ДВАДЕСЕТ ЧЕТВРТИ ТУРНИР ГРАДОВА

Јесење коло. Припремна варијанта, 20 октобар 2002.

8-9 разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена.)

1 (4 поена). У конвексном 2002-углу је повучено неколико дијагонала, које се не секу унутар тог 2002-угла. Тиме је тај многоугао разложен на 2000 троуглова. Да ли је могуће да тачно једна половина тих троуглова за све три своје странице има дијагонале тог многоугла?

2 (5 поена). Саша и Маша су замислили по један природан број и саопштили га Васи. Васа је на једном листу папира записао збир та два броја, а на другом листу папира њихов производ. Потом је један од тих листова сакрио, а други (на којем је био записан број 2002) је показао Саши и Маши. Када је видео тај број, Саша је рекао да не зна који је број замислила Маша. Чувши то, Маша је рекла да не зна који број је замислио Саша. Који број је замислила Маша?

3. а) (1 поен). Одељење је радило контролну вежбу. Познато је да је барем две трећине задатака на тој контролној вежби било тешко: сваки од тих задатака није решило барем две трећине ученика. Познато је такође да је барем две трећине ученика добро урадило контролну вежбу: сваки такав ученик је урадио барем две трећине задатака са контролне вежбе. Да ли је то могуће?

б) (2 поена). Да ли ће се променити одговор на постављено питање ако се свуда у услову задатка две трећине замене са три четвртине?

в) (2 поена). Да ли ће се променити одговор на постављено питање ако се свуда у услову задатка две трећине замене са седам десетина?

4 (5 поена). На столу се налази 2002 картица на којима су написани бројеви 1, 2, 3, ..., 2002. Два играча узимају наизменично по једну картицу. Када буду узете све картице, победник је онај играч код кога је већа последња цифра збира бројева на одабраним картицама. Одредите који од играча може увек да победи независно од тога како супарник игра и објасните како он при том треба да игра.

5 (5 поена). Дат је угао и тачка A унутар њега. Да ли је могуће повући три праве кроз тачку A, тако да на сваком од кракова угла једна од пресечних тачака тих правих са краком лежи у средишту међу другим двема тачкама пресека правих с тим истим краком.

ДВАДЕСЕТ ЧЕТВРТИ ТУРНИР ГРАДОВА

Јесење коло. Припремна варијанта, 20. октобар 2002.

10-11 разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

1 (4 поена). Саша и Маша су замислили по један природан број и саопштили га Васи. Васа је на једном листу папира записао збир та два броја, а на другом листу папира њихов производ. Потом је један од тих листова сакрио, а други (на којем је био записан број 2002) је показао Саши и Маши. Када је видео тај број, Саша је рекао да не зна који је број замислила Маша. Чувши то, Маша је рекла да не зна који број је замислио Саша. Који број је замислила Маша?

2. а) (1 поен). Одељење је радило контролну вежбу. Познато је да је барем две трећине задатака на тој контролној вежби било тешко: сваки од тих задатака није решило барем две трећине ученика. Познато је такође да је барем две трећине ученика добро урадило контролну вежбу: сваки такав ученик је урадио барем две трећине задатака са контролне вежбе. Да ли је то могуће?

б) (1 поена). Да ли ће се променити одговор на постављено питање ако се свуда у услову задатка две трећине замене са три четвртине?

в) (2 поена). Да ли ће се променити одговор на постављено питање ако се свуда у услову задатка две трећине замене са седам десетина?

3 (5 поена). Неколико правих, међу којима нема међусобно паралелних, деле раван на неколико области. Унутар једне од тих области је одабрана тачка A. Доказати да постоји тачка B са својством да свака од датих правих раздваја тачке A и B ако и само ако је област која садржи тачку A неограничена.

4 (5 поена). Нека су x, y, z произвољни бројеви из интервала $(0, \pi/2)$. Докажите неједнакост

$$\frac{x\cos x + y\cos y + z\cos z}{x+y+z} \le \frac{\cos x + \cos y + cos z}{3}.$$

5 (5 поена) У бесконачном низу чији су чланови природни бројеви сваки следећи број се добија тако што се претходном броју дода једна његова цифара која је различита од нуле. Доказати да ће се у том низу наћи бар један паран број.

DVADESET ČETVRTI TURNIR GRADOVA

Jesenje kolo. Osnovna varijanta, 27. oktobar 2002.

8-9. razred (mladji uzrast)

(Rezultat se računa na osnovu tri zadatka na kojima je dobijeno najviše poena.)

1. (4 poena) U banci radi 2002 zaposlenih. Svi zaposleni su došli na jubilej banke i bili su rasporedjeni za jedim okruglim stolom. Poznato je da se plate onih koji su susedi razlikuju za dva ili tri dolara. Kolika je najveća moguća razlika izmedju dve plate ako je poznato da svi zaposleni imaju različite plate?

2. (5 poena) Sve biljke koje rastu u Rusiji su numerisane brojevima od 2 do 20000 (bez preskakanja i ponavljanja brojeva). Za svaki par biljaka je izračunat najveći zajednički delitelj njima odgovarajućih brojeva, a sami brojevi su bili izgubljeni (zbog kvara na računaru). Da li je moguće da se svakoj biljci utvrdi njen broj?

3. (6 poena) Temena 50-ugla dele kružnicu na 50 lukova čije su dužine 1, 2, ..., 50 u nekom poretku. Poznato je da razlika dužina suprotnih lukova (onih koji odgovaraju suprotnim stranicama tog 50-ougla) iznosi 25. Dokažite da se u tom mnogouglu može naći par paralelnih stranica.

4. (6 poena) Unutar trougla ABC se nalazi tačka P takva da je ugao ABP jednak uglu ACP a ugao CBP jednak uglu CAP. Dokažite da je P tačka preseka visina datog trougla.

5. (7 poena) Konveksni n-tougao je razložen nekim svojim dijagonalama na trouglove (pri tome se dijagonale ne seku unutar mnogougla). Trouglovi su obojeni u belo ili crno tako da su svaka dva trougla koji imaju zajedničku stanicu obojena različitim bojama. Za svako n nadjite maksimum razlike broja belih trouglova i broja crnih trouglova.

6. (9 poena) Imamo veliki broj kartica, a na svakoj od njih je napisan jedan od brojeva od 1 do n. Znamo da je zbir brojeva na svim karticama jednak $k \cdot n!$, gde je k prirodan broj. Dokažite da se te kartice mogu rasporediti u k grupa tako da je u svakoj grupi zbir cifara na karticama te grupe jednak $n!$.

7. a) (5 poena) Električna mreža ima oblik rešetke 3×3: ukupno 16 čvorova (temena kvadratne mreže) koji su spojeni provodnicima (stranicama kvadrata te mreže). Moguće je da su neki od provodnika pregoreli. U jednom merenju je *moguće odabrati dva čvora i proveriti da li medju tim čvorovima ide tok struje* (to jest, da li postoji lanac provodnika koji nisu pregoreli koji spaja ta dva čvora). Poznato je da je mreža takva da postoji tok izmedju svaka dva čvora. Koji je najmanji broj merenja koji omogućava da se to pouzdano utvrdi?

b) (5 poena) Isto pitanje za mrežu koja ima oblik rešetke 5×5 (ukupno 36 čvorova).

DVADESET ČETVRTI TURNIR GRADOVA

Jesenje kolo. Osnovna varijanta, 27. oktobar 2002.

10-11. razred (stariji uzrast)

(Rezultat se računa na osnovu tri zadatka na kojima je dobijeno najviše poena.)

1. (5 poena) Sve biljke koje rastu u Rusiji su numerisane brojevima od 2 do 20000 (bez preskakanja i ponavljanja brojeva). Za svaki par biljaka je izračunat najveći zajednički delitelj njima odgovarajućih brojeva, a sami brojevi su bili izgubljeni (zbog kvara na računaru). Da li je moguće da se svakoj biljci utvrdi njen broj?

2. (6 poena) Kocka je presečena jednom ravni tako da je u preseku dobijen petougao. Dokažite da postoji stranica tog petougla čija se dužina razlikuje od jednog metra najmanje za 20 centimetara.

3. (6 poena) Konveksni n-tougao je razložen nekim svojim dijagonalama na trouglove (pri tome se dijagonale ne seku unutar mnogougla). Trouglovi su obojeni u belo ili crno tako da su svaka dva trougla koji imaju zajedničku stanicu obojena različitim bojama. Za svako n nadjite maksimum razlike broja belih trouglova i broja crnih trouglova.

4. (8 poena) Imamo veliki broj kartica, a na svakoj od njih je napisan jedan od brojeva od 1 do n. Znamo da je zbir brojeva na svim karticama jednak $k \cdot n!$, gde je k prirodan broj. Dokažite da se te kartice mogu rasporediti u k grupa tako da je u svakoj grupi zbir cifara na karticama te grupe jednak $n!$.

5. Dve kružnice se seku u tačkama A i B. Kroz tačku B prolazi prava, koja seče prvu i drugu kružnicu još i u tačkama K i M respektivno. Prava l_1 dodiruje prvu kružnicu u tački Q i paralelna je pravoj AM. Prava QA seče drugu kružnicu u tački R. Prava l_2 dodiruje drugu kružnicu u tački R. Dokazati da

a) (4 poena) je prava l_2 paralelna AK;

b) (4 poena) prave l_1, l_2 i KM imaju zajedničku tačku.

6. (8 poena) Posmatrajmo niz čija su prva dva člana brojevi 1 i 2, a svaki sledeći član niza je najmanji prirodan broj koji se još nije pojavio u nizu a nije uzajamno prost sa prethodnim članom niza. Dokazati da se svaki prirodan broj javlja u tom nizu.

7. a) (5 poena) Električna mreža ima oblik rešetke 3×3: ukupno 16 čvorova (temena kvadratne mreže) koji su spojeni provodnicima (stranicama kvadrata te mreže). Moguće je da su neki od provodnika pregoreli. U jednom merenju je moguće odabrati dva čvora i proveriti da li medju tim čvorovima ide tok struje (to jest, da li postoji lanac provodnika koji nisu pregoreli koji spaja ta dva čvora). Poznato je da je mreža takva da postoji tok izmedju svaka dva čvora. Koji je najmanji broj merenja koji omogućava sa se to pouzdano utvrdi?

b) (5 poena) Isto pitanje za mrežu koja ima oblik rešetke 7×7 (ukupno 64 čvora).

24. ТУРНИР ГРАДОВА

Пролећно коло.

Припремна варијанта, 23. фебруара 2003. год.

8–9. разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

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1. (4 поена). 2003 долара је распоређено у новчанике, а новчаници су смештени у џепове. Познато је да је бррој новчаника већи од броја долара у било ком џепу. Да ли је тачно да је број џепова већи од броја долара у неком од новчаника? (Није допуштено стављати новчанике један у други)

2. (4 поена). Два играча наизменично боје странице n-тоугла. Први може да обоји страницу која је суседна са 0 или 2 обојене странице, а други страницу која је суседна са једном обојеном страницом. Губи онај играч који не може да одигра потез. За које вредности n други играч може да победи независно од игре првог?

3. (4 поена). На крацима AB и BC једнакокраког троугла ABC налазе се тачке K и L респективно, такве да је $AK + LC = KL$. Кроз средиште дужи KL пролази права паралелна са BC, и та права сече страницу AC у тачки N. Наћи величину угла KNL.

4. (5 поена). У низу природних бројева сваки број, осим првог, је једнак збиру претходног броја и његове највеће цифре. Колико највише непарних узастопних бројева може садржати такав низ?

5. (5 поена). Може ли се табла величине 2003×2003 поплочати доминама величине 1×2, које је дозвољено стављати само хоризонтално, и правоугаоницима величине 1×3, које је дозвољено стављати само вертикално? (Две паралелне странице табле условно зовемо хоризонталним, а друге две вертикалним.)

24. ТУРНИР ГРАДОВА

Пролећно коло.

Припремна варијанта, 23. фебруара 2003. год.

10–11. разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

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1. (3 поена). 2003 долара је распоређено у новчанике, а новчаници су смештени у џепове. Познато је да новчаника има више него долара у било ком џепу. Да ли је тачно да џепова има више него долара у неком од новчаника?

2. (3 поена). Дато је 100 штапића, од којих се може саставити 100-угао. Да ли се може десити да се ни од ма којег мањег броја тих штапића не може саставити многоугао?

3. (4 поена). У троуглу ABC је одабрана тачка M тако да полупречници описаних кружница троуглова AMC, BMC и BMA нису мањи од полупречника кружнице описане око троугла ABC. Доказати да су сва четири полупречника једнака..

4. (5 поена). Сто бројева је поређано у растућем поретку: 00, 01, 02, 03, ... ,99. Затим су они испремештани тако да се сваки следећи број добија из претходног тако што се њему тачно једна цифра повећа или смањи за 1 (на пример, после 29 може се појавити само 19, 39 или 28, а 20 или 30 се не могу појавити). Колико највише бројева може остати на свом месту?

5. (5 поена). Дат је правоугаоник од картона чије су странице a cm и b cm, где је $b/2 < a < b$. Доказати да га је могуће разрезати на три дела од којих се може саставити квадрат.

24. ТУРНИР ГРАДОВА

Пролећно коло.

Основна варијанта, 2. марта 2003. год.

8–9. разред (млађи узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

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1. (4 поена). Васа напише на табли једначину $ax^2 + bx + c = 0$ са позитивним целобројним коефицијентима a, b и c. После тога Петар, ако жели, може да замени један или два знака "+" знаком "−". Ако добијена једначина има оба корена целобројна, онда побеђује Васа, а ако пак добијена једначина нема корена (решења) или је бар један од корена нецелобројан, онда побеђује Петар. Може ли Васа тако одабрати коефицијенте једначине да сигурно победи Петра?

2. (4 поена). Дат је троугао ABC. У њему је R - полупречник описане кружнице, r - полупречник уписане кружнице, a - дужина најдуже странице, h - дужина најмање висине. Доказати да је $R/r > a/h$.

3. На турниру је свака од 15 екипа одиграла са сваком другом екипом тачно један меч.

а) (4 поена) Доказати да су се барем у једном мечу сусреле екипе које су, до тог меча, у збиру одиграле непаран број мечева.

б) (3 поена) Може ли такав меч бити јединствен?

4. (7 поена) Чоколада облика једнакостраничног троугла станице дужине n издељена је браздама на мале троуглове чије су странице дужине 1 (свака страница је подељена на n једнаких делова, деоне тачке на сваком пару страница су спојене дужима паралелним трећој страници). Два играча играју игру. У једном потезу се може одломити троугаоно парче чоколаде (дуж бразде), појести га и остатак предати противнику. Онај који добије последње парче - троугао странице 1 - је победник. Ако играч не може да одигра потез, одмах губи игру. За свако n установите који од играча, први (онај који почиње) или други, може играти тако да увек победи (независно од игре противника).

5. (7 поена) Који је највећи број поља табле 9x9 која се могу разрезати по обе дијагонале, тако да се при томе табла не распадне на неколико делова?

6. (7 поена) Трапез са основицама AD и BC описан је око кружнице, E је тачка пресека његових дијагонала. Доказати да угао AED не може бити оштар.

24. ТУРНИР ГРАДОВА

Пролећно коло.
Основна варијанта, 2. марта 2003. год.
10–11. разред (старији узраст)

(Резултат се рачуна на основу три задатка на којима је добијено највише поена)

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1. (4 поена). Дата је тространа пирамида $ABCD$. У њој је R - полупречник описане сфере, r - радијус уписане сфере, a - дужина најдуже ивице h - дужина најмање висине (на једну од страна пирамиде). Доказати да је $R/r > a/h$.

2. (5 поена). Дат је полином $P(x)$ са реалним коефицијентима. Бесконачан низ различитих природних бројева a_1, a_2, a_3, ... је такав да је $P(a_1) = 0$, $P(a_2) = a_1$, $P(a_3) = a_2$, итд. Који степен може имати полином $P(x)$?

3. (5 поена). Може ли се површина коцке у потпуности покрити са три троугла без преклапања?

4. (6 поена). У кружницу је уписан правоугли троугао ABC чија је хипотенуза AB. Нека је K средиште лука BC који не садржи тачку A N - средиште дужи AC; M - тачка пресека полуправе KN са кружницом У тачкама A и C су конструисане тангенте на кружницу, које се секу у тачки E. Доказати да је угао EMK прав.

5. (6 поена). Бора је замислио цео број, већи од 100. Кира каже цео број d, већи од 1. Ако је Борин број дељив са d, онда Кира побеђује, а у супротном Бора одузима од свог броја број d и игра се наставља. Кира нема право да каже број које је раније рекла. Када Борин број постане негативан, Кира губи. Може ли Кира тако играти да сигурно победи?

6. (7 поена). У сваком пољу таблице величине 4x4 је записан знак "+" или "–". Дозвољено је истовремено мењати знак у произвољном пољу и свим пољима која имају заједничку страницу са тим пољем. Колико се различитих таблица може добити, више пута примењујући описану операцију?

7. (8 поена). Унутар квадрата је одабрано неколико тачака и те тачке су спојене дужима између себе и са теменима квадрата, тако да те дужи не секу једна другу (осим на крајевима). Тиме је квадрат подељен на троуглове и то тако да је свака одабрана тачка теме неког троугла, а ниједна се не налази на страници неког троугла. За сваку одабрану тачку, као и за свако теме квадрата, пребројане су одатле повучене дужи. Може ли се десити да сви таако добијени бројеви буду парни?

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2002.**

1 [**4**] In a convex 2002-gon several diagonals are drawn so that they do not intersect inside of the polygon. As a result, the polygon splits into 2000 triangles.

Is it possible that exactly 1000 triangles have diagonals for all of their three sides?

2 [**5**] Each of two children (John and Mary) selected a natural number and communicated it to Bill. Bill wrote down the sum of these numbers on one card and their product on another, hid one card and showed the other to John and Mary.

John looked at the number (which was 2002) and declared that he was not able to determine the number chosen by Mary. Knowing this, Mary said that she was also not able to determine the number chosen by John.

What was the number chosen by Mary?

3

a) [**1**] A test was conducted in a class. It is known that at least $\frac{2}{3}$ of the problems were hard: each such problem was not solved by at least $\frac{2}{3}$ of the students. It is also known that at least $\frac{2}{3}$ of students passed the test: each such student solved at least $\frac{2}{3}$ of the suggested problems. *Is this situation possible?*

b) [**2**] The same question with $\frac{2}{3}$ replaced by $\frac{3}{4}$.

c) [**2**] The same question with $\frac{2}{3}$ replaced by $\frac{7}{10}$.

4) [**5**] 2002 cards with the numbers 1, 2, 3,...,2002 written on them are put on a table face up. Two players in turns pick up a card from the table until all cards are gone. The player who gets the last digit of the sum of all numbers on his cards larger than his opponent, wins.

Who has a winning strategy and how one should play to win?

5) [**5**] An angle and a point A inside of it are given. Is it possible to draw through A three straight lines so that on either side of the angle one of three points of intersection of these lines be the midpoint between two other points of intersection with that side?

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Fall 2002.**

1 Consider a triangulation of 2002-gon satisfying the conditions. Triangles which contain at least one side of 2002-gon we call *exterior triangles*. So, our problem is reduced to the following question:

Is it possible to have exactly 1000 exterior triangles? (then we have exactly 1000 triangles which have diagonals for all three sides).

The answer is negative. Really, every exterior triangle contains at most 2 sides of 2002-gon and there should be at least 1001 of them. Contradiction.

2 Let j and m be numbers selected by J and M respectively. Note that $j|2002$; otherwise J would know that $m = 2002 - j$. Also $j \neq 2002$; otherwise $m = 1$ (since $m \neq 0$). So, $j \leq 1001$.

Further, the same is true for m. In addition, M knows that $j \leq 1001$. Therefore, $m = 1001$ (otherwise M would know $j = 2002 : m$).

So, $m = 1001$ is the only possible solution. One can check that it works.

3 Let N be the number of students in the class, M the number of the problems, P the number of passed students, H the number of hard problems. According to definition "a problem is hard" if it has not been solved by at least rN students; where $r = \frac{2}{3}, \frac{3}{4}, \frac{7}{10}$ in (a), (b),(c). Also, according to definition "a student passes" if he solves at least rM problems.

a) *It is possible.* Consider a class consisting of students S_1, S_2, S_3 and set of problems P_1, P_2, P_3. Let S_1 solve P_1 and P_3, S_2 solve P_2 and P_3 and S_3 solved neither P_1 nor P_2. Then S_1, S_2 pass and P_1, P_2 are hard problems.

b) *It is impossible.* Let us write down the results of the test ("+" or "−") into $N \times M$ table.

Let passed students be on the top and hard problems on the left of the table. Let us estimate K_+ and K_-, the numbers of "+" and "−" in the table. First,

$$K_+ \geq (\text{number of "+" got by students who passed}) \geq P \times rM \geq r^2 MN$$

and

$$K_- \geq (\text{number of "} - \text{" got for hard problems}) \geq H \times rN \geq r^2 MN.$$

Then $MN = K_+ + K_- \geq 2r^2 MN$ which is impossible for $r = \frac{3}{4}$.

c) *It is impossible.* Arguments of (b) do not work here since $2r^2 \le 1$. Now we denote by K_+ and K_- the numbers of "+" and "−" in the top-left $P \times H$ sub-table. Then

$$K_+ \ge (\text{minimal number of }" + "\text{ for hard problems got by students who passed}) \ge P \times \frac{4}{7} H$$

(a student cannot pass if he solves less than $\frac{4}{7}H$ of hard problems even if he solves all the easy problems, the number of which does not exceed $\frac{3}{7}M$). On the other hand,

$$K_- \ge (\text{minimal number of }" - "\text{ got by students who passed for hard problems}) \ge H \times \frac{4}{7} P.$$

So, $PH = K_+ + K_- \ge \frac{8}{7} PH$ which is impossible.

4 The First Player (FP) wins. Let us pair all the cards (numbers): we pair k with $1000 + k$, $k = 1, \ldots, 1000$. Also we pair 2001 with 2002. So, in each pair save the last one both cards have the same last digit.

FP starts and picks up 2002. From this moment his strategy is to pick up the other half of the pair chosen by SP. So, eventually SP is forced to pick up 2001. If cards are not gone, then FP takes any card leaving for SP to pick up the other half of the pair. At the end FP has the sum $\equiv 45000 + 2 \equiv 2$ (modulo 10) and SP has the sum $\equiv 1$ (modulo 10).

5 Let us assume that there are straight lines MZ, NY and LX passing through A such that $MN = NL$ and $XY = YZ$ where M, N, L are points on one side of the angle and X, Y, Z are points on the other side. Let us draw a straight line through X parallel to ML; it intersects lines NY and MZ at points N_1 and M_1 respectively. $\triangle AMN$ and $\triangle AM_1N_1$ are similar; so are $\triangle ANL$ and $\triangle XAN_1$. Then $XN_1 = N_1M_1$. Given the assumption $XY = YZ$ we have that lines $N_1Y \parallel M_1Z$, which is impossible since they intersect at A.

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Fall 2002.**

1 [**4**] Each of two children (John and Mary) selected a natural number and communicated it to Bill. Bill wrote down the sum of these numbers on one card and their product on another, hid one card and showed the other to John and Mary.

John looked at the number (which was 2002) and declared that he was not able to determine the number chosen by Mary. Knowing this, Mary said that she was also not able to determine the number chosen by John.

What was the number chosen by Mary?

2

a [**1**] A test was conducted in a class. It is known that at least $\dfrac{2}{3}$ of the problems were hard: each such problem was not solved by at least $\dfrac{2}{3}$ of the students. It is also known that at least 2/3 of students passed the test: each such student solved at least $\dfrac{2}{3}$ of the suggested problems.

Is this situation possible?

b [**1**] The same question with $\dfrac{2}{3}$ replaced by $\dfrac{3}{4}$.

c [**2**] The same question with $\dfrac{2}{3}$ replaced by $\dfrac{7}{10}$.

3 [**5**] Several straight lines such that no two of them are parallel, cut the plane into several regions. A point A is marked inside of one region. Prove that a point, separated from A by each of these lines, exists if and only if A belongs to unbounded region.

4 [**5**] Let x, y, z be any three numbers from the open interval $(0, \pi/2)$. Prove the inequality

$$\frac{x \cdot \cos x + y \cdot \cos y + z \cdot \cos z}{x + y + z} \leq \frac{\cos x + \cos y + \cos z}{3}.$$

5 [**5**] Each term of an infinite sequence of natural numbers is obtained from the previous term by adding to it one of its nonzero digits. Prove that this sequence contains an even number.

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Fall 2002.**

1 Let j and m be numbers selected by J and M respectively. Note that $j|2002$; otherwise J would know that $m = 2002 - j$. Also $j \neq 2002$; otherwise $m = 1$ (since $m \neq 0$). So, $j \leq 1001$.

Further, the same is true for m. In addition, M knows that $j \leq 1001$. Therefore, $m = 1001$ (otherwise M would know $j = 2002 : m$).

So, $m = 1001$ is the only possible solution. One can check that it works.

2 Let N be the number of students in the class, M the number of the problems, P the number of passed students, H the number of hard problems. According to definition "a problem is hard" if it has not been solved by at least rN students; where $r = \frac{2}{3}, \frac{3}{4}, \frac{7}{10}$ in (a), (b),(c). Also, according to definition "a student passes" if he solves at least rM problems.

a) *It is possible.* Consider a class consisting of students S_1, S_2, S_3 and set of problems P_1, P_2, P_3. Let S_1 solve P_1 and P_3, S_2 solve P_2 and P_3 and S_3 solved neither P_1 nor P_2. Then S_1, S_2 pass and P_1, P_2 are hard problems.

b) *It is impossible.* Let us write down the results of the test ("+" or "−") into $N \times M$ table.

Let passed students be on the top and hard problems on the left of the table. Let us estimate K_+ and K_-, the numbers of "+" and "−" in the table. First,

$$K_+ \geq (\text{number of "+" got by students who passed}) \geq P \times rM \geq r^2 MN$$

and

$$K_- \geq (\text{number of "−" got for hard problems}) \geq H \times rN \geq r^2 MN.$$

Then $MN = K_+ + K_- \geq 2r^2 MN$ which is impossible for $r = \frac{3}{4}$.

c) *It is impossible.* Arguments of (b) do not work here since $2r^2 \leq 1$. Now we denote by K_+ and K_- the numbers of "+" and "−" in the top-left $P \times H$ sub-table. Then

$$K_+ \geq (\text{minimal number of "+" for hard problems got by students who passed}) \geq P \times \frac{4}{7} H$$

(a student cannot pass if he solves less than $\frac{4}{7}H$ of hard problems even if he solves all the easy problems, the number of which does not exceed $\frac{3}{7}M$). On the other hand,

$$K_- \geq (\text{minimal number of "−" got by students who passed for hard problems}) \geq H \times \frac{4}{7} P.$$

So, $PH = K_+ + K_- \geq \frac{8}{7} PH$ which is impossible.

3 Let us assume that such point B exists (separated from A by each line). Then segment AB intersects all the lines and therefore ray $[BA)$ originated at B has no points of intersection beyond A. Therefore, A belongs to unbounded region.

Now, assume that A belongs to unbounded region. Our region is convex, bounded by two rays and maybe several segments. Note, that these rays are divergent. Therefore, one can draw a ray, originated at A and lying inside of our region. Without any loss of the generality we can assume that this ray is not-parallel to any of the lines; otherwise we can rotate it slightly. Then the opposite ray (originated at A) intersects all the lines and any point B beyond the last point of intersection satisfies the condition.

4 Since function $\cos x$ is a monotone decreasing on $(0, \pi/2)$ we have $(x-y)(\cos x - \cos y) \leq 0$ (equality holds only for $x = y$). Also $(x-z)(\cos z - \cos x) \leq 0$ and $(y-z)(\cos y - \cos z) \leq 0$. Adding these inequalities we get

$$2(x \cos x + y \cos y + z \cos z) \leq (y+z)\cos x + (x+z)\cos y + (y+x)\cos z$$

and therefore

$$3(x \cos x + y \cos y + z \cos z) \leq (x+y+z)(\cos z + \cos y + \cos x)$$

which implies our inequality.

5 Let $\{a_k\}$ be our sequence. Note that $1 \leq a_{k+1} - a_k \leq 9$. Then the segment $[9\ldots 989, 9\ldots 999]$ contains a term of our sequence; $a_k = 9\ldots 99r$. If r is even than a_k is even. If r is odd then a_{k+1} must be odd.

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2002.**

1 [**4**] There are 2002 employees in a bank. All the employees came to celebrate the bank's jubilee and were seated around one round table. It is known that the difference in salaries of any two employees sitting next to each other is 2 or 3 dollars. Find the maximal difference in salaries of two employees, if it is known that all the salaries are different.

2 [**5**] All the species of plants existing in Russia are catalogued (numbered by integers from 2 to 20 000; one after another, without omissions or repetitions). For any pair of species, the greatest common divisor of their catalogue numbers was calculated and recorded, but the catalogue numbers themselves were lost computer error). Is it possible to restore the catalogue number for each specie from that data?

3 [**6**] The vertices of a 50-gon divide a circumference into 50 arcs, whose lengths are 1, 2, 3,. . . , 50, in some order. It is known that lengths of any pair of "opposite" arcs (corresponding to opposite sides of the polygon) differ by 25. Prove that the polygon has two parallel sides.

4 [**6**] Point P is chosen in triangle ABC so that $\angle ABP$ is congruent to $\angle ACP$, while $\angle CBP$ is congruent to $\angle CAP$. Prove that P is the intersection point of the altitudes of the triangle.

5 [**7**] A convex N-gon is divided by diagonals into triangles so that no two diagonals intersect inside of the polygon. The triangles are painted in black and white so that any two triangles with common side are painted in different colors. For each N, find the maximal difference between the numbers of black and white triangles.

6 [**9**] There is a large pile of cards. On each card one of the numbers $1, 2, \ldots, n$ is written. It is known that the sum of all numbers of all the cards is equal to $k \cdot n!$ for some integer k. Prove that it is possible to arrange cards into k stacks so that the sum of numbers written on the cards in each stack is equal to $n!$.

7 a) [**5**] A power grid has the shape of a 3×3 lattice with 16 nodes (vertices of the lattice) joined by wires (along the sides of the squares). It may have happened that some of the wires are burned out. In one test technician can choose any pair of nodes and check if electrical current circulates between them (that is, check if there is a chain of intact wires joining the chosen nodes). Technician knows that current will circulate from any node to any other node. What is the least number of tests which is required to demonstrate this?

7 b) [**5**] The same question for a grid in the shape of a 5×5 lattice (36 nodes).

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Fall 2002.**

1 *Answer: \$ 3002.* First, let us prove that d (the difference in salaries) does not exceed 3002. Let us number employees in clock-wise direction starting from one with the minimal salary. Let n be the employee with the maximal salary. Then 1 and n are separated by $n-2$ employees in clock-wise and $(2002-n)$ counter-clockwise. So $d \leq 3(n-1)$ and $d \leq (2003-n)$. Then $d \leq 3(n-1+2003-n)/2 = 3003$. Note, that $d = 3003$ is only possible if the difference between any two neighbors is exactly 3, which contradicts to assumption that all employees have different salaries.

Let us construct an example with the difference 3002. Let $S(k)$ be a salary of k-th worker. Let $S(1) = 0$, $S(2) = 2$, $S(k) = S(k-1) + 3$ for $k = 3, 4, \ldots, 1002$, $S(1003) = S(1002) - 2$, $S(k) = S(k-1) - 3$ for $k = 1004, \ldots, 2002$. Then $S(1002) - S(1) = 3002$.

2 *The answer is negative.* It is sufficient to give an example of two numbers which have the same gcd with all the other numbers from 2 to 20,000.

Examples: a) 2^{13} and 2^{14};

b) $19,993$ and $19,997$; both numbers are primes because they have no prime divisors less than 142.

3 Let AB be an arc from A to B in clock-wise direction. For any ordered pair of opposite arcs AB and CD we define $d(AB)$ equal to the difference between arc DA and arc BC. Obviously $d(AB)$ is divisible by 50 (because the difference between two opposite arcs is ± 25 and we have 24 pairs).

Now let us switch to next pair of opposite arcs in clock-wise direction. Note that the increment of $d(AB)$ is either 50, or -50, or 0. Also note that $d(CD) = -d(AB)$. Therefore at some moment we reach a pair of opposite arcs with difference 0.

Then corresponding sides of polygon are parallel.

4 Let us encircle $\triangle ABC$. Let K be an intersection point of continuation of BP and encircle. Then $\angle ABK = \angle ACK$ and $\angle CBK = \angle CAK$ (subtended by the same arc). Then $\triangle APC \cong \triangle AKC$ (A-S-A). Therefore $PK \perp AC$. Similarly, we prove that $AP \perp BC$ as well.

5 Since in N-gon the sum of all angles equals $(N-2) \cdot 180°$, then N-gon is split into $(N-2)$ triangles by $(N-3)$ diagonals, not intersecting inside of N-gon. Side of each white (black) triangle we call white (black); so diagonals are both black and white.

Then, there are at least $(N-3)$ white (black) sides; therefore there are at least $\lceil \frac{1}{3}(N-3) \rceil$ triangles of each color. Let $R(N)$ be the difference in question. Let us consider 3 cases:

a) $N = 3k$. Then there are at least $k - 1$ black triangles, at most $2k - 1$ white triangles and thus $R(N) \leq k$.

b) $N = 3k + 1$. Then there are at least k black triangles, at most $2k - 1$ white triangles and thus $R(N) \leq k - 1$.

c) $N = 3k + 2$. Then there are at least k black triangles, at most $2k$ white triangles and thus $R(N) \leq k$.

Let us prove that all these estimates are sharp and equalities could be reached. For $N = 3, 4, 5$ ($k = 1$) one can check it easily. For larger N one can construct example by induction by k.

Let us assume that for some k we have corresponding N-gon with the required difference (white triangles are in excess). Then we add a pentagon (2 white and 1 black triangles) to N-gon matching black side of pentagon with the white one of N-gon. Then N increases by 3, k increases by 1 and $R(N)$ increases by 1.

6 Let us start from

Proposition. From any set $\{a_1, \ldots, a_n\}$ of n integers one can choose a number or several numbers with their sum divisible by n.

Proof. Let us assume that none of the numbers is divisible by n. Consider numbers $b_1 = a_1$, $b_2 = a_1 + a_2$, $\ldots$, $b_n = a_1 + a_2 + \ldots + a_n$. If none of them is divisible by n then at least two numbers b_j and b_l ($k < l$) have the same remainders. Then their difference $a_{j+1} + \ldots + a_l$ is divisible by n.

Let us apply an induction by n. If $n = 1$ then only number 1 is written on each card. So, every card by itself forms a required group (with sum 1!).

Assume that a main statement is proven for $(n - 1)$, meaning that if the sum of the numbers on all cards is $k \cdot (n - 1)!$ then cards could be arranged into k stacks with the sum of the numbers in each stuck equal $(n - 1)!$.

Lets call *a supercard* any group of cards with sum $l \cdot n$, $l = 1, \ldots, n-1$. We call l *a supercard value*. Any card with number n on it is a supercard of value 1. From the rest of cards with numbers $1, \ldots, n-1$ we form supercards by the following procedure: pick any n cards; due to proposition choose several with the sum divisible by n; they form a supercard by definition. This procedure stops when less than n cards are left. However, their sum must be divisible by n (since the total sum and sum on each supercard are divisible by n) meaning that leftovers also form a supercard (sum does not exceed $(n - 1)n$).

Now we have a pile of supercards with values $1, \ldots, n - 1$, the total sum of the values equals $(k \cdot n!)/n = k \cdot (n - 1)!$. Then according to induction assumption, we can split supercards into k stacks with the sum of the values in each equal $(n - 1)!$. Therefore the sum of cards (normal) in each stuck is $(n - 1)! \cdot n = n!$.

7 Solution for $(2k - 1) \times (2k - 1)$ lattice ($4k^2$ nodes).

For any test a technician chooses a pair of nodes. If the number of tests is less than $2k^2$, at least one node would not be tested. It could happen that this node is isolated but the rest of the wires are intact. So, at least $2k^2$ tests are needed.

Let us numerate the nodes along the main diagonal Δ of the grid from $1, \dots, 2k$. Let us test pairs of nodes $(1, k+1)$, $(2, k+2)$, $\dots (k, 2k)$ plus every pair of nodes which are symmetrical with respect to Δ $(k + k(2k-1) = 2k^2)$. Assume that all tests were successful. We need to prove that there is a link between every pair of nodes.

First, we prove that there is a link (connection) between every pair of nodes on Δ. Since nodes 1 and $k+1$ are linked, there exits a path π between them formed by intact wires. Consider a path π' symmetrical to π with respect to Δ. Notice that any node of π' is linked to the symmetrical node of π. Therefore every node of π' is linked to node 1 and therefore all nodes of π' are linked between themselves.

Note that node 2 is either encircled by $\pi \bigcup \pi'$ or belongs to both π and π'. Since nodes 2 and $k+2$ are linked then the intact path between them intersects $\pi \bigcup \pi'$ and therefore both 2 and $k+2$ are linked to 1. Similarly, all other diagonal nodes are linked to 1; therefore all of them are linked.

Now let us consider any non-diagonal node. It is linked with its symmetrical node; the intact path connecting them intersects Δ. This means that any two nodes are linked.

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2002.**

1 [**4**] All the species of plants existing in Russia are catalogued (numbered by integers from 2 to 20 000; one after another, without omissions or repetitions). For any pair of species, the greatest common divisor of their catalogue numbers was calculated and recorded, but the numbers themselves were lost (as the result of a computer error). Is it possible to restore the catalogue number for each specie from that data?

2 [**6**] A cube is cut by a plane so that the cross-section is a pentagon. Prove that the length of one of the sides of the pentagon differs from 1 meter by at least 20 centimeters.

3 [**6**] A convex N-gon is divided by diagonals into triangles so that no two diagonals intersect inside of the polygon. The triangles are painted in black and white so that any two triangles with common side are painted in different colors. For each N, find the maximal difference between the numbers of black and white triangles.

4 [**8**] There is a large pile of cards. On each card one of the numbers $\{1, 2, \ldots, n\}$ is written. It is known that the sum of all numbers of all the cards is equal to $k \cdot n!$ for some integer k. Prove that it is possible to arrange cards into k stacks so that the sum of numbers written on the cards in each stack is equal to $n!$.

5 Two circles intersect at points A and B. Through point B a straight line is drawn, intersecting the first and second circle at points K and M (different from B) respectively. Line ℓ_1 is tangent to the first circle at point Q and parallel to line AM. Line QA intersects the second circle at point R (different from A). Further, line ℓ_2 is tangent to the second circle at point R. Prove that

a) [4] ℓ_2 is parallel to AK;

b) [4] Lines ℓ_1, ℓ_2 and KM have a common point.

6 [**8**] A sequence with first two terms equal 1 and 2 respectively is defined by the following rule: each subsequent term is equal to the smallest positive integer which has not yet occurred in the sequence and is not coprime with the previous term. Prove that all positive integers occur in this sequence.

7 a) [4] A power grid has the shape of a 3×3 lattice with 16 nodes (vertices of the lattice) joined by wires (along the sides of the squares). It may have happened that some of the wires are burned out. In one test technician can choose any pair of nodes and check if electrical current circulates between them (that is, check if there is a chain of intact wires joining the chosen nodes). Technician knows that current will circulate from any node to any other node. What is the least number of tests which is required to demonstrate this?

7 b) [5] The same question for a grid in the shape of a 7×7 lattice (36 nodes).

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Fall 2002.**

1 *The answer is negative.* It is sufficient to give an example of two numbers which have the same gcd with all the other numbers from 2 to 20,000.

Examples: a) 2^{13} and 2^{14};

b) $19,993$ and $19,997$; both numbers are primes because they have no prime divisors less than 142.

2 Proof by a contradiction. Assume that pentagon has sides ranging from 0.8 to 1.2. To get a pentagon in cross-section of a cube, a plane has to cross five faces, two pairs of which are parallel. Therefore the pentagon has two pairs of parallel sides. Let us consider pentagon $BCDKL$ with $BC \parallel DK$ and $CD \parallel LB$. Then A be a point of intersection of BL and KD (extended). Note that $ABCD$ is a parallelogram. Due to triangle inequality $AL+AK > LK$, then $AB+AD > BL+LK+KD$. So, $BC+CD > BL+LK+KD$. Then even if BC and CD are two longest sides, $BC+CD \leq 2 \cdot 1.2 = 2.4$ while $BL+LK+KD \geq 3 \cdot 0.8 = 2.4$ which is contradiction.

3 Since in N-gon the sum of all angles equals $(N-2)\cdot 180°$, then N-gon is split into $(N-2)$ triangles by $(N-3)$ diagonals, not intersecting inside of N-gon. Side of each white (black) triangle we call white (black); so diagonals are both black and white.

Then, there are at least $(N-3)$ white (black) sides; therefore there are at least $\lceil \frac{1}{3}(N-3)\rceil$ triangles of each color. Let $R(N)$ be the difference in question. Let us consider 3 cases:

a) $N = 3k$. Then there are at least $k-1$ black triangles, at most $2k-1$ white triangles and thus $R(N) \leq k$.

b) $N = 3k+1$. Then there are at least k black triangles, at most $2k-1$ white triangles and thus $R(N) \leq k-1$.

c) $N = 3k+2$. Then there are at least k black triangles, at most $2k$ white triangles and thus $R(N) \leq k$.

Let us prove that all these estimates are sharp and equalities could be reached. For $N = 3,4,5$ $(k=1)$ one can check it easily. For larger N one can construct example by induction by k.

Let us assume that for some k we have corresponding N-gon with the required difference (white triangles are in excess). Then we add a pentagon (2 white and 1 black triangles) to N-gon matching black side of pentagon with the white one of N-gon. Then N increases by 3, k increases by 1 and $R(N)$ increases by 1.

4 Let us start from

Proposition. From any set $\{a_1, \ldots, a_n\}$ of n integers one can choose a number or several numbers with their sum divisible by n.

Proof. Let us assume that none of the numbers is divisible by n. Consider numbers $b_1 = a_1$, $b_2 = a_1 + a_2$, …, $b_n = a_1 + a_2 + \ldots + a_n$. If none of them is divisible by n then at least two numbers b_j and b_l $(k < l)$ have the same remainders. Then their difference $a_{j+1} + \ldots + a_l$ is divisible by n.

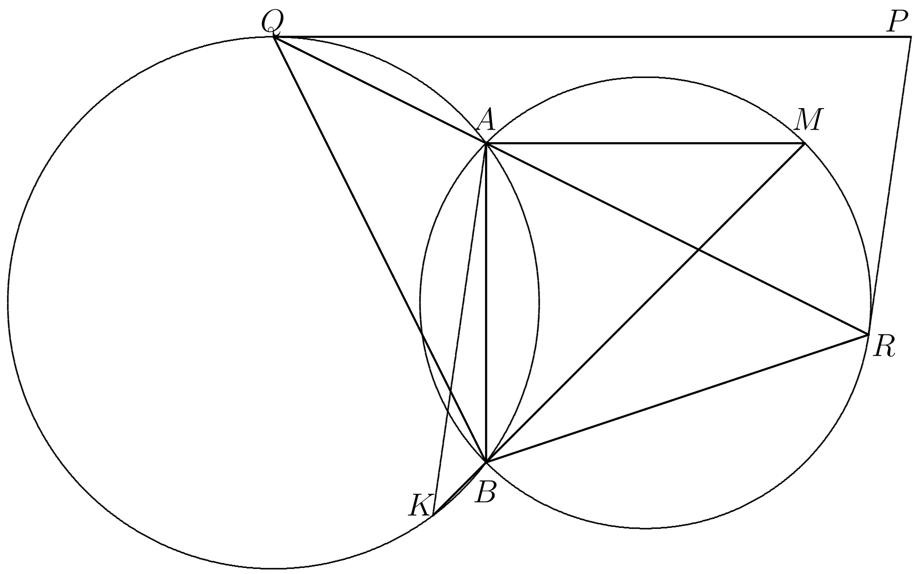
Let us apply an induction by n. If $n = 1$ then only number 1 is written on each card. So, every card by itself forms a required group (with sum 1!).

Assume that a main statement is proven for $(n-1)$, meaning that if the sum of the numbers on all cards is $k \cdot (n-1)!$ then cards could be arranged into k stacks with the sum of the numbers in each stuck equal $(n-1)!$.

Lets call *a supercard* any group of cards with sum $l \cdot n$, $l = 1, \ldots, n-1$. We call l *a supercard value*. Any card with number n on it is a supercard of value 1. From the rest of cards with numbers $1, \ldots, n-1$ we form supercards by the following procedure: pick any n cards; due to proposition choose several with the sum divisible by n; they form a supercard by definition. This procedure stops when less than n cards are left. However, their sum must be divisible by n (since the total sum and sum on each supercard are divisible by n) meaning that leftovers also form a supercard (sum does not exceed $(n-1)n$).

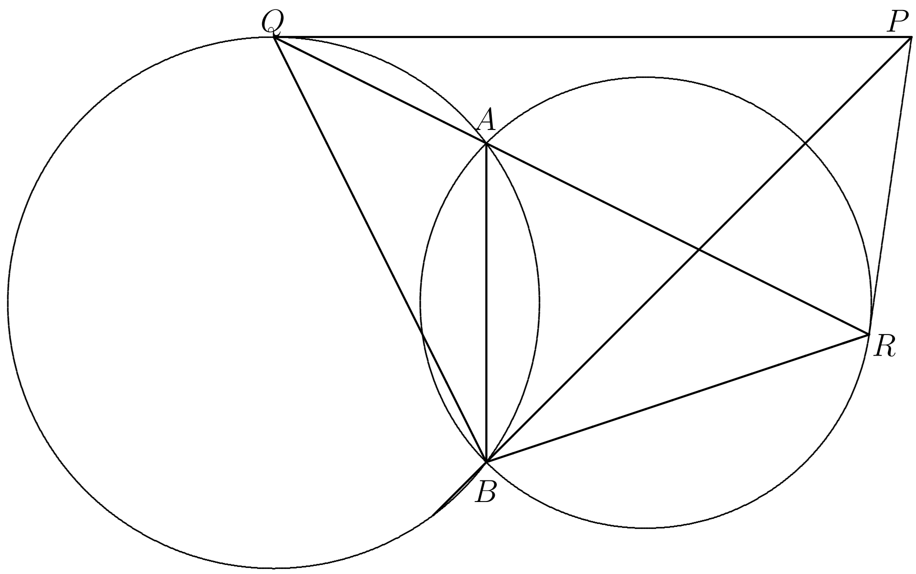
Now we have a pile of supercards with values $1, \ldots, n-1$, the total sum of the values equals $(k \cdot n!)/n = k \cdot (n-1)!$. Then according to induction assumption, we can split supercards into k stacks with the sum of the values in each equal $(n-1)!$. Therefore the sum of cards (normal) in each stuck is $(n-1)! \cdot n = n!$.

5 Denote the point of intersection of the two tangents by P.



(a) By Thales' Theorem, $\angle RBM = \angle RAM$. Since AM and QP are parallel, we have $\angle RAM = \angle RQP$. Since QP is tangent to the first circle, $\angle RQP = \angle QBA$. Similarly, $\angle ARP = \angle ABR = \angle ABM + \angle RAM = \angle ABM + \angle RAM = \angle QBM$. By Thales' Theorem, $\angle QAK = \angle QBK$. Hence $\angle QBM = 180° - \angle QBK = 180° - \angle QAK = \angle KAR$. From $\angle ARP = \angle KAR$, we conclude that AK and PR are parallel.

(b) We have $\angle QPR + \angle QBR = \angle QPR + \angle QBA + \angle RBA = \angle QPR + \angle AQP + \angle ARP = 180°$. Hence $BQPR$ is cyclic so that $\angle PBQ = \angle PRQ = \angle MBQ$ from (a). Hence P lies on MB.



6

Proposition 1. If p is prime and a sequence contains an infinite number of multiples of p then it contains all multiples of p.

Proof. Let us assume that for some k our sequence does not contain pk. If $p|a_n$ and $a_{n+1} \neq pk$ then $a_{n+1} < pk$. This could happen only for a finite number of terms multiple of p.

Proposition 2. Our sequence contains all even numbers.

Proof. It is enough to prove that our sequence contains an infinite number of even terms. Assume that it is not the case. Then for some n all terms starting from a_n are odd. Note, that sequence contains an infinite number of terms a_m (with $m \geq n$) such that $a_{m+1} > a_m$. Let $d = \gcd(a_m, a_{m+1})$, d is odd. Note that $a_m + d < a_{m+1}$ and is not coprime with a_m and therefore $a_m + d$ is a term of our sequence. Note that $a_m + d$ is even. Therefore our sequence contains an infinite number of even terms. Contradiction.

Proposition 3. Our sequence contains all odd numbers.

Proof. Let z be the smallest odd number which is skipped in our sequence. Note that the sequence contains all numbers $2kz$. Each such term should be followed by a term which is less than z. This could happen only for a finite number of terms.

7 Solution for $(2k-1)\times(2k-1)$ lattice ($4k^2$ nodes).

For any test a technician chooses a pair of nodes. If the number of tests is less than $2k^2$, at least one node would not be tested. It could happen that this node is isolated but the rest of the wires are intact. So, at least $2k^2$ tests are needed.

Let us numerate the nodes along the main diagonal Δ of the grid from $1,\ldots,2k$. Let us test pairs of nodes $(1,k+1)$, $(2,k+2)$, $\ldots(k,2k)$ plus every pair of nodes which are symmetrical with respect to Δ ($k+k(2k-1)=2k^2$). Assume that all tests were successful. We need to prove that there is a link between every pair of nodes.

First, we prove that there is a link (connection) between every pair of nodes on Δ. Since nodes 1 and $k+1$ are linked, there exits a path π between them formed by intact wires. Consider a path π' symmetrical to π with respect to Δ. Notice that any node of π' is linked to the symmetrical node of π. Therefore every node of π' is linked to node 1 and therefore all nodes of π' are linked between themselves.

Note that node 2 is either encircled by $\pi\bigcup\pi'$ or belongs to both π and π'. Since nodes 2 and $k+2$ are linked then the intact path between them intersects $\pi\bigcup\pi'$ and therefore both 2 and $k+2$ are linked to 1. Similarly, all other diagonal nodes are linked to 1; therefore all of them are linked.

Now let us consider any non-diagonal node. It is linked with its symmetrical node; the intact path connecting them intersects Δ. This means that any two nodes are linked.

Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Spring 2003.**

1 [4] 2003 dollars are placed into N purses, and the purses are placed into M pockets. It is known that N is greater than the number of dollars in any pocket. Is it true that there is a purse with less than M dollars in it?

2 [4] Two players in turns colour the sides of an n-gon. The first player colours any side that has 0 or 2 common vertices with already coloured sides. The second player colours any side that has exactly 1 common vertex with already coloured sides. The player who cannot move, loses. For which n the second player has a winning strategy?

3 [5] Points K and L are chosen on the sides AB and BC of the isosceles $\triangle ABC$ ($AB = BC$) so that $AK + LC = KL$. A line parallel to BC is drawn through midpoint M of the segment KL, intersecting side AC at point N. Find the value of $\angle KNL$.

4 [5] Each term of a sequence of natural numbers is obtained from the previous term by adding to it its largest digit. What is the maximal number of successive odd terms in such a sequence?

5 [5] Is it possible to tile 2003×2003 board by 1×2 dominoes placed horizontally and 1×3 rectangles placed vertically?

Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Spring 2003.**

1 Let S be an entire amount of money (\$2003),

a_i be amount of money in i-pocket, $i = 1, 2, \ldots, M$. Then

$$a_i < N, \qquad S = \sum_{i=1}^{M} a_i < MN. \tag{1}$$

Let us assume that each purse contains no less than M dollars in it. Let b_i be amount of money in i-purse. Then

$$b_i \geq M, \qquad S = \sum_{i=1}^{N} b_i \geq MN. \tag{2}$$

Contradiction.

2 Consider three cases:

(a) $n > 4$. Let us show that the first player has a winning strategy. On each of his subsequent moves, the first player colours a side which is one space away from one of already coloured sides. Note, that doing this, he creates a "store", which he can use later; however, the second player can not, because of the nature of requirement. So, in the end of the game, after the first player's move, we are left with cases:

(i) One uncoloured side is left (plus "store"). The second player has no move.

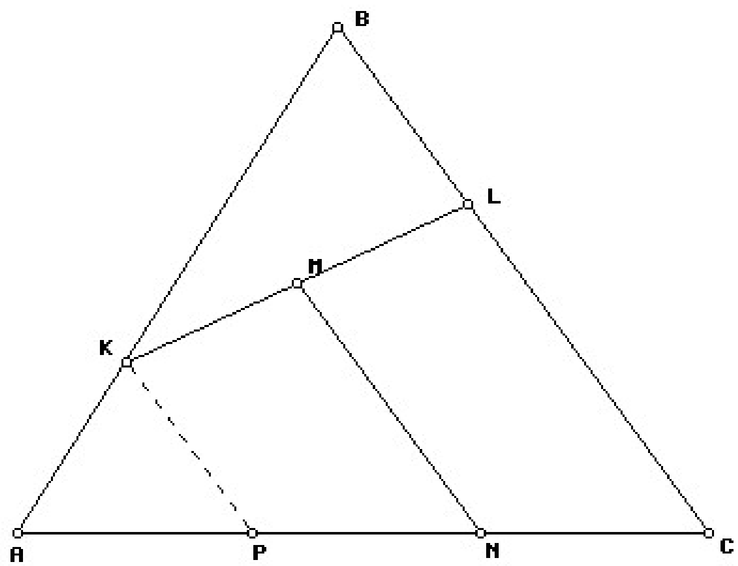
(ii) Two uncoloured sides are left (plus "store"). After the second player's move, the first player wins.

(iii) Three uncoloured sides are left (plus "store"). After the second player's move, the first player uses his "store", and wins on his next move.

(b) From above, we can see that the only chance for the second player to win is in the case (iii), when "store" is not yet created. It corresponds to the case $n = 4$. Really, the first player can not produce his second move and loses.

(c) $n = 3$. The first player wins.

3 Let us draw straight line $KP \parallel BC$ where P is a point on AC. Since $KLCP$ is a trapezoid, its midline $MN = \frac{1}{2}(KP + LC) = \frac{1}{2}(AK + LC) = \frac{1}{2}KL = KM = ML$. Then KL is a diameter of a circle passing through K, N, L and therefore $\angle KNL = 90°$.



4 We start from

Proposition. If a is an even number, then $5a \equiv 0 \pmod{10}$.

Proof is obvious.

Note, that in order to maintain the row of odd terms in a sequence, given the requirements, the last term's digit has to be odd and the term's largest digit even. Further, each addition could change the term's largest digit by at most 1. When it happens, the term's largest digit becomes odd and on next term the row of odd terms in a sequence is terminated. Since the term's largest digit stays the same through the row, the maximal number of terms cannot exceed five due to proposition. It is possible to have a row of five: 807, 815, 823, 831, 839.

5 The answer is negative.

Let us colour the board with black and white strips, black in excess. Note, that since dominoes placed horizontally and 1×3 rectangles placed vertically, each domino covers one black and one white square, meanwhile each rectangle covers three squares of the same colour.

Let us assume, that it is possible to tile 2003×2003 board by dominoes and rectangles. Let n be a number of dominoes. Then the numbers of black and white rectangles are equal to $(2003 \times 1002 - a)/3$ and $(2003 \times 1001 - a)/3$ respectively. Therefore, the difference between black and white rectangles is 2003 and has to be a multiple of 3. Contradiction.

Seniors
(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

O-Level Paper **Spring 2003.**

1 [**3**] 2003 dollars are placed into N purses, and the purses are placed into M pockets. It is known that N is greater than the number of dollars in any pocket. Is it (always) true that there is a purse with less than M dollars in it?

2 [**3**] 100-gon made of 100 sticks. Could it happen that it is not possible to construct a polygon from any lesser number of these sticks?

3 [**4**] Point M is chosen in $\triangle ABC$ so that the radii of the circumcircles of $\triangle AMC$, $\triangle BMC$, and $\triangle BMA$ are no smaller than the radius of the circumcircle of $\triangle ABC$. Prove that all four radii are equal.

4 [**5**] In the sequence 00, 01, 02, 03,. . . , 99 the terms are rearranged so that each term is obtained from the previous one by increasing or decreasing one of its digits by 1 (for example, 29 can be followed by 19, 39, or 28, but not by 30 or 20). What is the maximal number of terms that could remain on their places?

5 [**5**] Prove that one can cut $a \times b$ rectangle, $\frac{b}{2} < a < b$, into three pieces and rearrange them into a square (without overlaps and holes).

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Spring 2003.**

1 Let S be an entire amount of money ($2003),

a_i be amount of money in i-pocket, $i = 1, 2, \ldots, M$. Then

$$a_i < N, \qquad S = \sum_{i=1}^{M} a_i < MN. \tag{1}$$

Let us assume that each purse contains no less than M dollars in it. Let b_i be amount of money in i-purse. Then

$$b_i \geq M, \qquad S = \sum_{i=1}^{N} b_i \geq MN. \tag{2}$$

Contradiction.

2 Yes, it could happen.

Example. Consider a 100-gon with sides:

$$1, 1, 2, 2^2, \ldots, 2^{98}, 2^{99} - 1.$$

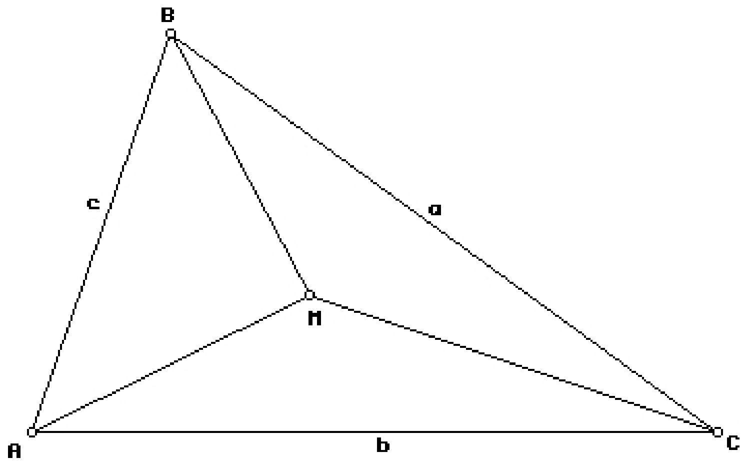
Since $1 + 1 + 2 + \ldots + 2^{98} = 2^{99} > 2^{99} - 1$ it is possible to construct 100-gon with these sides. On the other hand, one cannot construct a polygon from any lesser number of sides. Really, consider two cases:

(a) Side $(2^{99} - 1)$ is among selected.
Then even if the shortest side is absent, $1 + 2 + \ldots + 2^{98} = 2^{99} - 1$.

(b) The longest selected side is 2^k, $1 \leq k \leq 2^{98}$.
Then $1 + 1 + \ldots + 2^{k-1} = 2^k$.

3 Let $\angle AMC = \beta$, $\angle BMC = \alpha$, $\angle AMB = \gamma$, $AC = b$, $BC = a$, $AB = c$, R, r_1, r_2 and r_3 be the radii of the circumcircles of $\triangle ABC$, $\triangle AMC$, $\triangle BMC$ and $\triangle BMA$ respectively. Then formulae $b = 2R\sin\angle B$, $b = 2r_1 \sin\beta$ and condition $r_1 \geq R$ imply that $\sin\beta \leq \sin B$. Similarly, $\sin\alpha \leq \sin A$, $\sin\gamma \leq \sin C$.

Note that $\beta > B$, $\alpha > A$, $\gamma > C$.



Consider two cases:

(a) $\triangle ABC$ is acute.

Then $\beta > B$ and $\sin\beta \leq \sin B$ imply that $\beta \geq \pi - B$. Similarly, $\alpha \geq \pi - A$, $\gamma \geq \pi - C$. Then

$$2\pi = \alpha + \beta + \gamma \geq 3\pi - A - B - C = 2\pi$$

and therefore $\beta = \pi - B$, $\alpha = \pi - A$, $\gamma = \pi - C$ which imply $r_i = R$.

(b) $\triangle ABC$ is not acute.

Assume that $B \geq \frac{\pi}{2}$. Then $\beta > \frac{\pi}{2}$ and

$$2\pi = \alpha + \beta + \gamma > \frac{5\pi}{2} - A - C = \frac{3\pi}{2} + B.$$

Then $B < \frac{\pi}{2}$. Contradiction. This case is impossible.

4 The answer is 50.

Let b_k be a rearranged sequence. Note, that the given operation changes a parity of the next term. I.e., if sum of the digits of b_k is odd/even, then sum of the digits of b_{k+1} is even/odd respectively.

Let us assume that both b_k and b_{k+10} remain on their original places. Note, that the parities of b_k and b_{k+10} are always different. On the other hand, to get b_{k+10} from b_k, one need to change parity an even number of times; so the parities in question should be the same. This implies that a maximal number of terms which could remain on their places does not exceed 50.

Example, in which 50 is achieved:

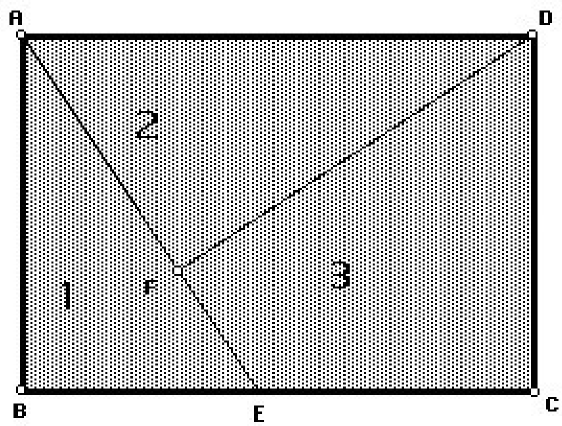
$$00 \nearrow 09, 19 \searrow 10, 20 \nearrow 29, 39 \searrow 30, 40 \nearrow 49, 59 \searrow 50, 60 \nearrow 69, 79 \searrow 70, 80 \nearrow 89, 99 \searrow 90$$

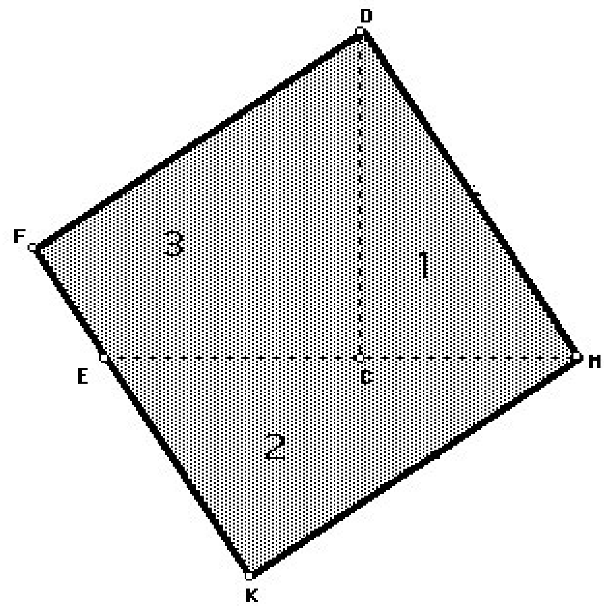
5 Note that $\frac{1}{2}b < a < b$ implies $a < \sqrt{ab} < b$. Let us choose point E on BC such that $AE = \sqrt{ab}$. It is possible due to inequality $BE = \sqrt{ab - a^2} < b$.

Let F be a point of intersection of AE and $DF \perp AE$. Calculating the area of $\triangle AED$ in two ways we get $\frac{1}{2}AE \cdot DF = \frac{1}{2}AD \cdot CD$. Then $FD = ab/\sqrt{ab} = \sqrt{ab}$.

Since $AF = \sqrt{b^2 - ab} < \sqrt{ab} = AE$ (due to inequality $b < 2a$) point F belongs to AE.

Now $\triangle ABE$, $\triangle AFD$ and quadrilateral $DFEC$ could be rearranged into a square by parallel translation of $\triangle ABE$ into $\triangle DCM$ and $\triangle ADF$ into $\triangle EMK$. One can justify it.





Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Spring 2003.**

1 [**4**] Johnny writes down quadratic equation

$$ax^2 + bx + c = 0$$

with positive integer coefficients a, b, c. Then Pete changes one, two, or none "+" signs to "−". Johnny wins, if both roots of the (changed) equation are integers. Otherwise (if there are no real roots or at least one of them is not an integer), Pete wins.

Can Johnny choose the coefficients in such a way that he will always win?

2 [**4**] $\triangle ABC$ is given. Prove that $R/r > a/h$, where R is the radius of the circumscribed circle, r is the radius of the inscribed circle, a is the length of the longest side, h is the length of the shortest altitude.

3 In a tournament, each of 15 teams played with each other exactly once. Let us call the game *"odd"* if the total number of games previously played by both competing teams was odd.

(a) [**4**] Prove that there was at least one "odd" game.

(b) [**3**] Could it happen that there was exactly one "odd" game?

4 [**7**] A chocolate bar in the shape of an equilateral triangle with side of the length n, consists of triangular chips with sides of the length 1, parallel to sides of the bar. Two players take turns eating up the chocolate.

Each player breaks off a triangular piece (along one of the lines), eats it up and passes leftovers to the other player (as long as bar contains more than one chip, the player is not allowed to eat it completely).

A player who has no move or leaves exactly one chip to the opponent, loses.

For each n, find who has a winning strategy.

5 [**7**] What is the largest number of squares on 9×9 square board that can be cut along their both diagonals so that the board does not fall apart into several pieces?

6 [**7**] A trapezoid with bases AD and BC is circumscribed about a circle, E is the intersection point of the diagonals. Prove that $\angle AED$ is not acute.

Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Spring 2003.**

1 ANSWER: Yes.

EXAMPLE. Consider quadratic equation $x^2 + 5x + 6 = 0$. It could be transformed into one of the following four equations:

(a) $x^2 + 5x + 6 = 0$ (roots $-2, -3$;

(b) $x^2 + 5x - 6 = 0$ (roots $-6, 1$);

(c) $x^2 - 5x + 6 = 0$ (roots $2, 3$);

(d) $x^2 - 5x - 6 = 0$ (roots $6, -1$).

2 The longest side of the triangle is a chord of a circumscribed circle and thus it does not exceeds its diameter: $a \le 2R$. Projection of incircle onto the shortest altitude is contained strictly inside of the projection of the triangle onto this altitude. So $2r < h$. Since all the numbers are positive we can multiply these inequalities: $2r \cdot a < h \cdot 2R$ which implies $a/h < R/r$.

3

(a) ANSWER: Yes. Let us assign to i-th team a number $a_i = 0$, if prior to the game it already played even numbers of games and $a_i = 1$ otherwise. Note, that a_i changes after each game in which i-th team participated.

Assume, that all games were "even", meaning that prior to the game both teams had the same parity.

Consider the sum $A = a_1 + a_2 + \cdots + a_{15}$ of the parities of all teams. After each game played by two teams with the same parity A changes by $\pm 2 \equiv 2 \mod 4$.

Initially we had $a_1 = a_2 = \cdots = a_{15} = 0$, therefore $A = 0$. In the end we have $a_1 = a_2 = \cdots = a_{15} = 0$ (each team played an even number of games (14)) and again $A = 0$.

Since the total number of games $15 \cdot 14/2 = 105$ is odd, so in the end of the tournament $A \equiv 2 \mod 4$.

Contradiction.

(b) ANSWER: Yes. We will construct an example of a tournament with one "odd" game. Let us consider a graph, in which vertices represent teams and edges represent games. It is enough to draw edges in such a way that every time (but one) we connect the vertices of the same parity. Let us split all the vertices into three sets of five: $A_1, A_2, \dots A_5$; $B_1, \dots, B_5$; $C_1, \dots, C_5$. We proceed in three steps:

(i) *Step 1*. Let us connect all vertices in each set in the following order: $1-2$, $3-4$, $2-3$, $2-5$, $1-5$, $1-3$, $1-4$, $4-5$, $2-4$, $3-5$. One can check that each time we connect vertices of the same parity and in the end of this step all vertices have parity 0.

(ii) *Step 2*. Now, consider a cycle $A_1B_1C_1A_2B_2C_2\dots A_5B_5C_5$. Let us connect vertices in order A_1-B_1, C_1-A_2, $\dots$, A_5-B_5 (the same parity 0), B_5-C_5 (opposite parities - *the only odd connection*), then C_5-A_1, B_1-C_1, $\dots$, C_4-A_5 (the same parity 1). Note, that now all the vertices have parity 0.

(iii) *Step 3*. Now consider 5 sequences of five connections:

$$A_1-B_1, A_2-B_2, \dots, A_5-B_5;$$
$$A_1-B_2, A_2-B_3, \dots, A_5-B_1;$$
$$A_1-B_3, A_2-B_4, \dots, A_5-B_2;$$
$$A_1-B_4, A_2-B_5, \dots, A_5-B_3;$$
$$A_1-B_5, A2-B_1, \dots, A_5-B_4.$$

We already made the first sequence. With each sequence the parities of vertices $A_1, \dots, B_5$ change; so after 4 sequences executed parities are restored to 0. Now all connections A_i-B_j are done.

In the same way we make remaining 20 connections of B_i-C_j and then remaining 20 connections C_i-A_j.

(b)′ *Second solution*. We construct an example for each $n=4k-1$, applying induction by k. For $k=1, n=3$ we make connections $1-2$, $2-3$, $3-1$ with only second connection odd.

Let us assume that the statement has been proven for $n=4k-1$; we will prove it for $n=4k+3$, proceeding from k to $k+1$. So, we add extra 4 points. Already we have n old points connected between themselves with one odd connection. Now all these vertices are even because each of them is connected with $n-1=4k-2$ others. Let us split old points in $k-1$ quartets and one triplet. Consider an old quartet $Q_1, \dots, Q_4$ and a new one $N_1, \dots, N_4$ and make the following 4 sequences of 4 connections each:

$$Q_1-N_1, Q_2-N_2, Q_3-N_3, Q_4-N_4;$$
$$Q_1-N_2, Q_2-N_3, Q_3-N_4, Q_4-N_1;$$
$$Q_1-N_3, Q_2-N_4, Q_3-N_1, Q_4-N_2;$$
$$Q_1-N_4, Q_2-N_1, Q_3-N_2, Q_4-N_3.$$

After each sequence the parities of all points in both quartets change and in the end they are restored. Let us repeat this procedure, connecting points $N_1, \dots, N_4$ with all old points except T_1, T_2, T_3 (last triplet).

Then we make connections T_1-N_1, T_2-N_2, T_3-N_3 (all points but N_4 become odd). Now connect:

$$N_2-N_3, N_3-N_4, N_4-T_3, N_2-N_4, N_4-T_2, N_1-N_2, N_1-N_4, N_4-T_1, N_1-N_3.$$

One can check easily that all these connections are even. Each new points is connected with other points.

4 Answer: if n is prime, Second Player has a winning strategy; otherwise First Player has.

(i) Let n be a prime number. Let First Player eat a triangle with side k. Leftover is a trapezoid with sides $(k, n-k, n, n-k)$. Denote $a = \max(k, n-k)$ $b = \min(k, n-k)$. Note that $a \neq b$ because $\gcd(a, b) = \gcd(n, n-k) = 1$. Second Player eats a triangle with the side $n-k$, leaving the parallelogram with sides a and b.

(A) Now, if First Player eats triangle with side less than b, then Second Player repeats his move symmetrically (with respect to the center of the parallelogram), and wins since First Player has no move.

(B) If First Player eats triangle with side b, leftover is the trapezoid with sides $(a-b, b, a, b)$, where $\gcd(a-b, b) = \gcd(a, b) = 1$. The game is over when $a = b = 1$, meaning that the last triangular chip is left after First Player's move. Therefore, Second Player wins.

(ii) Let n be a composite number, p any prime divisor of n, $n = kp$. First Player eats triangle with side p. Consider two cases:

(A) If Second Player eats triangle with a side, not equal to $n-p$, then First Player eats triangle with side 1 and wins.

(B) If Second Player eats a triangle with the side $n-p$, then leftover is a parallelogram with sides p and $(k-1)p$. First Player eats the triangle with side p. Again, if Second Player eats triangle with side, not equal to p, then First Player eats triangle with side 1 and wins. So, in the end, after First Player's move, leftover is a triangle with side p. We are in the situation (A) now; however, Second Player has the first move, therefore, he loses.

5 Answer:21

(I) *Example: Fig. 1*

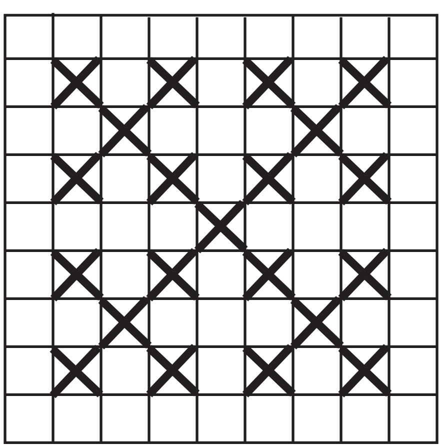


Fig. 1

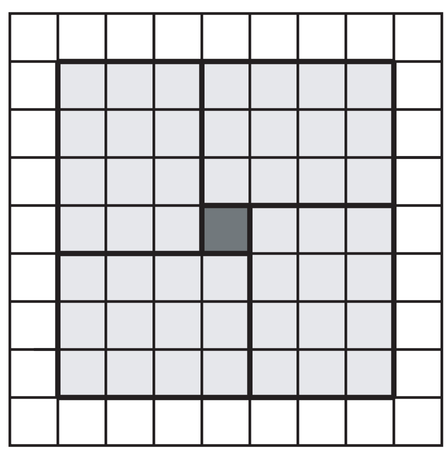


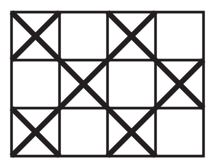
Fig. 2

(II) *Let us prove, that 21 is a maximum.*

First, note that cutting any square on the border results in the board falling apart. Cutting any two adjacent squares also results in failure. Let us divide 7×7 board without central square into four rectangles 3×4 as on Fig. 2. Let us show, that no more then five squares can be cut in each rectangle. Assume, that it is possible to cut at least six. Since row of 3×4 rectangle contains no more than two squares cut, so we have exactly two squares cut in each row. Consider two cases:

(i) The first line is cut (X- -X). Then in the second row only one square could be cut.

(ii) The first row is cut (X-X-) or (-X-X) then second line is cut (-X-X) or (X-X-) and the third line is cut like (X-X-) or (-X-X) again:



However, this results in the board falling apart. Contradiction.

(II)$'$ *Second proof that 21 is a maximum.* First of all, we cut all 81 squares. Let us prove that one needs to repair at least 60 squares in order to restore integrity of the board. Really, all squares are cut, the board splits into 180 pieces (9 triangles along each border and one diamond at each pair of adjacent squares; there are 8 pairs of adjacent squares in each row and column of the board; so we get $\big(4 \times 9 + (9 + 9) \times 8 = 180\big)$ of pieces.

Repairing one square we join no more than 4 different pieces, decreasing their total number by no more than 3. So, to get 1 piece we need to repair at least $\left\lceil \dfrac{179}{3} \right\rceil = 60$ squares.

6 Let O be the center of incircle, K and L tangency points with sides AD and BC respectively.

Solution 1. We start from two following statements:

Lemma 1. *Points K, E, O and L are colinear.*

Proof (see Fig.1 on next page). Note that OK and OL are perpendicular to bases of the trapezoid and thus are parallel. So, O belongs to KL. One can assume with no loss of the generality that $AD > BC$ (if $AD = BC$ our trapezoid is a rhombus and $\angle AED = 90°$).

Let N be a point of intersection of AB and CD. Let K' be a point of tangency of incircle of $\triangle BCN$ with side BC. From the property of tangents (drawn from the same point to the circle) we have

$$BK + BN = CK + CN$$
$$K'C + BN = p$$

where p is a half-perimeter of $\triangle BCN$. So, $BK = CK'$.

Note that $\triangle BEC \simeq \triangle DEA$ ($BC \parallel AD$). Then $\dfrac{BE}{ED} = \dfrac{BC}{AD}$ and therefore $\dfrac{BE}{ED} = \dfrac{BK}{LD}$. This implies that $\triangle BKE \simeq \triangle DLE$ ($\angle KBE = \angle LDE$ and $\dfrac{BE}{ED} = \dfrac{BK}{LD}$). Then $\angle BEK = \angle DEL$ which means that points K, E, L are colinear. $\square$

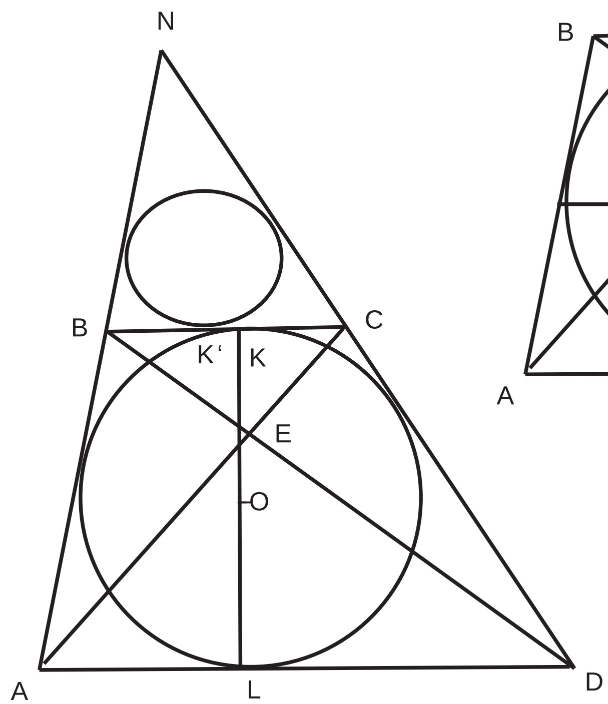


Fig. 1

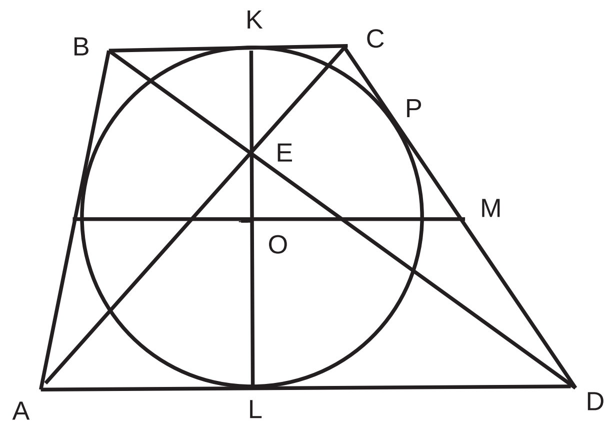


Fig. 2

Lemma 2. *Let S be a midpoint of side PQ of $\triangle PQR$. If $RS = \frac{1}{2}PQ$ then $\angle PRS = 90°$. If $RS < \frac{1}{2}PQ$ then $\angle PRS > 90°$.*

Proof. Consider a circumference with diameter PQ. Then R belongs to this circumference in the former case and lies inside of it in the latter case. $\square$

Let M be a midpoint of CD (see Fig. 2); then OM is a midline of trapezoid $KCDL$, and therefore it is parallel to its bases and is equal to $(KC + LD)/2 = (PC + PD)/2 = CD/2$ where P is a point of tangency with CD. Then by lemma 2 $\angle COD = 90°$ and O belongs to a circumference with diameter CD and a center M. Since $MO \parallel CK$, therefore $MO \perp KL$, we conclude that KL is tangent to this circle at O. Then all points of KL (but O) are outside of this circumference. Therefore $\angle DEC$ does not exceed $90°$, so $AED \geq 90°$.

Solution 2. Extending AD beyond A (see Fig. 3), we choose point D' such that $AD' = BC$. Also extending BC beyond B we choose point C' such that $BC' = AD$. Then $CC'D'D$ is a parallelogram. Select point N on CC', such that $C'N = D'A = BC$.

Since $AC'BD$ is a parallelogram ($C'B = AD$, $C'B \parallel AD$) then $C'A \parallel BD$. Therefore $\angle BEC = \angle C'AC$. So we need to prove that $\angle C'AC \geq 90°$. Let M be a midpoint of CC'; then M is a midpoint of NB. Then $CC' = AD + BC = AB + CD$ (property of circumscribed quadrilateral), and $CD = AN$ (because $ANC'D'$ is a parallelogram) and $AB + AN \geq 2AM$ (triangle inequality). Then $\angle C'AC \geq 90°$ due to lemma 2.

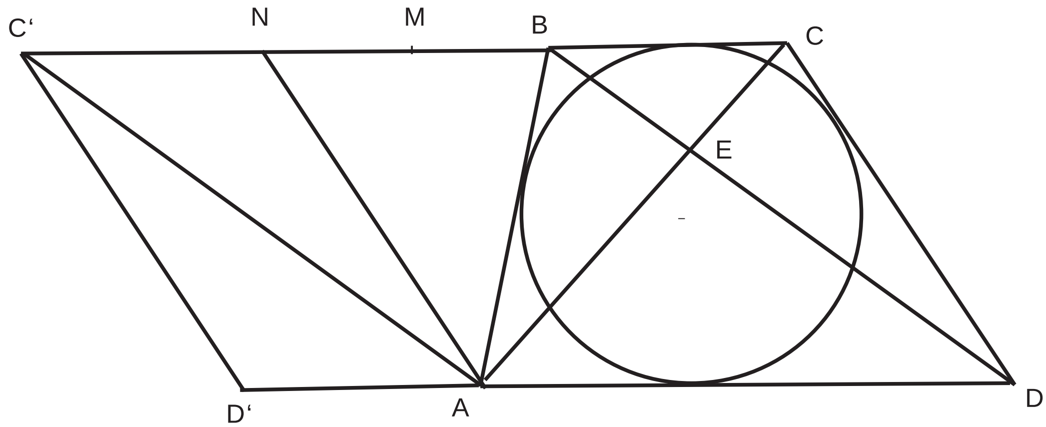


Fig. 3

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Spring 2003.**

1 [**4**] A triangular pyramid $ABCD$ is given. Prove that $R/r > a/h$, where R is the radius of the circumscribed sphere, r is the radius of the inscribed sphere, a is the length of the longest edge, h is the length of the shortest altitude (from a vertex to the opposite face).

2 [**5**] $P(x)$ is a polynomial with real coefficients such that $P(a_1) = 0$, $P(a_{i+1}) = a_i$ $(i = 1, 2, \dots)$ where $\{a_i\}_{i=1,2,\dots}$ is an infinite sequence of distinct natural numbers. Determine the possible values of degree of $P(x)$.

3 [**5**] Can one cover a cube by three paper triangles (without overlapping)?

4 [**6**] A right $\triangle ABC$ with hypothenuse AB is inscribed in a circle. Let K be the midpoint of the arc BC not containing A, N the midpoint of side AC, and M a point of intersection of ray KN with the circle. Let E be a point of intersection of tangents to the circle at points A and C.

Prove that $\angle EMK = 90°$.

5 [**6**] Prior to the game John selects an integer greater than 100.

Then Mary calls out an integer d greater than 1. If John's integer is divisible by d, then Mary wins. Otherwise, John subtracts d from his number and the game continues (with the new number). Mary is not allowed to call out any number twice. When John's number becomes negative, Mary loses. Does Mary have a winning strategy?

6 [**7**] The signs "+" or "-" are placed in all cells of a 4×4 square table. It is allowed to change a sign of any cell altogether with signs of all its adjacent cells (i.e. cells having a common side with it). Find the number of different tables that could be obtained by iterating this procedure.

7 [**8**] A square is triangulated in such way that no three vertices are colinear. For every vertex (including vertices of the square) the number of sides issuing from it is counted. Can it happen that all these numbers are even?

Keep the problem set.
Visit: `http://www.math.toronto.edu/oz/turgor/`

Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Spring 2003.**

1 **Solution 1.** The longest edge of the pyramid is a chord of the circumscribed sphere and thus it does not exceed diameter of the sphere: $a \le 2R$. Projection of insphere onto the shortest altitude of the pyramid is strictly contained in the projection of the pyramid onto this altitude. So, $2r < h$. Multiplying inequalities we get $2r \cdot a < h \cdot 2R$, which is equivalent to $\dfrac{a}{h} < \dfrac{R}{r}$.

Solution 2. Let us calculate the volume of the pyramid in two ways: $V = \frac{1}{3} H_j S_j$ and $V = \frac{1}{3} r(S_1 + S_2 + S_3 + S_4)$, where S_j is the area of j-th face, and H_j is a corresponding altitude. Thus $H_j = 3V/S_j$ and $h = 3V/S_{\max}$, where $S_{\max} = \max_j S_j$ is the area of the face with the largest area.

Therefore, $r = 3V/(S_1+S_2+S_3+S_4)$. Note that $(S_1+S_2+S_3+S_4) > 2S_{\max}$. Really, if we project the pyramid onto one of its faces (treated as a base) then a projections of the lateral faces will cover the base. Since area of projection is less than the area of the face itself (because none of lateral faces is parallel to the base) we get our inequality.

Then
$$\frac{R}{r} = \frac{R(S_1 + S_2 + S_3 + S_4)}{3V} > \frac{2RS_{\max}}{3V} = \frac{2R}{h} \ge \frac{a}{h}.$$

2 ANSWER: $\deg P = 1$.

SOLUTION. We consider a more general problem when a_i are integers (not necessarily positive).

(i) $\deg P = 0$ then $P = c = \text{const}$ and all $a_i = P(a_{i+1})$ are equal which contradicts conditions.

(ii) $\deg P = 1$ is possible: for example, $a_i = i$, $P(x) = x - 1$.

(iii) $m = \deg P \geq 2$. Let us prove that such sequence $\{a_i\}$ does not exists.

LEMMA. *If $m \geq 2$ then there exists a constant C such that* $\forall x : |x| \geq C \quad |P(x)| > |x|$.

PROOF. Let $P(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$ with $b_m \neq 0$. Then for $|x| \geq 1$

$$|P(x)| \geq |b_m| \cdot |x|^m - \big(|b_{m-1}| + |b_{m-2} + \cdots + |b_0|\big)\Big)|x|^{m-1} \geq \\ |x|^{m-1}\Big(|b_m| \cdot |x| - \big(|b_{m-1}| + |b_{m-2} + \cdots + |b_0|\big)\Big)$$

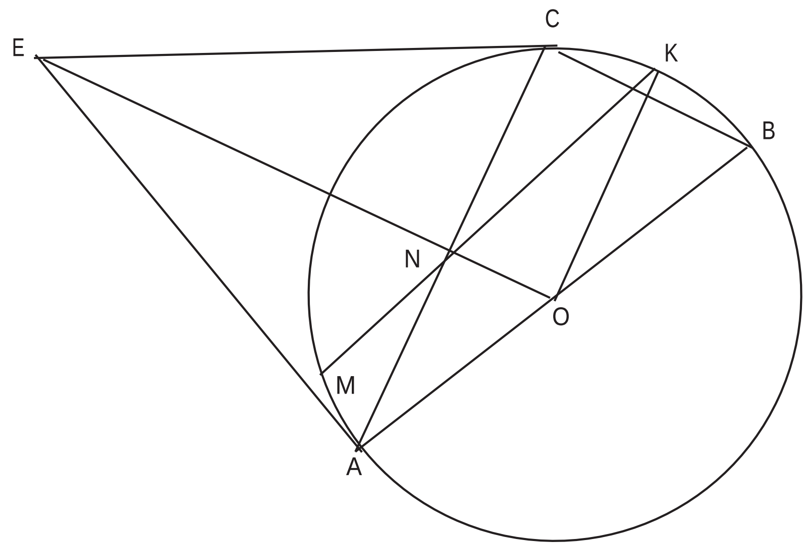
which is larger than $|x|^{m-1}$ as $|x| \geq \big(|b_{m-1}| + |b_{m-2} + \cdots + |b_0| + 1\big)/|b_m|$ and in turn $|x|^{m-1} \geq |x|$. $\square$

Since a_i are distinct integers, for any C there exists M such that $\forall i \geq M \quad |a_i| \geq C$. Then according to Lemma, for $i \geq M \quad |a_i| = |P(a_{i+1})| \geq |a_{i+1}|$ and therefore $|a_i|$ are bounded. Contradiction.

3 First let us notice that no vertex can be covered by an interior of a triangle. So, it should be covered by edges. Note that if an interior of edge covers a vertex, the sum of adjacent angles covered by triangle is exactly 180°. At the same time the sum of angles adjacent to vertex of cube is 270°. Therefore, at least 90° at each vertex should be covered by angles of triangles. So angles of triangles cover at least $8 \cdot 90°$ and there should be at least $8 \cdot 90°/180° = 4$ of triangles.

Consider T-shaped envelope of a cube, consisting of two rectangles. Each of them can be covered by 2 triangles. So, it is possible to cover a cube by 4 triangles.

4



Let O be a center of the circle. Since $\triangle ABC$ is a right triangle, O is a midpoint of hypotenuse AB. Then $\angle NOK$ is a right angle. Really, midline NO of $\triangle ABC$ is parallel to BC and $OK \perp BC$ (arcs CK and KB are equal).

Note that right triangles $\triangle ECO$ and $\triangle EAO$ are congruent (by side and hypotenuse). So EO is a bisector of $\angle AEC$.

Further, $\triangle AEEC$ is isosceles ($AE = EC$ as tangents to the circle). Then median EN is also a bisector. Therefore, EN and EO are both bisectors of the same $\angle AEC$; so E, N, O are colinear.

Furthermore, A, E, C and O belong to the same circumference ($\angle ECO = \angle EAO = 90°$). By power of the point we have

$$AN \times NC = EN \times NO,$$
$$AN \times NC = MN \times NK$$

which imply that

$$MN \times NK = EN \times NO,$$

meaning that M, K, E and O belong to the same circumference (by power of the point).

Then $\angle EMK = \angle EOK$ (subtended by the same arc). However, $\angle EOK = 90°$; therefore $\angle EMK = 90°$.

5 Answer: Mary has a winning strategy.

Consider John's number modulo 6.

Mary calls 2. If John continues to play, then his number was odd: $J \equiv 1, 3, 5 \mod 6$. His new number $J_1 = J - 2 \equiv 1, 3, 5 \mod 6$ is also odd.

Mary calls 3. So, if $J_1 \equiv 3 \mod 6$, Mary wins on her second move. So, after two moves John's number is $J_2 = J_1 - 3 \equiv 2, 4 \mod 6$ or $J_2 \equiv 2, 4, 8, 10 \mod 12$.

Mary calls 4. John continues to play, if $J_2 \equiv 2, 10 \mod 12$ or $J_3 = J_2 - 4 \equiv 10, 6 \mod 12$.

Mary calls 6. If $J_3 \equiv 6 \mod 12$ then $J_4 \equiv 0 \mod 12$, meaning that Mary wins. So, $J_4 \equiv 4 \mod 12$.

Mary calls 16. $J_5 \equiv 0 \mod 12$ and Mary wins. Note, that John's last number is not negative, for the most he subtracted is $2+3+4+6+16 = 31$.

There are other sequences of numbers of Mary's moves.

6 Answer: 2^{12}.

Let A be a 4×4-table consisting of "+" and "–".

Since it is allowed to change a sign in any cell (altogether with signs of all adjacent cells), we have 16 elementary transformations T_{ij} ($i, j = 1, \ldots, 4$); all other transformations are compositions of elementary ones.

Note, that elementary transformations commute: if from table A we get table V applying some sequence of elementary transformations, then applying to A the same sequence, but in different order, we get V again. Also note, that changing sign in a cell (and in its neighboring cells) of table A twice we will get A again; therefore every elementary transformation needs to be applied no more than once.

Let T be a 4×4-matrix of transformation consisting of "0" and "1". The number 0(1) in cell (i, j) shows that elementary transformation T_{ij} is applied 0(1) times.

It is clear, that if table A and matrix T are given, then the resulting table V is uniquely defined. Note, that if we apply two transformations with matrices T and S, then resulting transformation corresponds to matrix $T+S$ (corresponding elements are added modulo 2).

(i) First, let us get an upper estimate. One can check that the following matrices do not change a table:

$$\begin{bmatrix}0110\\1001\\1001\\0110\end{bmatrix},\begin{bmatrix}1111\\0110\\0110\\1111\end{bmatrix},\begin{bmatrix}1001\\1111\\1111\\1001\end{bmatrix},\begin{bmatrix}0000\\0000\\0000\\0000\end{bmatrix},H=\begin{bmatrix}0001\\0011\\0101\\1110\end{bmatrix},G=\begin{bmatrix}0010\\0111\\1000\\1011\end{bmatrix};$$

from matrix H we can get 3 more matrices with the same property by 90° rotations; from matrix G we can get 7 more matrices with the same property by rotations and a mirror reflection. Altogether, we have at least 16 (2^4) matrices P_α ($\alpha = 1, \ldots, 16$), which preserve tables.

Now let us divide all transformation matrices into equivalence classes in the following way: $T \sim S$ if applied to table A both produce the same result. Note that for any matrix of transformation S and any $\alpha = 1, \ldots, 16$ we have $S \sim S + P_\alpha$. So each equivalence class contains at least 2^4 elements and since there are 2^{16} matrices of transformations, there are at most $2^{16}/2^4 = 2^{12}$ different equivalence classes. This means that table A can generate no more than 2^{12} different tables.

(ii) Let us get a lower estimate. Let us color our table as a chess board with white top-left corner.

1. Note that any table could be transformed into a table with "$-$" in all black cells (if some black cell contains "+" we can change it to "$-$" without affecting all other black cells).

2. Now we show how with some special transformations we can make "$-$" in 4 white cells of the lower half of our table without affecting black cells. Let us consider the following matrices of transformations:

$$S_1=\begin{bmatrix}0100\\1110\\0100\\0000\end{bmatrix},\quad S_2=\begin{bmatrix}1100\\1000\\0000\\0000\end{bmatrix},\quad S_3=\begin{bmatrix}0100\\1110\\0101\\0011\end{bmatrix},\quad S_3=\begin{bmatrix}1100\\1010\\0111\\0010\end{bmatrix}.$$

One can check that applying transformations with matrices S_1, S_2, S_3, S_4 we change signs only in cells, marked by 1 (all of them are white):

$$I_1 = \begin{bmatrix} 0000 \\ 0101 \\ 0000 \\ 0100 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1010 \\ 0000 \\ 1000 \\ 0000 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 0000 \\ 0100 \\ 0000 \\ 0001 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1000 \\ 0000 \\ 0010 \\ 0000 \end{bmatrix}.$$

Note that each of matrices I_j has exactly one "1" in the lower half-table. Therefore, if table A has "+" in some white cells of lower half-table, we can change them into "−" applying corresponding transformations S_i without affecting black cells.

Now, we have "−" in all cells except 4 white cells in the upper half-table. Thus we can transform A into one of 16 tables of this type; call them canonical tables.

Inversely, if one can reduce table A to canonical table V, one can restore A from V by the same transformation. We already proved in (i) that each table can be transformed into no more than 2^{12} tables; since there is 2^{16} tables and only 2^4 canonical tables, each canonical table can be transformed into exactly 2^{12} tables.

Therefore, *every table can be transformed into exactly* 2^{12} *tables.*

7 Answer: No.

Solution. Let us introduce *degree of vertex* P, the number of segments issued from P.

Let us assume that degrees of all vertices are even.

Lemma. *Let degrees of all vertices be even. Then one could paint all the triangles into two colors so that every two triangles with a common side would have different colors.*

Proof. Let us consequently paint adjacent triangles into opposite colors, every time connecting the centers of consequent triangles by a curve passing through their common side.

Assume that on some step we painted a triangle and found that an adjacent triangle had been already painted into the same color. Connecting centers of conflicting triangles we get a closed path, intersecting an odd number of segments; each of them is intersected only once. This path bounds some region D.

Consider directed segments issued from vertices belonging to D. Their total number i equals the sum of degrees of vertices belonging to D and is even by assumption. On the other hand, the number of directed segments with both ends in D is also even because each such directed segment is paired with the opposite one. Therefore the total number of (directed) segments intersecting our path must be also even.

This contradiction proves lemma. □

Let us paint triangles according to Lemma. Due to the assumption that vertices of the square have even degrees as well, all the "boundary" triangles are painted in the same color, say, white.

Let W and B be the numbers of white and black triangles respectively. We assume that every inner segment has two sides; one is colored in black and the other in white colors. Then the total number of white sides is $3W$ while the total number of black sides is $3B$. Note that xactly 4 white sides do not have black counterparts; they are sides of the square. So, $3(W - B) = 4$ which is impossible. Contradiction.