

C4. A sequence of $2n+1$ non-negative integers a_1, $a_2,\ldots$, a_{2n+1} is given. There's also a sequence of $2n+1$ consecutive cells enumerated from 1 to $2n+1$ from left to right, such that initially the number a_i is written on the i-th cell, for $i = 1,2, \ldots 2n+1$. Starting from this initial position, we repeat the following sequence of steps, as long as it's possible:

Step 1: Add up the numbers written on all the cells, denote the sum as s.

Step 2: If s is equal to 0 or if it is larger than the current number of cells, the process terminates. Otherwise, remove the s-th cell, and shift all cells that are to the right of it one position to the left. Then go to Step 1.

Example: $(1,0,1,\underline{2},0) \to (1,\underline{0},1,0) \to (1,\underline{1},0) \to (\underline{1},0) \to (0)$.

A sequence a_1, $a_2,\ldots$, a_{2n+1} of non-negative integers is called *balanced*, if at the end of this process there's exactly one cell left, and it's the cell that was initially enumerated by $(n+1)$, i.e. the cell that was initially in the middle.

Find the total number of balanced sequences as a function of n.

Proposed by North Macedonia

Solution. The answer is: $C_n \cdot C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

We divide the proof into several steps. First, some terminology: the last (rightmost) n cells will be called the *back* cells and the front (leftmost) n cells will be called the *front* cells. The central, $(n+1)$-st, cell will be called the *middle* cell.

Claim 1. All the back cells must be removed before any front cell is removed.

Proof. Assume for contradiction that this is not the case. Then there must be a point in time where a front cell is deleted and then immediately after a back cell is deleted. Let us say that the deleted front cell was at position i. So all back cells have positions greater or equal to $i+2$. After the cell is deleted all back cells have positions greater or equal to $i+1$. But since we deleted cell i, then the total sum is i and this does not increase. So at the next step we delete a cell at position at most i, a contradiction. $\square$

Claim 2. The middle cell must contain the number 0, i.e., $a_{n+1} = 0$.

Proof. Consider the last step in the process where we have total of 2 cells. One of these is the middle cell, and by Claim 1 the other must be one of the front cells. I.e. we have (x, a_{n+1}). On the next move, we remove x, which means that $x + a_{n+1} = 1$. So $a_{n+1} = 0$ or $a_{n+1} = 1$. But after that we cannot remove a_{n+1}, which means that $a_{n+1} \neq 1$. So $a_{n+1} = 0$. $\square$

Now, let's define a *self-destructing* sequence to be one with no surviving cells at the end of the process. For example, $(0,1,2)$ is self-destructing because $(0,1,2) \to (0,1) \to (1) \to ()$.
Let $\mathcal{S}_n$ be the set of self-destructing sequences of length n. For example, $\mathcal{S}_2 = \{(0,1),(1,1)\}$. It is clear that the front cells form a self-destructing sequence, i.e., $(a_1, a_2, \cdots a_n) \in \mathcal{S}_n$. The back cells also have certain self-destructing quality, which is made more precise in Claim 3 below.

Claim 3. Fix the front sequence $\varphi = (a_1, a_2, \cdots, a_n)$. Let $\mathcal{B}_\phi$ be the set of all possible back sequences of length n that can be appended to φ (with a 0 between them) to get a balanced sequence. Then there is a bijection $f\colon \mathcal{S}_n \mapsto \mathcal{B}_\phi$.

Proof. Let $c = n+1-\sum_{i=1}^{n} a_i$ and consider a particular $\sigma = (s_1, s_2, \ldots, s_n) \in \mathcal{S}_n$. Let ℓ be the initial index of the last surviving cell in σ. Then $f(\sigma) = (s_1, s_2, \ldots, s_\ell + c, s_{\ell+1}, \ldots, s_n)$ defines a bijection $\mathcal{S}_n \mapsto \mathcal{B}_\phi$.

Indeed we claim that the k-th deleted cell in σ is the k-th deleted cell in $\overline{\varphi 0 f(\sigma)}$ for each $k = 1, \ldots, n$. Indeed after some deletions let S be the total sum remaining in σ. Then the total sum remaining in $\overline{\varphi 0 f(\sigma)}$ is $-\sum_{i=1}^{n} a_i + 0 + S + c = S + n + 1$. So we delete next the cell in position S in σ if and only if we delete the cell in position $S + n + 1$ in $\overline{\varphi 0 f(\sigma)}$.

So $\overline{\phi 0 f(\sigma)}$ is clearly a balanced sequence: we first eliminate all cells in the back, then the front. In the same manner it follows that every balanced sequence in of this form. □

So far we have shown that the total number of balanced sequences is $|\mathcal{S}_n|^2$. It remains to calculate the size $|\mathcal{S}_n|$.

Claim 4. Let $\mathcal{T}_n$ be the set of $2n$-sequences consisting of n zeros and n ones such that in each initial segment the number of 1's does not surpass the number of 0's. Then $|\mathcal{S}_n| = |\mathcal{T}_n|$.

Proof. Let $[n] = \{1, 2, \ldots, n\}$, and let us also consider the set $\mathcal{F}_n$ of non-decreasing mappings $f : [n] \to [n]$ such that $f(i) \leqslant i$ for each $i \in [n]$. The claim will follow once we show that $|\mathcal{S}_n| = |\mathcal{F}_n|$ and that $|\mathcal{F}_n| = |\mathcal{T}_n|$.

In order to demonstrate that $|\mathcal{S}_n| = |\mathcal{F}_n|$, observe that there is an obvious bijective correspondence $a \mapsto f$ between the sets $\mathcal{S}_n$ and $\mathcal{F}_n$. Indeed, reversing the self-destructing process for an n-sequence $a = (a_1, a_2, \ldots, a_n) \in \mathcal{S}_n$, simply define $f(i)$ to be the (partial) sum of the existing terms after the i-th backward step.

As for $|\mathcal{T}_n| = |\mathcal{F}_n|$, note the following bijective correspondence $t \mapsto f$ between the sets $\mathcal{T}_n$ and $\mathcal{F}_n$. Let $f(i)$ equal $1 + \#(i)$, where $\#(i)$ is defined to be the total number of $1's$ appearing in t before the i-th zero.

Finally, it is a known fact that $|\mathcal{B}_n|$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. (The essential idea of the textbook proof of this fact uses the so-called *reflection principle* of A. D. André.)

C5. Angel has a warehouse, which initially contains 100 piles of 100 pieces of rubbish each. Each morning, Angel either clears every piece of rubbish from a single pile, or one piece of rubbish from each pile. However, every evening, a demon sneaks into the warehouse and adds one piece of rubbish to each non-empty pile, or creates a new pile with one piece. What is the first morning when Angel can guarantee to have cleared all the rubbish from the warehouse?

Proposed by United Kingdom

Solution 1. We will show that he can do so by the morning of day 199 but not earlier.

If we have n piles with at least two pieces of rubbish and m piles with exactly one piece of rubbish, then we define the value of the pile to be

$$V = \begin{cases} n & m = 0\,, \\ n + \frac{1}{2} & m = 1\,, \\ n+1 & m \geqslant 2\,. \end{cases}$$

We also denote this position by (n, m). Implicitly we will also write k for the number of piles with exactly two pieces of rubbish.

Angel's strategy is the following:

(i) From position $(0, m)$ remove one piece from each pile to go position $(0, 0)$. The game ends.

(ii) From position $(n, 0)$, where $n \geqslant 1$, remove one pile to go to position $(n-1, 0)$. Either the game ends, or the demon can move to position $(n-1, 0)$ or $(n-1, 1)$. In any case V reduces by at least $1/2$.

(iii) From position $(n, 1)$, where $n \geqslant 1$, remove one pile with at least two pieces to go to position $(n-1, 1)$. The demon can move to position $(n, 0)$ or $(n-1, 2)$. In any case V reduces by (at least) $1/2$.

(iv) From position (n, m), where $n \geqslant 1$ and $m \geqslant 2$, remove one piece from each pile to go to position $(n-k, k)$. The demon can move to position $(n, 0)$ or $(n-k, k+1)$. In any case V reduces by at least $1/2$. (The value of position $(n-k, k+1)$ is $n + \frac{1}{2}$ if $k = 0$, and $n - k + 1 \leqslant n$ if $k \geqslant 1$.)

So during every day if the game does not end then V is decreased by at least $1/2$. So after 198 days if the game did not already end we will have $V \leqslant 1$ and we will be in one of positions $(0, m), (1, 0)$. The game can then end on the morning of day 199.

We will now provide a strategy for demon which guarantees that at the end of each day V has decreased by at most $1/2$ and furthermore at the end of the day $m \leqslant 1$.

(i) If Angel moves from $(n, 0)$ to $(n-1, 0)$ (by removing a pile) then create a new pile with one piece to move to $(n-1, 1)$. Then V decreases by $1/2$ and and $m = 1 \leqslant 1$

(ii) If Angel moves from $(n, 0)$ to $(n-k, k)$ (by removing one piece from each pile) then add one piece back to each pile to move to $(n, 0)$. Then V stays the same and $m = 0 \leqslant 1$.

(iii) If Angels moves from $(n, 1)$ to $(n-1, 1)$ or $(n, 0)$ (by removing a pile) then add one piece to each pile to move to $(n, 0)$. Then V decreases by $1/2$ and $m = 0 \leqslant 1$.

(iv) If Angel moves from $(n, 1)$ to $(n-k, k)$ (by removing a piece from each pile) then add one piece to each pile to move to $(n, 0)$. Then V decreases by $1/2$ and $m = 0 \leqslant 1$.

Since after every move of demon we have $m \leqslant 1$, in order for Angel to finish the game in the next morning we must have $n = 1, m = 0$ or $n = 0, m = 1$ and therefore we must have $V \leqslant 1$.

But now inductively the demon can guarantee that by the end of day N, where $N \leqslant 198$ the game has not yet finished and that $V \geqslant 100 - N/2$.

Solution 2.

Define Angel's score S_A to be $S_A = 2n + m - 1$. The Angel can clear the rubbish in at most $\max\{S_A, 1\}$ days. The proof is by induction on (n, m) in lexicographic order.

Angel's strategy is the same as in Solution 1 and in each of cases (ii)-(iv) one needs to check that S_A reduces by at least 1 in each day. (Case (i) is trivial as the game ends in one day.)

Now define demon's score S_D to be $S_D = 2n - 1$ if $m = 0$ and $S_D = 2n$ if $m \geqslant 1$. The claim is the if $(n, m) \neq (0, 0)$, then the demon can ensure that Angel requires S_D days to clear the rubbish.

Again, demon's strategy is the same as in the Solution by PSC and in each of cases (i)-(iv) one needs to check that S_D reduced by at most 1 in each day.

Comment. If we start from position (n, m), then the number N of days required is

$$N = \begin{cases} 2n - 1 & \text{if } m = 0\,, \\ 2n & \text{if } m = 1\,, \\ 2n & \text{if } m \geqslant 2, \text{ and } k \geqslant 1\,, \\ 2n + 1 & \text{if } m \geqslant 2, \text{ and } k = 0\,. \end{cases}$$

C6. There is a population P of 10000 bacteria, some of which are friends (friendship is mutual), so that each bacterion has at least one friend and if we wish to assign to each bacterion a coloured membrane so that no two friends have the same colour, then there is a way to do it with 2021 colours, but not with 2020 or less.

Two friends A and B can decide to *merge* in which case they become a single bacterion whose friends are precisely the union of friends of A and B. (Merging is not allowed if A and B are not friends.) It turns out that no matter how we perform one merge or two consecutive merges, in the resulting population it would be possible to assign 2020 colours or less so that no two friends have the same colour. Is it true that in any such population P every bacterium has at least 2021 friends?

Proposed by Bulgaria

Solution 1. The answer is affirmative.

We will use the terminology of graph theory. Here the vertices of our main graph G are the bacteria and there is an edge between two precisely when they are friends. The degree $d(v)$ of a vertex v of G is the number of neighbours of v. The minimum degree $\delta(G)$ of G is the smallest amongst all $d(v)$ for vertices v of G. The chromatic number $\chi(G)$ of G is the number of colours needed in order to colour the vertices such that neighbouring vertices get distinct colours.

It suffices to establish the following:

Claim. Let k be a positive integer and let G be a graph on $n > k$ vertices with $\delta(G) \geqslant 1$ and $\chi(G) = k$. Suppose that merging one pair or two pairs of vertices results in a graph G' with $\chi(G') \leqslant k-1$. Then $\delta(G) \geqslant k$.

We establish this in a series of claims.

Claim 1. $\delta(G) \geqslant k-1$.

Proof. Suppose for contradiction that we have a vertex v of degree $r \leqslant k-2$ and denote its neighbours by $v_1, \ldots, v_r$. (Note that, by assumption, v has at least one neighbour.)

Suppose we merge v with v_i. We denote the new vertex by v_0, and we colour the obtained graph in $k-1$ colours. Note that at most $r \leqslant k-2$ colours can appear in the set $S_1 = \{v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r\}$. Therefore we can get a $(k-1)$-colouring of G by assigning the colour of v_0 to v_i and an unused colour (from the $k-1$ available) to v, thus contradicting the assumption that $\chi(G) = k$. $\square$

So from now on we may assume that there is a vertex v of G with $\deg(v) = k-1$, as otherwise the proof is complete. We denote its neighbours by $v_1, \ldots, v_{k-1}$.

Claim 2. The set of neighbours of v induces a complete graph.

Proof of Claim 2. Suppose $v_iv_j \notin E(G)$. Merge v with v_i, giving a next vertex w, and then merge w with v_j, denoting the newest vertex by v_0. Then colour the resulting graph in $k-1$ colours. Note that at most $k-2$ colours can appear in the set $S_2 = \{v_0, v_1, \ldots, v_{k-1}\} \setminus \{v_i, v_j\}$. So we can get a $(k-1)$-colouring of G by assigning the colour of v_0 to v_i and v_j and an unused colour (from the $k-1$ available) to v, thus contradicting the assumption that $\chi(G) = k$. $\square$

Claim 3. For every edge uw, both u and w belong in the set $\{v, v_1, \ldots, v_{k-1}\}$.

Proof. Otherwise merge u and w and call the new vertex z. If $u, w \notin \{v, v_1, \ldots, v_{k-1}\}$ then by Claim 2 the resulting graph contains a complete graph on $\{v, v_1, \ldots, v_{k-1}\}$ and so its chromatic number is at least k, a contradiction. If one of u, w belongs in the set $\{v, v_1, \ldots, v_{k-1}\}$, say

$u = v_i$, then the resulting graph contains a complete graph on $\{v, v_1, \ldots, v_{k-1}, z\} \setminus \{v_i\}$. This is again a contradiction. $\square$

From Claim 3 we see that G consists of a complete set on k vertices together with $n - k > 0$ isolated vertices. This is a contradiction as $\delta(G) \geqslant 1$.

Remark. We do not know if the result is best possible or whether it can be improved to show $\delta(G) \geqslant 2022$.

GEOMETRY

G1. Let ABC be a triangle with $AB < AC < BC$. On the side BC we consider points D and E such that $BA = BD$ and $CE = CA$. Let K be the circumcenter of triangle ADE and let F, G be the points of intersection of the lines AD, KC and AE, KB respectively. Let ω_1 be the circumcircle of triangle KDE, ω_2 the circle with center F and radius FE, and c_3 the circle with center G and radius GD.

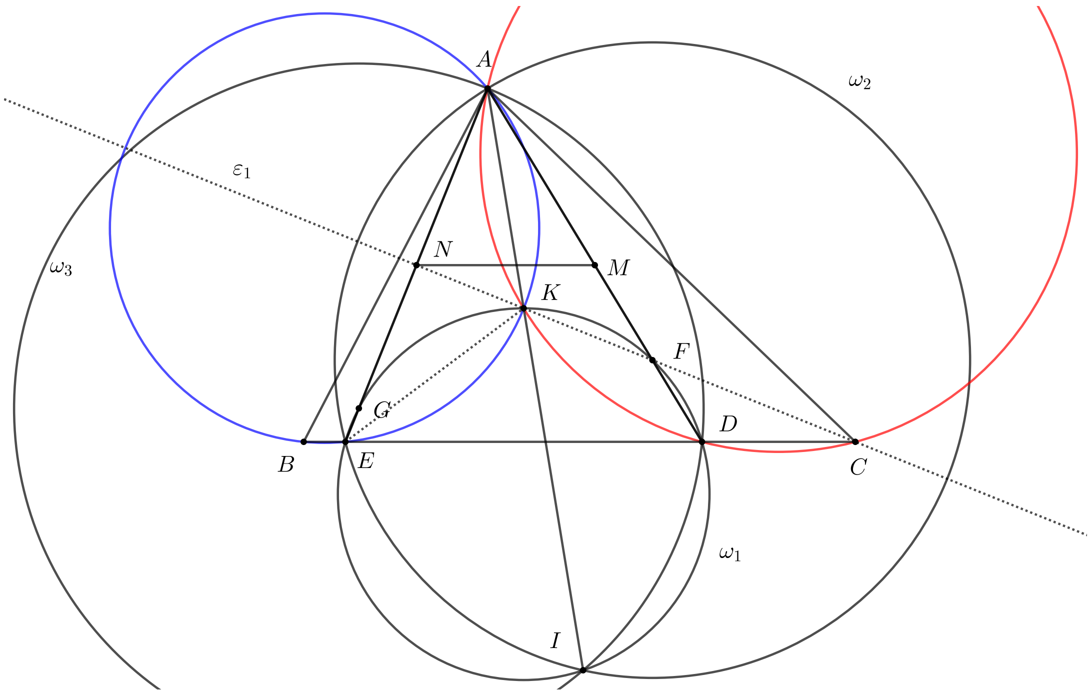
Prove that ω_1, ω_2 and ω_3 pass through the same point and that this point of intersection lies on the line AK.

Proposed by Greece

Solution 1. Since the triangles BAD, KAD and KDE are isosceles, then $\angle BAD = \angle BDA$ and $\angle KAD = \angle KDA$ and $\angle KDE = \angle KED$. Therefore,

$$\angle BAK = \angle BAD - \angle KAD = \angle BDA - \angle KDA = \angle KDE = \angle KED = 180^\circ - \angle BEK\,.$$

So the points B, E, K, A are concyclic. Similarly the points C, D, K, A are also concyclic.



Let M, N be the midpoints of AD and AE respectively. Since the triangle ACE is isosceles, the perpendicular bisector of AE, say ε_1, passes through the points C, K and N. Similarly, the perpendicular bisector of AD, say ε_2, passes through the points B, K and M. Therefore the points F, G lie on ε_1 and ε_2 respectively. Thus, using also the fact that $AKDC$ is a cyclic quadrilteral we get that

$$\angle FDC = \angle ADC = \angle AKC = \angle EKC = \angle EKF\,.$$

So the point F lies on the circle ω_1. Similarly G also lies on ω_1.

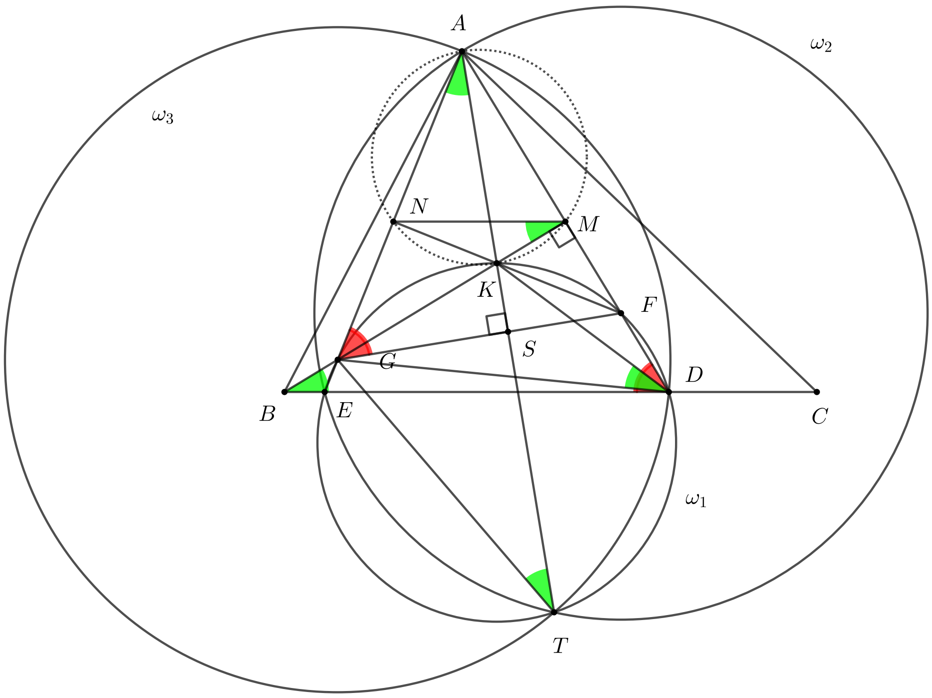
Let I be the point of intersection of the line AK with ω_1. The triangles AKF and EKF are equal, so $\angle KAF = \angle KEF$. Since also K, E, F, I all belong on ω_1 then

$$\angle KAF = \angle KEF = \angle FIK\,.$$

It follows that $FI = FA = FE$. Therefore I lies on ω_2 as well. Similarly it also lies on ω_3. So the circles $\omega_1, \omega_2, \omega_3$ all pass through I which lies on line AK.

Solution 2. Let M the midpoint of AD. Then BM is the perpendicular bisector of AD, because the triangle ABD is isosceles. KM is also the perpendicular bisector of AD, because the point K is the circumcenter of the triangle AED. So points B, G, K, M are collinear and GM is also the perpendicular bisector of AD. Therefore $GD = GA$ and so A belongs on ω_3. Similarly A belongs on ω_2.

Since ADG is isosceles with $GA = GD$, it follows that $\angle EGD = 2\angle GAD = 2\angle EAD$. Since AFE is isosceles with $FA = FE$, it follows that $\angle EFD = 2\angle FAE = 2\angle EAD$. We also have $EKD = 2\angle EAD$ as K is the circumcenter of the triangle EAD. From the last three equalities it follows that F, G belong on ω_1.



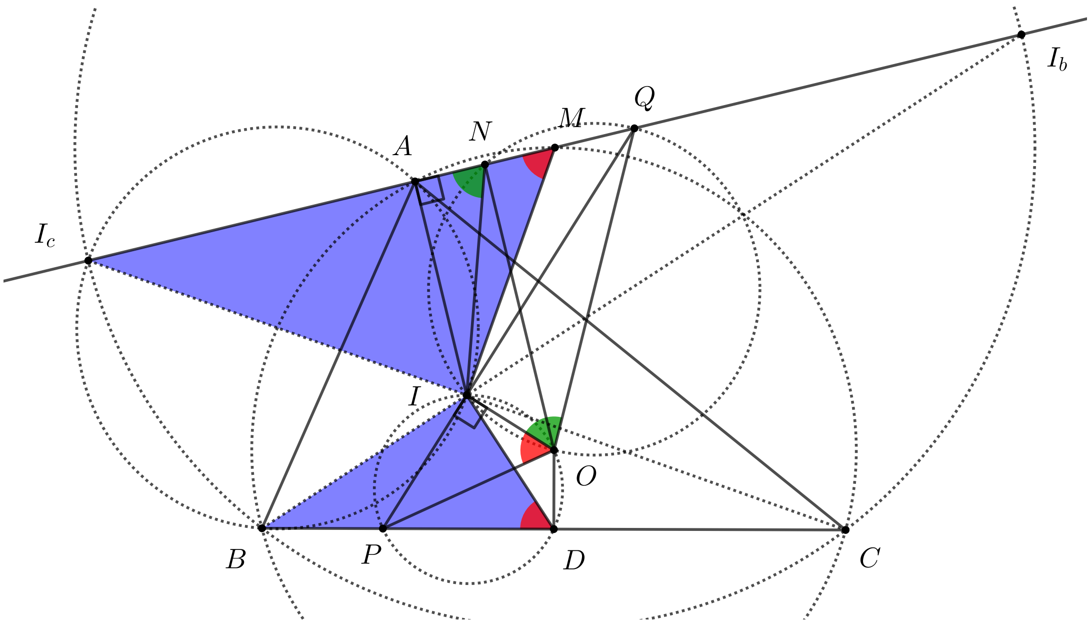
Let $T \neq A$ be the second point of intersection of the circles ω_2, ω_3 and let $S = AT \cap FG$. Let N be the midpoint of AE. Since $\angle AMK = \angle ANK = 90°$, then the points A, M, K, N are concyclic and therefore $\angle NAK = \angle NMK$. Since NM is parallel to ED (M, N midpoints of AD, AE) then $\angle NMK = \angle DBM = 90° - \angle MDB$. Since also D, E, G, F are concyclic, then $\angle MDB = \angle FGN = 90° - \angle GAS$. From the above, it follows that $\angle NAK = \angle GAS$ and so A, K, S are collinear. By definition of S, we get that T also belongs on the same line.

Since GF is the perpendicular bisector of AT then $\angle GAK = \angle GAS = \angle GTS = \angle GTK$. But since GK is the perpendiuclar bisector of AD we also have $\angle GAK = \angle GDK$. Thus $\angle GTK = \angle GDK$ showing that T belongs to ω_1 as well.

G2. Let I and O be the incenter and the circumcenter of a triangle ABC, respectively, and let s_a be the exterior bisector of angle $\angle BAC$. The line through I perpendicular to IO meets the lines BC and s_a at points P and Q, respectively. Prove that $IQ = 2IP$.

Proposed by Serbia

Solution. Denote by I_b and I_c the respective excenters opposite to B and C. Also denote the midpoint of side BC by D, the midpoint of the arc BAC by M, and the midpoint of segment AM by N. Recall that M is on the perpendicular bisector of BC, i.e. on line OD. Points I, O, D, P lie on the circle with diameter OP, whereas points I, O, Q, N lie on the circle with diameter OQ. Thus $\angle IOP = \angle IDP$ and $\angle IOQ = 180^\circ - \angle INQ = \angle INA$. So the triangles IAN and QIO are similar.



On the other hand, points B, C, I_b, I_c are on the circle with diameter I_bI_c, so the triangles IBC and II_cI_b are similar. We have $\angle II_cA = \angle CI_cI_b = \angle CBI_b = \frac{1}{2}\beta$. Since also $\angle IBA = \frac{1}{2}\beta = \angle II_cA$ then we deduce (the known fact) that I_c, A, I, B are concyclic. Thus $\angle BI_cA = 180^\circ - AIB = \frac{1}{2}(\alpha + \beta)$. Since also $I_cMB = AMB = ACB = \gamma$, then we also have that $\angle I_cBM = \angle BI_cA = \frac{1}{2}(\alpha + \beta)$. We deduce that $I_cM = MB = MC = I_bM$, i.e. M is the midpoint of I_bI_c.

It follows that the triangles IBD and II_cM are similar, so $\angle IOP = \angle IDP = \angle IMA$. Thus the triangles OIP and MAI are similar. Therefore

$$\frac{IQ}{IO} = \frac{IA}{AN} = \frac{2IA}{AM} = \frac{2IP}{IO}.$$

Thus $IQ = 2IP$.

G3. Let ABC be a triangle with $AB < AC$. Let ω be a circle passing through B, C and assume that A is inside ω. Suppose X, Y lie on ω such that $\angle BXA = \angle AYC$ and X lies on the opposite side of AB to C while Y lies on the opposite side of AC to B.

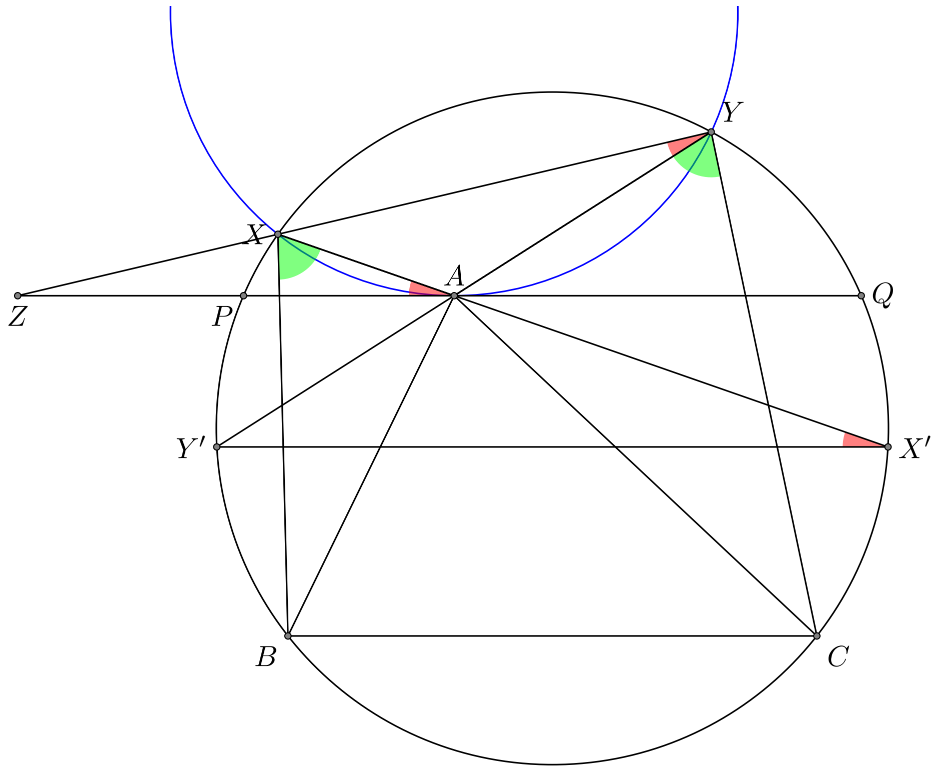
Show that, as X, Y vary on ω, the line XY passes through a fixed point.

Proposed by United Kingdom

Solution 1. Extend XA and YA to meet ω again at X' and Y' respectively. We then have that:

$$\angle Y'YC = \angle AYC = \angle BXA = \angle BXX'.$$

so $BCX'Y'$ is an isosceles trapezium and hence $X'Y' \parallel BC$.



Let ℓ be the line through A parallel to BC and let ℓ intersect ω at P, Q with P on the opposite side of AB to C. As $X'Y' \parallel BC \parallel PQ$ then

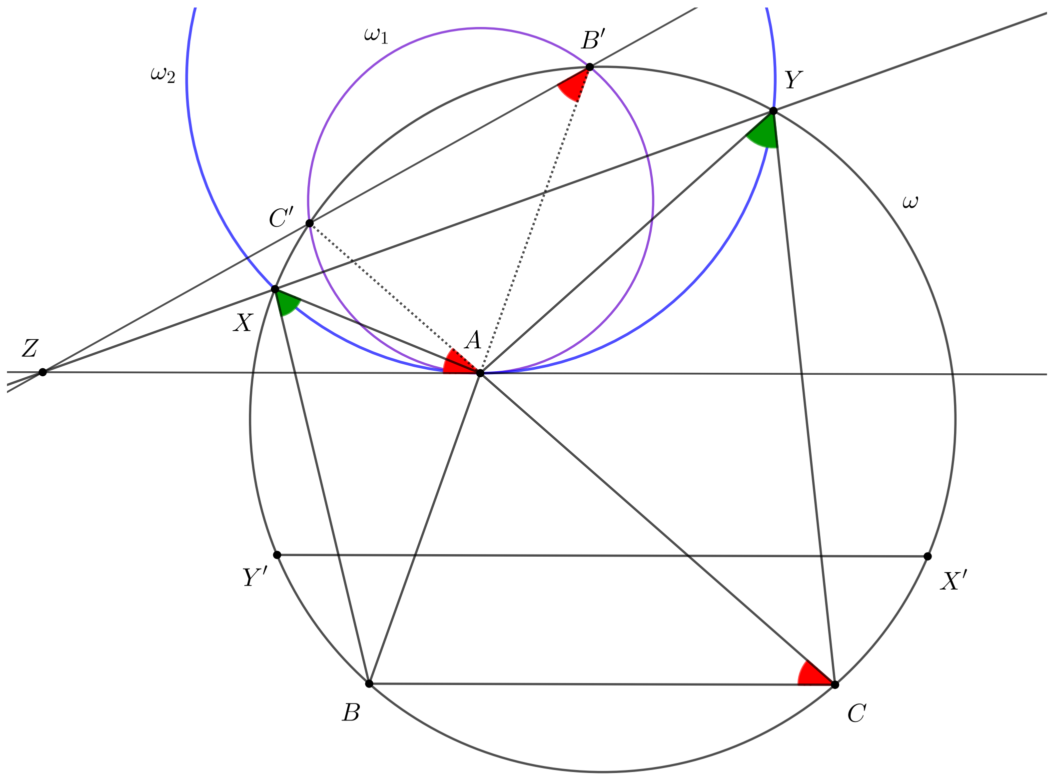
$$\angle XAP = \angle XX'Y' = \angle XYY' = \angle XYA$$

which shows that ℓ is tangent to the circumcircle of triangle AXY. Let XY intersect PQ at Z. By power of a point we have that

$$ZA^2 = ZX \cdot ZY = ZP \cdot ZQ.$$

As P, Q are independent of the positions of X, Y, this shows that Z is fixed and hence XY passes through a fixed point.

Solution 2. Let B' and C' be the points of intersection of the lines AB and AC with ω respectively and let ω_1 be the circumcircle of the triangle $AB'C'$. Let ε be the tangent to ω_1 at the point A. Because $AB < AC$ the lines $B'C'$ and ε intersects at a point Z which is fixed and independent of X and Y.



We have

$$\angle ZAC' = \angle C'B'A = \angle C'B'B = \angle C'CB \,.$$

Therefore, $\varepsilon \parallel BC$.

Let X', Y' be the points of intersection of the lines XA,YA with ω respecively. From the hypothesis we have $\angle BXX' = \angle Y'YC$. Therefore

$$\widehat{BX'} = \widehat{Y'C} \implies \widehat{BC} + \widehat{CX'} = \widehat{Y'B} + \widehat{BC} \implies \widehat{CX'} = \widehat{Y'B}$$

and so $X'Y' \parallel BC \parallel \varepsilon$. Thus

$$\angle XAZ = \angle XX'Y' = \angle XYY' = \angle XYA \,.$$

From the last equality we have that ε is also tangent to the circmucircle ω_2 of the triangle XAY.

Consider now the radical centre of the circles $\omega, \omega_1, \omega_2$. This is the point of intersection of the radical axes $B'C'$ (of ω and ω_1), ε (of ω_1 and ω_2) and XY (of ω and ω_2).

This must be point Z and therefore the variable line XY passes through the fixed point Z.

Remark: The condition that $AB < AC$ ensures that the point Z exists (rather than being at infinity). If $XY \parallel \ell \parallel BC$ then $AX = AY$ and $XB = YC$ so, as $\angle BXA = \angle AYC$, we would have $\triangle AXB \cong \triangle AYC$ and hence $AB = AC$.

G4. Let ABC be a right-angled triangle with $\angle BAC = 90^\circ$. Let the height from A cut its side BC at D. Let I, I_B, I_C be the incenters of triangles ABC, ABD, ACD respectively. Let also E_B, E_C be the excenters of ABC with respect to vertices B and C respectively. If K is the point of intersection of the circumcircles of $E_C I B_I$ and $E_B I C_I$, show that KI passes through the midpoint M of side BC.

Proposed by Greece

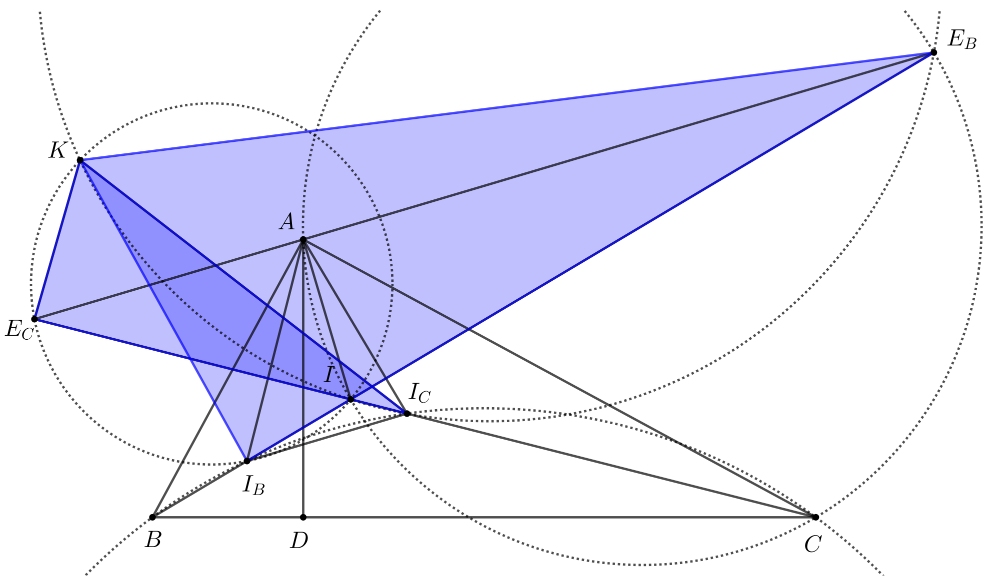
Solution. Since $\angle E_C BI = 90^\circ = ICE_B$, we conclude that $E_C BCE_B$ is cyclic. Moreover, we have that

$$\angle BAI_B = \frac{1}{2}\angle BAD = \frac{1}{2}\widehat{C}\,,$$

so $AI_B \perp CI$. Similarly $AI_C \perp BI$. Therefore is the orthocenter of triangle $AI_B I_C$. It follows that

$$\angle II_B I_C = 90^\circ - \angle AI_C I_B = \angle IAI_C = 45^\circ - \angle I_C AC = 45^\circ - \frac{1}{2}\widehat{B} = \frac{1}{2}\widehat{C}\,.$$

Therefore $I_B I_C CB$ is cyclic. Since $AE_B CI$ is also cyclic (on a circle of diameter IE_B) then



$$\angle E_C E_B B = \angle ACI = \frac{1}{2}\widehat{C} = \angle II_B I_C\,,$$

therefore $I_B I_C \parallel E_B E_C$.

From the inscribed quadrilaterals we get that

$$\angle KI_C I = \angle KE_B I \quad \text{and} \quad KE_C I = \angle KI_B I\,,$$

which implies that the triangles $KE_C I_C$ and $KI_B E_B$ are similar. So

$$\frac{d(K, E_C I_C)}{d(K, E_B I_B)} = \frac{E_C I_C}{E_B I_B}\,.$$

But $I_B I_C \parallel E_B E_C$ and $I_B I_C CB$ is cyclic, therefore

$$\frac{E_C I_C}{E_B I_B} = \frac{II_C}{II_B} = \frac{IB}{IC}\,.$$

We deduce that

$$\frac{d(K, IC)}{d(K, IB)} = \frac{IB}{IC},$$

i.e. the distances of K to the sides IC and IB are inversly analogous to the lenghts of these sides. So by a well known property of the median, K lies on the median of the triangle IBC. (The last property of the median can be proved either by the law of sines, or by taking the distances of the distances of the median M to the sides and prove by Thales theorem that M, I, K are collinear.)

G5. Let ABC be an acute triangle with $AC > AB$ and circumcircle Γ. The tangent from A to Γ intersects BC at T. Let M be the midpoint of BC and let R be the reflection of A in B. Let S be a point so that $SABT$ is a parallelogram and finally let P be a point on line SB such that MP is parallel to AB.

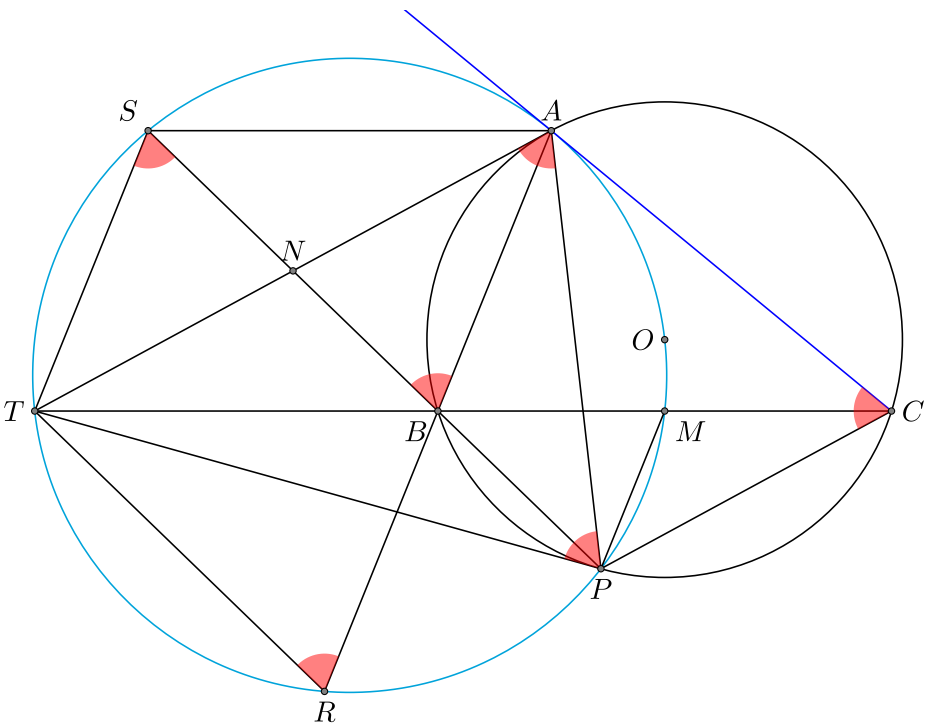
Given that P lies on Γ, prove that the circumcircle of $\triangle STR$ is tangent to line AC.

Proposed by United Kingdom

Solution 1. Let N be the midpoint of BS which, as $SABT$ is a parallelogram, is also the midpoint of TA. Using $ST \parallel AB \parallel MP$ we get:

$$\frac{NB}{BP} = \frac{1}{2} \cdot \frac{SB}{BP} = \frac{TB}{2 \cdot BM} = \frac{TB}{BC}$$

which shows that $TA \parallel CP$.



Let Ω be the circle with diameter OT. As $\angle OMT = 90^\circ = \angle TAO$ we have that A, M lie on Ω. We now show that P lies on Ω. As $TA \parallel CP$ and TA is tangent to Γ we have that $AP = AC$, so

$$\angle TAP = \angle ACP = \angle CPA = \angle CBA = \angle TMP$$

where in the last step we used the fact that $MP \parallel AB$. This shows that P lies on Ω. Furthermore, this shows that $\angle OPT = 90^\circ$ and so TP is also tangent to Γ.

Now we show that R, S lie on Ω which would show that Ω is the circumcircle of triangle STR. For S, using $ST \parallel AB$ and that TA tangent to Γ we have

$$\angle TSP = \angle ABS = \angle ACP = \angle TAP\,.$$

For R, the homothety with factor 2 centred at A takes BN to RT. So $BN \parallel RT$ and hence

$$\angle ART = \angle ABS = \angle TAP = \angle APT\,,$$

where the last step follows from $TA = TP$ as they are both tangents to Γ.

Finally, we observe that as TA tangent to Γ then

$$\angle TAC = 180^\circ - \angle CBA = \angle ABT = \angle TSA$$

which, by the alternate segment theorem, means that line AC is tangent to Ω as required.

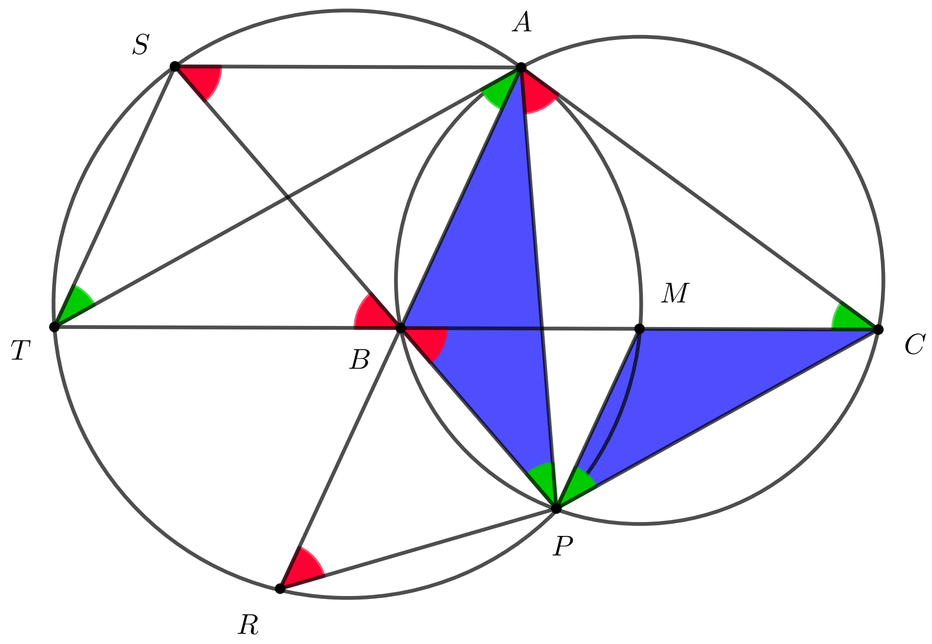
Solution 2. We have

$$\angle APS = \angle ACB = \angle TAB = \angle ATS\,,$$

so S, A, P, T are concyclic on a circle Ω. We also have

$$\angle PAC = \angle PBC = \angle SBT = \angle PSA$$

so AC is tangent to Ω. It remains to prove that R belongs on Ω.



As in Solution 1 we have that $TA \parallel CP$. Then

$$\angle CPM = \angle ATS = \angle APS\,.$$

Since also $\angle BAP = \angle BCP$, then the triangles APB and CPM are similar. But then the triangles BPC and RAP are also similar as $\angle RAP = \angle BCP$ and

$$\frac{RA}{AP} = \frac{2BA}{AP} = \frac{2MC}{CP} = \frac{BC}{CP}\,.$$

It now follows that

$$\angle ARP = \angle PBC = \angle ASP$$

and therefore R belongs to Ω as required.

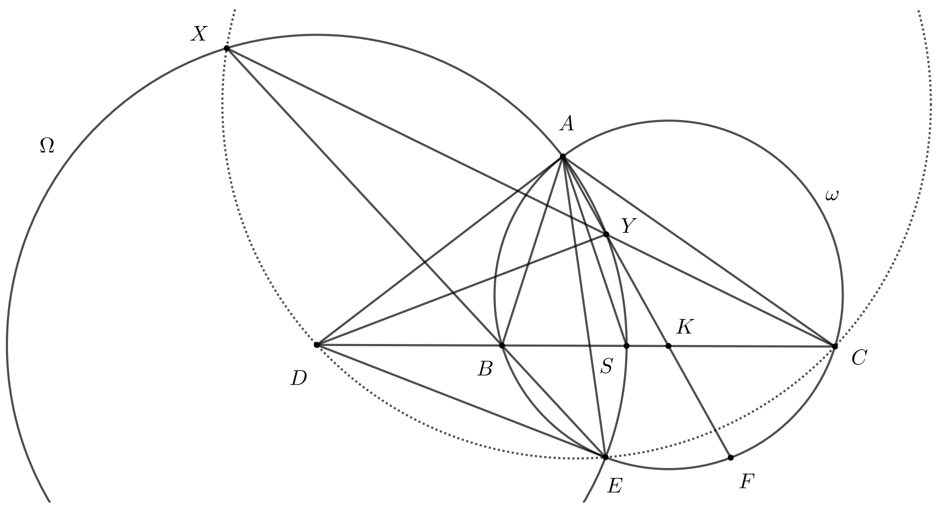
G6. Let ABC be an acute triangle such that $AB < AC$. Let ω be the circumcircle of ABC and assume that the tangent to ω at A intersects the line BC at D. Let Ω be the circle with center D and radius AD. Denote by E the second intersection point of ω and Ω. Let M be the midpoint of BC. If the line BE meets Ω again at X, and the line CX meets Ω for the second time at Y, show that A, Y and M are collinear.

Proposed by North Macedonia

Solution 1. Denote by S the intersection point of Ω and the segment BC. Because $DA = DS$, we have $\angle DSA = \angle DAS$. Now using that DA is tangent to ω we obtain:

$$\angle BAS = \angle DAS - \angle DAB = \angle DSA - \angle DCA = \angle CAS\,.$$

This means that the line AS is the angle bisector of $\angle BAC$.



Notice that DE is also tangent to ω, because it is the second intersection point of ω and Ω. From here, and from $DE = DX$, we see that

$$\angle DCE = \angle BCE = \angle BED = \angle DXE\,.$$

It follows that $CEDX$ is a cyclic quadrilateral.

Since D is the center of Ω, then $\angle EDY = 2\angle EXY$. Since $CEDX$ is cyclic, we also have

$$\angle SDE = \angle CDE = \angle CXE = \angle EXY\,.$$

Thus

$$2\angle SDE = 2\angle EXY = \angle EDY = \angle SDE + \angle SDY\,.$$

and so $\angle SDE = \angle SDY$. So we obtain

$$\angle SAE = \frac{1}{2}\angle SDE = \frac{1}{2}\angle SDY = \angle SAY\,.$$

Combining this with the fact that AS is the angle bisector of $\angle BAC$, we see that the lines AE and AY are symmetric with respect to the angle bisector of $\angle BAC$.

Now let F be the second intersection point of the line AY and the circumcircle ω. We have shown that $\angle BAE = \angle CAF$, which means that $BE = CF$ (two chords with the same corresponding central angle are equal). We similarly get $BF = CE$.

Since DA is tangent to ω, then $\angle BAD = \angle DCA$. Since also $\angle ADB = \angle CDA$ then the triangles DAB and DCA are similar. This gives.

$$\frac{AB}{AC} = \frac{AD}{CD}\,.$$

Similarly, the triangles DEB and DCE are similar, giving

$$\frac{BE}{CE} = \frac{ED}{CD}\,.$$

Combining these with $BE = CF$ and $BF = CE$ which we have shown above, and using that $DA = DE$ (tangents from the same point D), we get the relation

$$\frac{CF}{BF} = \frac{BE}{CE} = \frac{ED}{CD} = \frac{AD}{CD} = \frac{AB}{AC}\,.$$

Finally, let K be the intersection point of the line AY with the segment BC. We have

$$\frac{BK}{CK} = \frac{BK\sin(\angle BKA)}{BK\sin(\angle CKA)} = \frac{AB\sin(\angle BAK)}{AC\sin(\angle CAK)} = \frac{CF\sin(\angle BCF)}{BF\sin(\angle CBF)} = 1\,.$$

Thus $K = M$ and A, Y, M are collinear as required.

Solution 2. As in Solution 1, we let S be the intersection of Ω with BS and obtain that AS is the angle bisector of $\angle BAC$ and that AE and AY are symmetric with respect to AS.

Let $R = \sqrt{(AB)(AC)}$ and let Ψ be the map obtained by first inverting on the circle centered at A of radius R and the reflecting on AS.

By construction of Ψ we have $\Psi(B) = C$ and $\Psi(C) = B$. (After the inversion B maps to a point B' on AB such that $(AB)(AB') = R^2 = (AB)(AC)$. So after the reflection B' maps to C.) Since the inversion of any line not passing through A is a circle passing through A, then $\Psi(BC)$ is a circle passing through A. Since it also passes through B and C then $\Psi(BC) = \omega$.

Because DA is tangent to ω at A, and D is the center of Ω, the circles ω and Ω are orthogonal. Both reflection and inversion preserve orthogonality and both are involutions. This means that Ψ is an involution that preserves orthogonality. From here we conlude that the images $\Psi(\omega) = BC$ and $\Psi(\Omega)$ are orthogonal lines.

Since $\Psi(AS) = AS, \Phi(BC) = \omega$ and S belongs on BC, then $\Psi(S)$ is the intersection of AS with ω. Since AS is the angle bisector of triangle ABC, then $\Psi(S) = N$, the midpoint of the arc BC of ω not containing A.

Since S belongs on Ω and $\Psi(\Omega)$ and $\Psi(\omega)$ are orthogonal lines, then $\Psi(\Omega)$ is the line perpendicular to BC at N. It therefore contains the midpoint M of BC.

The intersection point E of ω and Ω maps to $\Psi(E)$, which is the intersection point of $\Psi(\omega) = BC$ and $\Psi(\Omega) = MN$, which must be equal to M, i.e. $\Psi(E) = M$. Because of this, we see that AE and AM are symmetric with respect to the angle bisector AS. Since also AE and AY are symmetric with respect to AS, it follows that A, M, Y are collinear as required.

G7. Let ABC be an acute scalene triangle. Its C-excircle tangent to the segment AB meets AB at point M and the extension of BC beyond B at point N. Analogously, its B-excircle tangent to the segment AC meets AC at point P and the extension of BC beyond C at point Q. Denote by A_1 the intersection point of the lines MN and PQ, and let A_2 be defined as the point, symmetric to A with respect to A_1. Define the points B_2 and C_2, analogously. Prove that $\triangle ABC$ is similar to $\triangle A_2B_2C_2$.

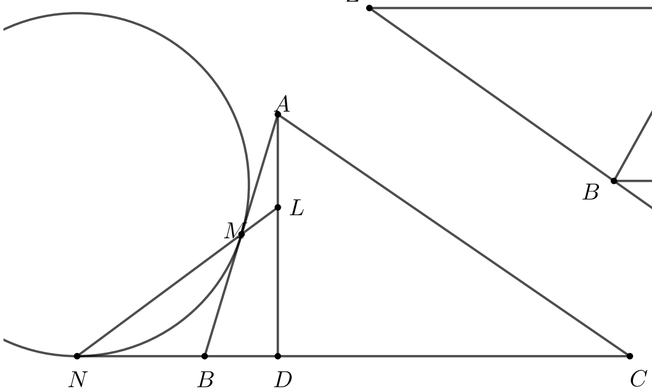
Proposed by Bulgaria

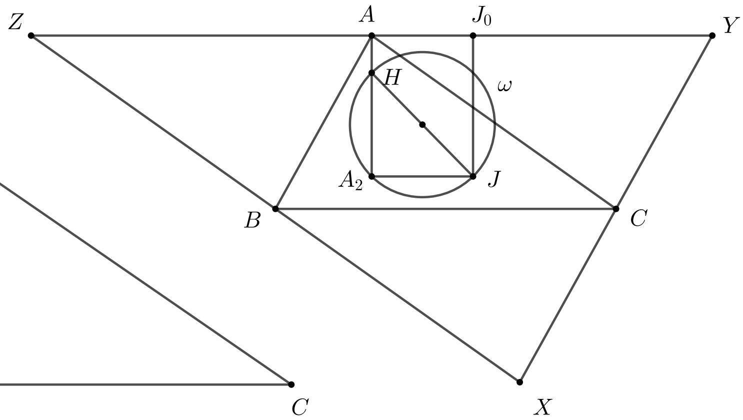
Solution 1. We shall use the standard notations for ABC, i.e. $\angle ABC = \beta$, $BC = a$ etc. We also write $s = \frac{a+b+c}{2}$ for the semiperimeter and r for the inradius.

Let MN intersect the altitude AD (D lies on BC) at the point L. We have that $\angle BAD = 90^\circ - \beta$ and $\angle AML = \angle BMN = \frac{\beta}{2}$. (Since BMN is an isosceles triangle with $\angle MBN = 180^\circ - \beta$.) It is known that $AM = s - b$ so by the Sine Law in the triangle AML we have

$$\frac{AM}{\sin\angle ALM} = \frac{AL}{\sin\angle AML} \implies \frac{s-b}{\sin(90^\circ + \frac{\beta}{2})} = \frac{AL}{\sin\frac{\beta}{2}} \implies AL = (s-b)\tan\frac{\beta}{2} = r\,.$$

Analogously we see that if PQ intersects AD at L', then $AL' = r$. Therefore L and L' coincide and since $A_1 = MN \cap PQ$ by definition, we conclude that $L = L' = A_1$. In particular, we can now view the point A_2 as the point on the A-altitude such that $AA_2 = 2r$. Analogously B_2 and C_2 lie on the B-altitude and C-altitude, respectively, and $BB_2 = CC_2 = 2r$.





Now let X be the reflection of A on the midpoint of BC and define XYZ analogously. So XYZ is the triangle whose midpoints of sides are A, B and C. Let J be the incenter of this triangle. As the triangles XYZ and ABC are similar with ratio 2, the inradius of XYZ is equal to $2r$. So if JJ_0 is perpendicular to YZ (with J_0 on YZ), then AA_2 and JJ_0 are parallel (both perpendicular to YZ) and equal, hence AA_2JJ_0 is a rectangle and in particular A_2 is the foot of the perpendicular from J to the A-altitude of ABC. It follows that A_2, B_2 and C_2 lie on the circle ω with diameter JH.

Now we finish with a simple angle chasing. The circle k gives $\angle A_2B_2C_2 = \angle A_2HC_2 = \angle 180^\circ - \angle AHC = \angle ABC$; similarly for the angles at A_2 and C_2. The desired similarity follows.

Solution 2. As in Solution 1, we have that A_2, B_2, C_2 belong on the corresponding altitudes with $AA_2 = BB_2 = CC_2 = 2r$. We present an approach with complex numbers (and minimal calculations) which can also complete the proof.

Set the incenter I of the triangle ABC to be the origin. We may assume that $r = 1$. We write a, b, c to denote A', B', C'. Point A is the intersection of the tangents to the unit circle (incircle) at B' and C' and is therefore represented by the complex number $2bc/(b + c)$. Analogously the points B and C are represented by $2ac/(a + c)$ and $2ab/(a + b)$ respectively.

Since $AA_2 = r = 2$ and AA_2 is parallel to IA', we have that A_2 is represented by the complex number

$$\frac{2bc}{b+c} + 2a = \frac{2(ab + bc + ca)}{b+c} .$$

Now since $|c| = 1$, then

$$(AB) = \left|\frac{bc}{b+c} - \frac{ac}{a+c}\right| = \left|\frac{b-a}{(a+c)(b+c)}\right| .$$

We also have

$$(A_2B_2) = \left|\frac{2(ab+bc+ca)}{b+c} - \frac{2(ab+bc+ca)}{a+c}\right| = 2|ab+bc+ca|(A_2B_2) .$$

Analogously we get

$$\frac{A_2B_2}{AB} = \frac{B_2C_2}{BC} = \frac{C_2A_2}{CA} = 2|ab+bc+ca| .$$

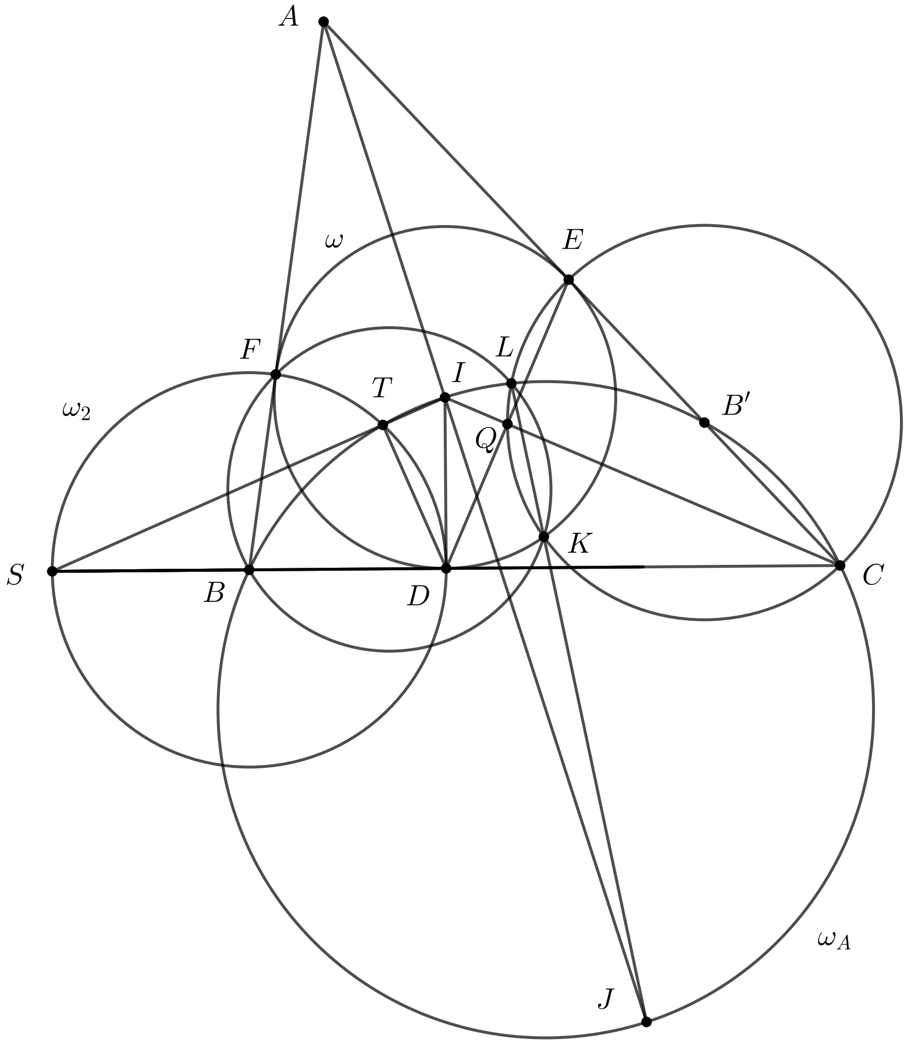
So the triangle $A_2B_2C_2$ is similar to the triangle ABC.

G8. Let ABC be a scalene triangle and let I be its incenter. The projections of I on BC, CA and AB are D, E and F respectively. Let K be the reflection of D over the line AI, and let L be the second point of intersection of the circumcircles of the triangles BFK and CEK. If $\frac{1}{3}BC = AC - AB$, prove that $DE = 2KL$.

Proposed by Romania

Solution. Writing $AE = AF = x, BF = BD = y$ and $CE = CD = z$, the condition $\frac{1}{3}BC = AC - AB$ translates to $y + z = 3(z - y)$ giving $z = 2y$, i.e. $CD = 2BD$.

Letting B' be the reflection of B on AI we have that B' belongs on AC with $B'E = BF = BD = \frac{1}{2}CD = \frac{1}{2}CE$ therefore B' is the midpoint of CE.



Under reflection on AI, the circumcircle ω of triangle DEF remains fixed. Its tangent BD maps to $B'K$. So $B'K$ is tangent to ω. Since $B'E$ is tangent to ω, then $B'E = B'K = B'C$. Thus CKE is a right-angled triangle with diameter CE. If Q is the midpoint of DE then, since $CD = CE$, we have that $\angle CQE = 90^\circ$ and therefore the points C, K, Q, L, E are concyclic.

Observe that

$$\begin{aligned}\angle BLC &= \angle BLK + \angle CLK = \angle BFK + \angle CEK = (180^\circ - \angle AFK) + (180^\circ - \angle AEK)\\ &= \angle BAC + \angle FKE = \angle BAC + \angle FDE = \angle BAC + \left(90^\circ - \frac{1}{2}\angle BAC\right)\\ &= 90^\circ + \frac{1}{2}\angle BAC = \angle BIC\,.\end{aligned}$$

So L belongs on the circumcircle of triangle BIC, i.e. on the A-excircle ω_A of triangle ABC.

Let J be the A-excenter of triangle ABC and recall that it is the antipodal point of I on ω_A. Then

$$\angle CLJ = \angle CBJ = 90^\circ - \frac{1}{2}\angle ABC = \angle BFD = \angle CEK = \angle CLK\,.$$

So K, L, J are collinear and therefore $\angle ILK = 90^\circ$.

Let T be the reflection of L on AI. Since L belongs on the circle with centre B' containing E and K, then L belongs on the circle ω_2 with centre B containing F and D. Let S be the intersection of IT and BC. Since $KL \perp IL$, then $DT \perp IT$. It follows that $\angle IDT = 90^\circ - \angle DIS = \angle ISD$. Since ID is tangent on ω_2, then S belongs on ω_2. Then $SD = 2BD = DC$ and so the triangles IDC and IDS are equal. Their height DT and DQ must be equal. Therefore $DE = 2DQ = 2DT = 2KL$ as required.

NUMBER THEORY

N1. Let $n \geqslant 3$ be an integer and let

$$M=\left\{\frac{a_1+a_2+\cdots+a_k}{k}: 1 \leqslant k \leqslant n \text{ and } 1 \leqslant a_1<\cdots<a_k \leqslant n\right\}$$

be the set of the arithmetic means of the elements of all non-empty subsets of $\{1,2,\ldots,n\}$.

Find $\min\{|a-b| : a, b \in M \text{ with } a \neq b\}$.

Proposed by Romania

Solution. We observe that M is composed by rational numbers of the form $a=\frac{x}{k}$, where $1 \leqslant k \leqslant n$. As the arithmetic mean of $1,\ldots,n$ is $\frac{n+1}{2}$, if we look at these rational numbers in their irreducible form, we can say that $1 \leqslant k \leqslant n-1$.

A non-zero difference $|a-b|$ with $a, b \in M$ is then of form

$$\left|\frac{x}{k}-\frac{y}{p}\right|=\frac{|p_0 x-k_0 y|}{[k,p]},$$

where $[k,p]$ is the l.c.m. of k,p, and $k_0=\frac{[k,p]}{k}, p_0=\frac{[k,p]}{p}$. Then $|a-b| \geqslant \frac{1}{[k,p]}$, as $|p_0 x-k_0 y|$ is a non-zero integer. As

$$\max\{[k,p] | 1 \leqslant k<p \leqslant n-1\}=(n-1)(n-2),$$

we can say that $m=\min\limits_{\substack{a,b \in M \\ a \neq b}}|a-b| \geqslant \frac{1}{(n-1)(n-2)}$.

To reach this minimum, we seek $x \in\{3,4,\ldots,2n-1\}$ and $y \in\{1,2,\ldots,n\}$ for which

$$\left|\frac{\frac{n(n+1)}{2}-x}{n-2}-\frac{\frac{n(n+1)}{2}-y}{n-1}\right|=\frac{1}{(n-1)(n-2)},$$

meaning

$$\left|\frac{n(n+1)}{2}-(n-1)x+(n-2)y\right|=1.$$

If $n=2k$, we can choose $x=k+3$ and $y=2$ and if $n=2k+1$ we can choose $x=n=2k+1$ and $y=k$. Therefore, the required minimum is $\frac{1}{(n-1)(n-2)}$.

Comment. For $n \geqslant 5$, the only other possibilities are to take $x=3k-1, y=2k-1$ if $n=2k$ and to take $x=2k+3, y=k+2$ if $n=2k+1$. (For $n=3,4$ there are also examples where one of the sets is of size n.)

N2. Denote by $\ell(n)$ the largest prime divisor of n. Let $a_{n+1} = a_n + \ell(a_n)$ be a recursively defined sequence of integers with $a_1 = 2$. Determine all natural numbers m such that there exists some $i \in \mathbb{N}$ with $a_i = m^2$.

Solution. We will show that all such numbers are exactly the prime numbers.

Let $p_1, p_2, \ldots$ be the sequence of prime numbers. We will prove the following:

Claim: Assume $a_n = p_i p_{i+1}$. Then for each $k = 1, 2, \ldots, p_{i+2} - p_i$ we have that $a_{n+k} = (p_i + k)p_{i+1}$.

Proof. By induction on k. Since $\ell(a_n) = p_{i+1}$, then $a_{n+1} = p_i p_{i+1} + p_{i+1} = (p_i + 1)p_{i+1}$. Assume now that $a_{n+r} = (p_i + r)p_{i+1}$ for some $r < p_{i+2} - p_i$. For the inductive step, it is enough to show that $\ell(a_{n+r}) = p_{i+1}$ as then we would have $a_{n+r} = (p_i + r)p_{i+1} + p_{i+1} = (p_i + r + 1)p_{i+1}$. Assume for contradiction that $\ell(a_{n+r}) \neq p_{i+1}$. Since $p_{i+1} | a_{n+r}$, then we must have that $\ell(a_{n+r}) > p_{i+1}$. Since also $a_{n+r} = (p_i + r)p_{i+1}$, then $\ell(p_i + r) > p_{i+1}$ and therefore $\ell(p_i + r) \geqslant p_{i+2}$. This is impossible as $p_i + r < p_{i+2}$. $\square$

Since $a_1 = 2, a_2 = 4, a_3 = 6 = 2 \cdot 3 = p_1 p_2$, from the above claim, by induction, we can break up the sequence into pieces of the form $p_i p_{i+1}, (p_i + 1)p_{i+1}, \ldots, p_{i+2} p_{i+1}$ for $i = 1, 2, \ldots$, together with the initial piece $2, 4$.

We immediately see that for each prime p, the number p^2 appears in the sequence. It remains to show that no other square number appears in the sequence.

Assume for contradiction that another square appears in $p_i p_{i+1}, (p_i + 1)p_{i+1}, \ldots, p_{i+2} p_{i+1}$ for some i. Since all elements of this piece are multiples of p_{i+1}, if a square appears in this sequence, it must be a multiple of p_{i+1}^2. So the smallest possible square different from p_{i+1}^2 is $4p_{i+1}^2$. It is enough to show that $4p_{i+1}^2 > p_{i+2} p_{i+1}$. This is equivalent to showing that $p_{i+2} < 4p_{i+1}$ which follows from Bertrand's postulate.

N3. Let n be a positive integer. Determine, in terms of n, the greatest integer which divides every number of the form $p+1$, where $p \equiv 2 \bmod 3$ is a prime number which does not divide n.

Proposed by Bulgaria

Solution. Let k be the greatest such integer. We will show that $k = 3$ when n is odd and $k = 6$ when n is even.

We will say that a number p is nice if p is a prime number of the form 2 mod 3 which does not divide N.

Note first that if $3|p+1$ for every nice number p and so k is a multiple of 3.

If n is odd, then $p = 2$ is nice, so we must have $k|3$. From the previous paragraph we get that $k = 3$.

If n is even, then $p = 2$ is not nice, therefore every nice p is of the form 5 mod 6. So in this case $6|p+1$ for every nice number p.

It remains to show that (if n is even then)

(i) There is a nice p such that $4 \nmid p+1$.

(ii) There is a nice p such that $9 \nmid p+1$.

(iii) There is a nice p such that for every prime $q \neq 2, 3$ we have that $q \nmid p+1$.

For (i), by Dirichlet's theorem on arithmetic progressions, there are infinitely many primes of the form $p \equiv 5 \bmod 12$. Any one of them which is larger than n will do.

For (ii), by Dirichlet's theorem on arithmetic progressions, there are infinitely many primes of the form $p \equiv 2 \bmod 9$. Any one of them which is larger than n will do.

For (iii), by Dirichlet's theorem on arithmetic progressions, there are infinitely many primes of the form $p \equiv 2 \bmod 3q$. Any one of them which is larger than n will do.

Remark. In the proposal, the statement of Dirichlet's theorem on Arithmetic Progressions was given as known. Even though this makes the problem fairer we omitted it because we feel that it also makes it easier.

N4. Can every positive rational number q be written as

$$\frac{a^{2021}+b^{2023}}{c^{2022}+d^{2024}},$$

where a, b, c, d are all positive integers?

Proposed by United Kingdom

Solution. The answer is yes. Set $a = x^{2023}, b = x^{2021}$ and $c = y^{2024}, d = y^{2022}$ for some integers x, y and let $q = \frac{m}{n}$ in lowest terms. Then we could try to solve

$$\frac{a^{2021}+b^{2023}}{c^{2022}+d^{2024}} = \frac{2x^{2021\times 2023}}{2y^{2022\times 2024}} = \frac{x^{2021\times 2023}}{y^{2022\times 2024}} = \frac{m}{n}\,.$$

Consider setting $x = m^{x_1}n^{x_2}$ and $y = m^{y_1}n^{y_2}$. Then by considering powers of m and powers of n separately, it would be sufficient to solve the pair of equations

$$2021 \times 2023x_1 - 2022 \times 2024y_1 = 1, \quad \text{and} \quad 2021 \times 2023x_2 - 2022 \times 2024y_2 = -1\,.$$

We know that these equations have solutions in positive integers so long as 2021×2023 and 2022×2024 are coprime. Amongst integers which differ by at most three, the only possible common prime factors are 2 and 3. Clearly 2 is not a common prime factor of the products, nor is 3, since only one of the four factors is divisible by 3. So these two integers are coprime, and the equations have solutions.

N5. A natural number n is given. Determine all $(n-1)$-tuples of nonnegative integers $a_1, a_2, \dots, a_{n-1}$ such that

$$\left[\frac{m}{2^n-1}\right] + \left[\frac{2m+a_1}{2^n-1}\right] + \left[\frac{2^2m+a_2}{2^n-1}\right] + \left[\frac{2^3m+a_3}{2^n-1}\right] + \cdots + \left[\frac{2^{n-1}m+a_{n-1}}{2^n-1}\right] = m$$

holds for all $m \in \mathbb{Z}$.

Proposed by Serbia

Solution 1. We will show that there is a unique such n-tuple: $a_k = 2^{n-1} + 2^{k-1} - 1$ for $k = 1, \dots, n-1$.

Write $N = 2^n - 1$ and $f_k(x) = \left[\frac{2^k x + a_k}{N}\right]$ for $k = 0, 1, \dots, n-1$, where $a_0 = 0$. Since

$$\sum_{k=0}^{n-1} f_k(m) - \sum_{k=0}^{n-1} f_k(m-1) = 1\,,$$

for each $m \in \mathbb{Z}$, there is exactly one k for which $f_k(m) = f_k(m-1) + 1$. We work modulo N. The last equality holds if and only if $2^k m + a_k \in \{0, 1, \dots, 2^k - 1\}$. I.e. if and only if

$$2^k m \in \{-a_k, 1-a_k, \dots, 2^k - 1 - a_k\}\,.$$

Multiplying with 2^{n-k}, and noting that $2^n \equiv 1 \bmod N$, we get the following:

For each $m \in \mathbb{Z}$ there is a unique $k \in \{0, 1, \dots, n-1\}$ such that $m \in B_k$ (modulo N) where

$$B_k = \{b_k, b_k + 2^{n-k}, \dots, b_k + (2^k - 1)2^{n-k}\}$$

with $b_k = -2^{n-k} a_k$. Therefore the problem condition is equivalent to $\bigcup_{k=0}^{n-1} B_k$ being a partition of $\{0, 1, \dots, N-1\}$.

For a number b and set a $A \subseteq \mathbb{Z}$ we write $b + A = \{b + a : a \in A\}$. With this notation, $B_{n-1} = b_{n-1} + \{0, 2, 4, \dots, 2^n - 2\}$. The set $B_{n-2} = b_{n-2} + \{0, 4, 8, \dots, 2^n - 4\}$ is contained in $\overline{B_{n-1}} = b_{n-1} + \{1, 3, \dots, 2^n - 3\}$, implying $b_{n-2}, b_{n-2} + 2^n - 4 \in \overline{B_{n-1}}$, which holds only if $b_{n-2} \equiv b_{n-1} + 1$. Further, the set $B_{n-3} = b_{n-3} + \{0, 8, 16, \dots, 2^n - 8\}$ is contained in $\overline{B_{n-1} \cup B_{n-2}} = b_{n-1} + \{3, 7, \dots, 2^n - 5\}$, so we must have $b_{n-3} \equiv b_{n-1} + 3$. Similarly, $b_{n-4} \equiv b_{n-1} + 7$ etc. In general, $b_{n-k} \equiv b_{n-1} + 2^{k-1} - 1$ for $k = 1, \dots, n-1$. It follows that $b_0 \equiv b_{n-1} + 2^{n-1} - 1$. On the other hand, we have $b_0 = 0$, which gives $b_{n-1} \equiv 1 - 2^{n-1}$ and therefore $b_k \equiv 2^{n-1-k} - 2^{n-1}$. Thus $a_k \equiv -2^k b_k \equiv 2^{n+k-1} - 2^{n-1} \equiv 2^{n-1} + 2^{k-1} - 1$ for $k = 1, \dots, n-1$.

Finally, $\sum_k f_k(0) = 0$ implies $a_k < N$ for all k, so we conclude that $a_k = 2^{n-1} + 2^{k-1} - 1$ for each $k = 1, 2, \dots, n-1$.

Solution 2. We will use the identity

$$[x] + \left[x + \frac{1}{N}\right] + \left[x + \frac{2}{N}\right] + \cdots + \left[x + \frac{N-1}{N}\right] = [Nx]$$

which holds for every $x \in \mathbb{R}$ and every $N \in \mathbb{N}$. (One can check this by noting that the difference between the two sides of the identity is periodic with period $1/N$ and that the identity clearly holds for $x \in [0, \frac{1}{N})$.)

Writing $a_0 = 0$ and $N = 2^n - 1$ we observe that

$$m = \sum_{k=0}^{n-1} \left[\frac{2^k m + a_k}{N}\right] = \sum_{r=0}^{2^k-1} \sum_{r=0}^{2^k-1} \left[\frac{m + \frac{a_k}{2^k}}{N} + \frac{r}{2^k}\right] = \sum_{k=0}^{n-1} \sum_{r=0}^{2^k-1} \left[\frac{m + \frac{a_k + rN}{2^k}}{N}\right]\,. \tag{1}$$

It follows that $c_{r,k} = \left[\frac{a_k+rN}{2^k}\right]$ are all distinct modulo N for $k = 0, 1, \ldots, n-1$ and $r = 0, 1, \ldots, 2^k - 1$. Indeed if two (or more) of them are congruent to t, then writing $f(t)$ for the right hand side of (1) we get $1 = f(-t) - f(-t-1) \geqslant 2$, a contradiction.

Since $N = 2^n - 1$, then $c_{r,k} = r2^{n-k} + d_{r,k}$, where $d_{r,k} = \left[\frac{a_k - r}{2^k}\right]$. Because $c_{0,0} = 0$, then $c_{0,k} \neq 0$ for each $k \neq 0$ giving $a_k \geqslant 2^k$ for each $k \geqslant 1$. Setting $m = 0$ in the original equation gives $a_k < N$ for each k and so $d_{0,k} \leqslant 2^{n-k} - 1$ for each k. Furthermore

$$2^{n-k} - 1 \geqslant d_{0,k} \geqslant d_{1,k} \geqslant \cdots \geqslant d_{2^k-1,k} \geqslant d_{2^k,k} = d_{0,k} - 1 \geqslant 0\,. \tag{2}$$

In particular $0 \leqslant c_{r,k} = r2^{n-k} + d_{r,k} \leqslant (2^n - 2^{n-k}) + (2^{n-k} - 1) = N$. For $k = 0, 1, 2, \ldots, n-1$ define $A_k = \{c_{r,k} : r = 0, 1, \ldots, 2^k - 1\}$. From the above, since $A_0 = \{0\}$, we must have that $A_1 \cup A_2 \cup \cdots \cup A_{n-1} = \{1, 2, \ldots, N-1\}$.

For a natural number t let $v_2(t)$ be as usual the largest exponent such that $2^{v_2(t)} | t$. Let

$$f(t) = n - v_2(t) - 1\,, \quad g(t) = \frac{t - 2^{v_2(t)}}{2^{1+v_2(t)}}\,, \quad \text{and} \quad h(t) = 2^{f(t)} - 1 - g(t)\,.$$

Note that $f(t)$ uniquely determines $v_2(t)$ and together with $g(t)$ they uniquely determine t. Similarly $h(t)$ and $g(t)$ uniquely determine t.

Claim. For each $t \in \{1, 2, \ldots, 2^{n-1} - 1\}$ we have:

(i) $d_{g(t),f(t)} = 2^{v_2(t)}$,

(ii) $d_{h(t),f(t)} = 2^{v_2(t)} - 1$,

(iii) $c_{g(t),f(t)} = t$,

(iv) $c_{h(t),f(t)} = N - t$.

Proof of Claim. We proceed by induction on t. For $t = 1$ we have $v_2(1) = 0, f(1) = n-1, g(1) = 0$ and $h(1) = 2^{n-1} - 1$. From (2) we have $1 \geqslant d_{0,n-1}$ and $d_{0,n-1} - 1 \geqslant 0$ proving (i). Also, $c_{g(1),f(1)} = c_{0,n-1} = d_{0,n-1} = 1$ proving (iii). From (2) we have $1 \geqslant d_{2^{n-1}-1,n-1} \geqslant 0$. But $c_{2^{n-1}-1,n-1} = 2^n - 2 + d_{2^{n-1}-1,n-1} = N - 1 + d_{2^{n-1}-1,n-1}$. Since $c_{2^{n-1}-1,n-1} \leqslant N - 1$ we deduce both (ii) and (iv).

Assume now that the result is true for $t = s - 1$. We will prove the result for $t = s$.

Case 1: If $s - 1 = 2u$ is even, then $v_2(s) = 0$, so $f(s) = n-1, g(s) = u$ and $h(s) = 2^{n-1} - 1 - u$.

By the induction hypothesis, since all the $c_{r,k}$'s are distinct, we must have

$$s \leqslant c_{g(s),f(s)} = 2u + d_{g(s),f(s)} = s - 1 + d_{g(s),f(s)}$$

and

$$N - s \geqslant c_{h(s),f(s)} = 2^n - 2 - 2u + d_{h(s),f(s)} = N - s + d_{h(s),f(s)}\,.$$

From the above we must have $d_{g(s),f(s)} \geqslant 1$ and $d_{h(s),f(s)} \leqslant 0$. But from (2) any two $d_{r,k}$'s for fixed k differ by at most 1. This can only be achieved if we have equalities everywhere proving (i)-(iv).

Case 2: If $s - 1 = 2u + 1$ is odd, then we write $s = 2u + 2 = 2^v w$ for some odd w. Then $v_2(s) = v$ and so $k = f(s) = n - 1 - v$ and $r = g(s) = (w-1)/2$. Also $h(s) = 2^k - 1 - r$. By the induction hypothesis we must have

$$s \leqslant c_{r,k} = r2^{n-k} + d_{r,k} = 2^v(w-1) + d_{r,k} = s - 2^v + d_{r,k}$$

and

$$\begin{aligned} N - s \geqslant c_{h(s),k} &= (2^k - 1 - r)2^{n-k} + d_{h(s),k} \\ &= 2^n - 2^{v+1} - s + 2^v + d_{h(s),k} \\ &= N + 1 - s - 2^v + d_{h(s),k}\,. \end{aligned}$$

From the above we must have $d_{r,k} \geqslant 2^v$ and $d_{h(s),k} \leqslant 2^v - 1$. As in Case 1 we must have equalities everywhere proving (i)-(iv). □

For $t = 2^{n-1} - 2^{n-k-1}$ we have $v_2(t) = n - k - 1, f(t) = k, g(t) = 2^{k-1} - 1$ and $h(t) = 2^k - 1 - (2^{k-1} - 1) = 2^{k-1}$. Thus from (ii) and (iv) we get

$$\left\lceil \frac{a_k - (2^{k-1} - 1)}{2^k} \right\rceil = 2^{n-k-1} \quad \text{and} \quad \left\lceil \frac{a_k - 2^{k-1}}{2^k} \right\rceil = 2^{n-k-1} - 1\,.$$

This is only possible if $a^k = 2^k \cdot 2^{n-k-1} + (2^{k-1} - 1) = 2^{n-1} + 2^{k-1} - 1$ as required.

N6. Let a, b and c be positive integers satisfying the equation $(a,b)+[a,b]=2021^c$. If $|a-b|$ is a prime number, prove that the number $(a+b)^2+4$ is composite.

Proposed by Serbia

Solution. We write $p=|a-b|$ and assume for contradiction that $q=(a+b)^2+4$ is a prime number.

Since $(a,b)\mid[a,b]$, we have that $(a,b)\mid 2021^c$. As (a,b) also divides $p=|a-b|$, it follows that $(a,b)\in\{1,43,47\}$. We will consider all 3 cases separately:

(1) If $(a,b)=1$, then $1+ab=2021^c$, and therefore

$$q=(a+b)^2+4=(a-b)^2+4(1+ab)=p^2+4\cdot 2021^c\,. \tag{1}$$

(a) Suppose c is even. Since $q\equiv 1 \bmod 4$, it can be represented uniquely (up to order) as a sum of two (non-negative) squares. But (1) gives potentially two such representations so in order to have uniqueness we must have $p=2$. But then $4|q$ a contradiction.

(b) If c is odd then $ab=2021^c-1\equiv 1 \bmod 3$. Thus $a\equiv b \bmod 3$ implying that $p=|a-b|\equiv 0 \bmod 3$. Therefore $p=3$. Without loss of generality $b=a+3$. Then $2021^c=ab+1=a^2+3a+1$ and so

$$(2a+3)^2=4a^2+12a+9=4\cdot 2021^c+5\,.$$

So 5 is a quadratic residue modulo 47, a contradiction as

$$\left(\frac{5}{47}\right)=\left(\frac{47}{5}\right)=\left(\frac{2}{5}\right)=-1\,.$$

(2) If $(a,b)=43$, then $p=|a-b|=43$ and we may assume that $a=43k$ and $b=43(k+1)$, for some $k\in\mathbb{N}$. Then $2021^c=43+43k(k+1)$ giving that

$$(2k+1)^2=4k^2+4k+4-3=4\cdot 43^{c-1}\cdot 47-3\,.$$

So -3 is a quadratic residue modulo 47, a contradiction as

$$\left(\frac{-3}{47}\right)=\left(\frac{-1}{47}\right)\left(\frac{3}{47}\right)=\left(\frac{47}{3}\right)=\left(\frac{2}{3}\right)=-1\,.$$

(3) If $(a,b)=47$ then analogously there is a $k\in\mathbb{N}$ such that

$$(2k+1)^2=4\cdot 43^c\cdot 47^{c-1}-3\,.$$

If $c>1$ then we get a contradiction in exactly the same way as in (2). If $c=1$ then $(2k+1)^2=169$ giving $k=6$. This implies that $a+b=47\cdot 6+47\cdot 7=47\cdot 13\equiv 1 \bmod 5$. Thus $q=(a+b)^2+4\equiv 0 \bmod 5$, a contradiction.

N7. A *super-integer* triangle is defined to be a triangle whose lengths of all sides and at least one height are positive integers. We will deem certain positive integer numbers to be *good* with the condition that if the lengths of two sides of a super-integer triangle are two (not necessarily different) good numbers, then the length of the remaining side is also a good number. Let 5 be a good number. Prove that all integers larger than 2 are good numbers.

Proposed by Serbia

Solution. Evidently, all right-angle triangles with integer sides are super-integer triangles. We will use the following notation $(a, b, c\,\{h\})$ to denote a super-integer triangle whose sides are a, b and c and the height of integer length is h. The height will be written in curly brackets next to the corresponding side and it will be omitted for right-angled triangles. It also follows that if (a, b, c) is an super-integer triangle, then so is (ka, kb, kc), where k is a positive integer.

Note. In all cases of right-angled triangles one can check directly that they are right-angled by Pythagoras' Theorem or use the standard result that $(d(m^2 - n^2), 2dmn, d(m^2 + n^2))$ is a right-angled triangle. For non-right angled triangled we will use Heron's formula that the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$ where s is the semiperimeter. For the triangle to be super-integer we need that $s(s-a)(s-b)(s-c)$ is a perfect square, say $s = m^2$, and that $2m$ is a multiple of a or b or c. We will only make implicit use of the above.

From $(5, 5, 6\,\{4\})$ and $(5, 5, 8\,\{3\})$ it follows that 6 and 8 are good. From $(6, 8, 10)$ it then follows that 10 is also good.

It thus follows if a is good that $2a$ is also good. Indeed consider a sequence of super-integer triangles showing that if 5 is good then a is good. Then the sequence of super-integer triangles of double the size of their edges show that since 10 is good then $2a$ is good.

It easily follows that 12, 16, 20 and 24 are good. From $(5, 12, 13)$ it follows that 13 and therefore also 26 are good. From $(11\,\{12\}, 13, 20)$ and $(21\,\{12\}, 13, 20)$ it follows that 11 and 21 are good. From $(20, 21, 29)$ it follows that 29 is good. From $(6\,\{20\}, 25, 29)$ it follows that 25 is good.

We will say that a positive integer is nice if it is either good or equal to 1 or 2.

Claim 1. If a is good and b is nice then ab is good.

Proof of Claim. The claim is trivial if $b = 1$ and we already proved the case $b = 2$. So assume that b is good. Pick a sequence of super-integer triangles which shows that if 5 is good then b is good. Then the sequence of super-integer triangles 5 times the size of their edges shows that since 25 is good then $5b$ is also good. Now pick a sequence of super-integer triangles which shows that if 5 is good then b is good. Then the sequence of super-integer triangles b times the size of their edges shows that since $5a$ is good then ab is also good. $\square$.

Next, from $(15, 20, 25)$ and $(7, 24, 25)$ we get that 15, 7 and therefore 14 are good. From $(9, 12, 15)$ and $(8, 15, 17)$ we get that 9, 17 and therefore 18 are good and finally from $(3\,\{24\}, 25, 26)$ and then $(3, 4, 5)$ we get that 3 and 4 are good.

We now have that all integers from 3 to 18 are good. To prove that the remaining integers larger than 18 are good, we will proceed by strong induction. Assume that all integers from 3 to $n - 1$ are good for $n \geqslant 19$.

Case 1. If $n = 2m$ is even, then $3 \leqslant m \leqslant n-1$ so m is good. By Claim 1, $n = 2m$ is also good.

Case 2. If n is odd and composite, say $n = ab$, with $a, b > 1$, then $3 \leqslant a, b \leqslant n - 1$ so a, b are good. By Claim 1, $n = ab$ is also good.

Case 3. If n is an odd prime of the form $4k+1$, then by Fermat sum of two squares theorem we can write $n = a^2 + b^2$. We may assume $a > b$. ($a \neq b$ as n is prime.) Consider the triangle $(a^2 - b^2, 2ab, a^2 + b^2)$. This is a super-integer triangle since it is a right-angled triangle. We have $3 \leqslant a^2 - b^2 \leqslant n - 1$ so $a^2 - b^2$ is good. We also have $3 \leqslant 2ab < a^2 + b^2 = n$ so $2ab$ is also good. Thus $n = a^2 + b^2$ is good as well.

Case 4. Assume n is an odd prime of the form $4k+3$. Note that $4k+4$ is good by Case 1 as $2k+2 < 4k+3$. We also have that $4k+5$ is good either by Case 2 (if it is composite) or by Case 3 (if it is prime) except if $4k+5$ is a prime equal to a^2+1. (Because in this case, to use Case 3 we would need that $a^2 - 1 = n$ is good which is what we are trying to prove. But in this exceptional case $n = a^2 - 1 = (a-1)(a+1)$ is not prime.

We will make use of the following Claim:

Claim 2. Let a, b, ℓ be positive integers such that $\ell > 1$ and $a \neq b$. If $\ell - 1, |a-b|, a, b$ are nice, and $\ell, a+b, a^2\ell + b^2$ are good, then $a^2\ell^2 + b^2$ is good.

Proof of Claim. By Claim 1, the numbers $|a^2 - b^2| = |a-b|(a+b)$ and $2ab$ are good. From the right-angled triangle $(2ab, |a^2 - b^2|, a^2 + b^2)$ it follows that $a^2 + b^2$ is good. So by Claim 1 $\ell(a^2 + b^2)$ is good. By Claim 1 $(\ell - 1)(a^2\ell + b^2)$ is also good. Finally, from the triangle $((\ell-1)(a^2\ell + b^2)\,\{2\ell ab\}, \ell(a^2 + b^2), a^2\ell^2 + b^2)$, we get that $a^2\ell^2 + b^2$ is good. □

From Claim 2 with $a = 2, b = 1$ and $\ell = k+1$ to obtain that

$$2^2(k+1)^2 + 1^2 = 4k^2 + 8k + 5 = 4(k+1) + (2k+1)^2$$

is good. From Claim 2 with $a = 2, b = 2k+1$ and $\ell = k+1$ we obtain that

$$2^2(k+1)^2 + (2k+1)^2 = (2k+2)^2 + (2k+1)^2$$

is good. Since from Claim 1, $2(2k+1)(2k+2)$ is good, then from the right-angled triangle $(4k+3, 2(2k+1)(2k+2), (2k+2)^2 + (2k+1)^2)$ we finally deduce that $4k+3$ is good as required.