PROBLEM SHORTLIST

(with solutions)

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Contents

LINEAR ALGEBRA

Problem 1. Let $A \in \mathcal{M}_2(\mathbb{R})$. Suppose

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_2 \end{pmatrix}$$

satisfies

$$a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 < \frac{1}{5}.$$

Show that $I + A$ is invertible.

Solution. We have

$$\det(I + A) = 1 + a_{11} + a_{22} + a_{11}a_{22} - a_{12}a_{21}.$$

Since

$\pm ab \geq -\frac{1}{2}(a^2 + b^2)$ for all $a, b \in \mathbb{R}$, we get

$$\det(I + A) \geq 1 + a_{11} + a_{22} - \frac{1}{2}(a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2) > 1 + a_{11} + a_{22} - \frac{1}{10}.$$

Also, $a_{11}^2 < \frac{1}{5}$, so $|a_{11}| < \frac{1}{\sqrt{5}}$, and similarly for a_{22}, therefore

$$\det(I + A) > 1 - \frac{2}{\sqrt{5}} - \frac{1}{10} > 0$$

so $I + A$ is invertible.

Remark. The problem is a particular case of a well known result in matrix theory: if $\|\cdot\|$ is a sub-multiplicative norm (that is, $\| XY \| \leq \| X \| \cdot \| Y \|$ for all matrices *X*, *Y*) and $\| A \| < 1$, then $I_n + A$ is invertible.

Problem 2. Let

$$A=\begin{pmatrix} 1 & \cos\varphi & 0 \\ \sin\varphi & 1 & -\cos\varphi \\ 0 & \sin\varphi & 1 \end{pmatrix}, \quad \varphi \neq \frac{m\pi}{4}, \quad m\in\mathbb{Z}.$$

Calculate A^n.

Solution. We present the matrix A in view of

$$A=I_3+H,$$

where I_3 is the identity matrix and

$$H=\begin{pmatrix} 0 & \cos\varphi & 0 \\ \sin\varphi & 0 & -\cos\varphi \\ 0 & \sin\varphi & 0 \end{pmatrix}.$$

Then

$$H^2=\begin{pmatrix} \sin\varphi\cos\varphi & 0 & -\cos^2\varphi \\ 0 & 0 & 0 \\ \sin^2\varphi & 0 & -\sin\varphi\cos\varphi \end{pmatrix} \quad \text{and} \quad H^3=O_3,$$

where O_3 is the zero matrix. We expand

$$\begin{aligned} A^n &=\left(I_3+H\right)^n=\sum_{k=0}^{n}\binom{n}{k}H^k I_3^{n-k}=\binom{n}{0}H^0+\binom{n}{1}H+\binom{n}{2}H^2 \\ &=I_3+nH+\frac{n(n-1)}{2}H^2 \\ &=\begin{pmatrix} 1+\dfrac{n(n-1)}{4}\sin 2\varphi & n\cos\varphi & -\dfrac{n(n-1)}{2}\cos^2\varphi \\ n\sin\varphi & 1 & -n\cos\varphi \\ \dfrac{n(n-1)}{2}\sin^2\varphi & n\sin\varphi & 1-\dfrac{n(n-1)}{4}\sin 2\varphi \end{pmatrix}. \end{aligned}$$

<u>Problem 3.</u> Consider the $n \times n$ matrix

$$A_n = \begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 & 1 \\ 1 & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & 0 & ... & 1 & 1 \end{bmatrix}.$$

Find the values of n, for which A_n is invertible.

<u>Solution.</u> We first need to find the characteristic polynomial of A_n.

$$\chi_{A_n}(x) = \begin{vmatrix} 1-x & 0 & 0 & 0 & ... & 0 & 1 \\ 1 & 1-x & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 1-x & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & 1-x & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 1-x & 0 \\ 0 & 0 & 0 & 0 & ... & 1 & 1-x \end{vmatrix}$$

$$= (1-x)^n + (-1)^{n+1}, \quad \text{for } n \in \mathbb{N}.$$

For $n = 2k$, it is $\chi_{A_n}(x) = (1-x)^{2k} + (-1)^{2k+1}$, and $x \mid \chi_{A_n}(x)$, so A_n is not invertible (it has the zero as an eigenvalue).

For $n = 2k+1$, it is $\chi_{A_n}(x) = (1-x)^{2k+1} + (-1)^{2k+2}$, and $x \nmid \chi_{A_n}(x)$. So, A_n is invertible for n odd.

Problem 4. Consider a natural number $n \geq 1$ and a continuous real valued function f defined on the interval $[a, b]$. Show that there is only one polynomial function p of degree $\leq n$ such that

$$p(a) = f(a), \tag{1}$$

and

$$\int_a^b \left(f(x) - p(x) \right) q(x) \mathrm{d}x = 0, \tag{2}$$

for any polynomial function q of degree $\leq n-1$.

Solution. On the vector space of continuous real valued functions defined on $[a, b]$ take the inner product

$$\langle f, g \rangle = \int_a^b f(x) g(x) \mathrm{d}x,$$

so that (2) can be written as

$$\langle f - p, q \rangle = 0. \tag{3}$$

Denote by P_n the vector space of polynomial functions of degree $\leq n$. In P_n take a basis of orthogonal polynomials,

$$\varphi_0, \varphi_1, ..., \varphi_n \ \left(\deg \varphi_0 < \deg \varphi_1 < ... < \deg \varphi_n \right).$$

Represent p with respect to this basis as

$$p = \sum_{i=0}^{n} \alpha_i \varphi_i,$$

and write (3) in the equivalent form

$$\langle f - \sum_{i=0}^{n} \alpha_i \varphi_i, \varphi_j \rangle = 0, \quad j = 0, 1, ..., n-1,$$

which implies

$$\alpha_i = \frac{\langle f, \varphi_i \rangle}{\langle \varphi_i, \varphi_i \rangle}, \quad i = 0, 1, ..., n-1.$$

Now, condition (1) emerges as

$$\sum_{i=0}^{n} c_i \varphi_i(a) = f(a). \tag{4}$$

Since all the roots of φ_n are in (a, b), we have $\varphi_n(a) \neq 0$, so that (4) yields

$$c_n = \frac{1}{\varphi_n(a)} \left(f(a) - \sum_{i=0}^{n-1} c_i \varphi_i(a) \right).$$

<u>Problem 5.</u> Calculate the determinant

$$\Delta_n = \begin{vmatrix} \binom{2n-2}{n-1} & \binom{2n-3}{n-1} & \binom{2n-4}{n-1} & \dots & \binom{n+1}{n-1} & \binom{n}{n-1} & 1 \\ \binom{2n-3}{n-2} & \binom{2n-4}{n-2} & \binom{2n-5}{n-2} & \dots & \binom{n}{n-2} & \binom{n-1}{n-2} & 1 \\ \binom{2n-4}{n-3} & \binom{2n-5}{n-3} & \binom{2n-6}{n-3} & \dots & \binom{n-1}{n-3} & \binom{n-2}{n-3} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \binom{n+1}{2} & \binom{n}{2} & \binom{n-1}{2} & \dots & \binom{4}{2} & \binom{3}{2} & 1 \\ \binom{n}{1} & \binom{n-1}{1} & \binom{n-2}{1} & \dots & \binom{3}{1} & \binom{2}{1} & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{vmatrix}.$$

<u>Solution.</u> We transform the identity $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$ into $\binom{n+1}{k}-\binom{n}{k-1}=\binom{n}{k}$. We present the determinant Δ_n in view of

$$\Delta_n = \begin{vmatrix} \binom{2n-2}{n-1} & \binom{2n-3}{n-1} & \binom{2n-4}{n-1} & \dots & \binom{n+1}{n-1} & \binom{n}{n-1} & 1 \\ \binom{2n-3}{n-2} & \binom{2n-4}{n-2} & \binom{2n-5}{n-2} & \dots & \binom{n}{n-2} & \binom{n-1}{n-2} & 1 \\ \binom{2n-4}{n-3} & \binom{2n-5}{n-3} & \binom{2n-6}{n-3} & \dots & \binom{n-1}{n-3} & \binom{n-2}{n-3} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \binom{n+1}{2} & \binom{n}{2} & \binom{n-1}{2} & \dots & \binom{4}{2} & \binom{3}{2} & 1 \\ \binom{n}{1} & \binom{n-1}{1} & \binom{n-2}{1} & \dots & \binom{3}{1} & \binom{2}{1} & 1 \\ \binom{n-1}{0} & \binom{n-2}{0} & \binom{n-3}{0} & \dots & \binom{2}{0} & \binom{1}{0} & 1 \end{vmatrix}.$$

Subtracting the adjacent rows we obtain

$$\Delta_n = \begin{vmatrix} \binom{2n-3}{n-1} & \binom{2n-4}{n-1} & \binom{2n-5}{n-1} & \dots & \binom{n}{n-1} & \binom{n-1}{n-1} & 0 \\ \binom{2n-4}{n-2} & \binom{2n-5}{n-2} & \binom{2n-6}{n-2} & \dots & \binom{n-1}{n-2} & \binom{n-2}{n-2} & 0 \\ \binom{2n-5}{n-3} & \binom{2n-6}{n-3} & \binom{2n-7}{n-3} & \dots & \binom{n-2}{n-3} & \binom{n-3}{n-3} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \binom{n}{2} & \binom{n-1}{2} & \binom{n-2}{2} & \dots & \binom{3}{2} & \binom{2}{2} & 0 \\ \binom{n-1}{1} & \binom{n-2}{1} & \binom{n-3}{1} & \dots & \binom{2}{1} & \binom{1}{1} & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{vmatrix}.$$

Expanding the determinant with respect to the last column we have

$$\Delta_n = \Delta_{n-1} = \begin{vmatrix} \binom{2n-3}{n-1} & \binom{2n-4}{n-1} & \binom{2n-5}{n-1} & \dots & \binom{n+1}{n-1} & \binom{n}{n-1} & 1 \\ \binom{2n-4}{n-2} & \binom{2n-5}{n-2} & \binom{2n-6}{n-2} & \dots & \binom{n}{n-2} & \binom{n-1}{n-2} & 1 \\ \binom{2n-5}{n-3} & \binom{2n-6}{n-3} & \binom{2n-7}{n-3} & \dots & \binom{n-1}{n-3} & \binom{n-2}{n-3} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \binom{n+1}{3} & \binom{n}{3} & \binom{n-1}{3} & \dots & \binom{5}{3} & \binom{4}{3} & 1 \\ \binom{n}{2} & \binom{n-1}{2} & \binom{n-2}{2} & \dots & \binom{4}{2} & \binom{3}{2} & 1 \\ \binom{n-1}{1} & \binom{n-2}{1} & \binom{n-3}{1} & \dots & \binom{3}{1} & \binom{2}{1} & 1 \end{vmatrix}.$$

Note that Δ_{n-1} is the upper right $(n-1)\times(n-1)$ block of Δ_n. Subtracting the adjacent rows again we have

$$\Delta_{n-1}=\begin{vmatrix} \binom{2n-4}{n-1} & \binom{2n-5}{n-1} & \binom{2n-6}{n-1} & \cdots & \binom{n}{n-1} & \binom{n-1}{n-1} & 0 \\ \binom{2n-5}{n-2} & \binom{2n-6}{n-2} & \binom{2n-7}{n-2} & \cdots & \binom{n-1}{n-2} & \binom{n-2}{n-2} & 0 \\ \binom{2n-6}{n-3} & \binom{2n-7}{n-3} & \binom{2n-8}{n-3} & \cdots & \binom{n-2}{n-3} & \binom{n-3}{n-3} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{n}{3} & \binom{n-1}{3} & \binom{n-2}{3} & \cdots & \binom{4}{3} & \binom{3}{3} & 0 \\ \binom{n-1}{2} & \binom{n-2}{2} & \binom{n-3}{2} & \cdots & \binom{3}{2} & \binom{2}{2} & 0 \\ \binom{n-1}{1} & \binom{n-2}{1} & \binom{n-3}{1} & \cdots & \binom{3}{1} & \binom{2}{1} & 1 \end{vmatrix}.$$

After an expansion the determinant Δ_{n-1} becomes

$$\Delta_{n-1}=\begin{vmatrix} \binom{2n-4}{n-1} & \binom{2n-5}{n-1} & \binom{2n-6}{n-1} & \cdots & \binom{n+1}{n-1} & \binom{n}{n-1} & 1 \\ \binom{2n-5}{n-2} & \binom{2n-6}{n-2} & \binom{2n-7}{n-2} & \cdots & \binom{n}{n-2} & \binom{n-1}{n-2} & 1 \\ \binom{2n-6}{n-3} & \binom{2n-7}{n-3} & \binom{2n-8}{n-3} & \cdots & \binom{n-1}{n-3} & \binom{n-2}{n-3} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{n+1}{4} & \binom{n}{4} & \binom{n-1}{4} & \cdots & \binom{6}{4} & \binom{5}{4} & 1 \\ \binom{n}{3} & \binom{n-1}{3} & \binom{n-2}{3} & \cdots & \binom{5}{3} & \binom{4}{3} & 1 \\ \binom{n-1}{2} & \binom{n-2}{2} & \binom{n-3}{2} & \cdots & \binom{4}{2} & \binom{3}{2} & 1 \end{vmatrix},$$

which means that

$$\Delta_n=\Delta_{n-1}=\Delta_{n-2}\ .$$

Continue with the same arguments we have

$$\Delta_n=\Delta_{n-1}=\Delta_{n-2}=\Delta_{n-3}=\ldots=\Delta_2=\begin{vmatrix} \binom{n}{n-1} & 1 \\ \binom{n-1}{n-2} & 1 \end{vmatrix}=\binom{n}{n-1}-\binom{n-1}{n-2}=\binom{n-1}{n-1}=1\,.$$

<u>Problem 6.</u> Consider an $n\times n$ symmetric matrix *A* with real entries a_{ij}, and let λ_1 be the largest eigenvalue of *A*.

a) Prove that $a_{ii} \le \lambda_1, \ \ \forall i = 1,2,...,n.$

b) Show that, if for a some $i \in \{1,2,...,n\}$

$$a_{ii} = \lambda_1$$

holds, then

$$a_{ij} = 0 \quad \text{for} \quad j \ne i,\ j \in \{1,2,...,n\}.$$

<u>Solution.</u> Denote by

$\lambda_1 \ge \lambda_2 \ge ... \ge \lambda_n$ - the eigenvalues of *A*,

$M_{n,1}(\mathbb{R})$ - the vector space of $n\times 1$ column matrices with real entries,

$e_1, e_2, ..., e_n$ - the canonical basis of $M_{n,1}(\mathbb{R})$, and by

(u,v) - the Euclidean inner product on $M_{n,1}(\mathbb{R})$:

$$(u,v) = \sum_{i=1}^{n} u_i v_i, \quad \text{with} \quad u = (u_1,...,u_n)^T, v = (v_1,...,v_n)^T.$$

As any symmetric matrix, A can be expressed as

$$A = QDQ^T, \tag{1}$$

where Q is an orthogonal matrix and *D* is the diagonal matrix $D = \text{diag}(\lambda_1,...,\lambda_n)$.

a) Using (1) we deduce

$$a_{ii} = (Ae_i, e_i) = (DQ^T e_i, Q^T e_i) = \lambda_1 x_1^2 + ... + \lambda_n x_n^2,$$

with $x = Q^T e_i$. Since $\| x \| = 1$, relation (2) immediately implies

$$a_{ii} \le \lambda_1 (x_1^2 + ... + x_n^2) = \lambda_1 \| x \|^2 = \lambda_1. \tag{2}$$

Obs. A straightforward answer to question a) uses the inequality

$$\lambda_n \| x \|^2 \le (Ax, x) \le \lambda_1 \| x \|^2,$$

valid for any $n\times 1$ column matrix *x*: here one takes $x = e_i$.

b) Assume $a_{ii} = \lambda_1$ for a some *i*, that is, see (2),

$$\lambda_1 x_1^2 + ... + \lambda_n x_n^2 = \lambda_1. \tag{3}$$

Let r denote the multiplicity of the eigenvalue λ_1.

For $\| x \| = 1$, assumption (3) can be written as

$$\lambda_1 x_1^2 + ... + \lambda_1 x_r^2 + \lambda_2 x_{r+1}^2 + ... + \lambda_n x_n^2 = \lambda_1 (x_1^2 + ... + x_n^2),$$

implying

$$(\lambda_2 - \lambda_1) x_{r+1}^2 + ... + (\lambda_n - \lambda_1) x_n^2 = 0,$$

and hence $x_{r+1} = ... = x_n = 0.$ This gives $Dx = \lambda_1 x$. Finally,

$$a_{ij} = (Ae_i, e_j) = (QDQ^T e_i, e_j) = (QDx, e_j) = \lambda_1 (Qx, e_j) = \lambda_1 (e_i, e_j) = \lambda_1 \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. Consequently, $a_{ij} = 0$ for $j \ne i$.

Problem 7. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ be two matrices satisying the conditions $A^2 = A,\ \ B^2 = B$ and $\operatorname{rank} A = \operatorname{rank} B$.

Prove that *A* and *B* are similar matrices. (*A* and *B* are similar matrices if there exists a nonsingular matrix $C \in \mathcal{M}_n(\mathbb{R})$, such that $A = C^{-1}BC$.)

Solution. First of all, let us remark that the eigenvalues of the matrix *A* can be 0 or 1, only. Let *A* be an eigenvalue of *A* and *X* be the correesponding eigenvector.

We have:

$$AX = \lambda X \qquad \Rightarrow \quad A^2X = \lambda AX \quad \overset{A^2=A}{\Rightarrow} \quad AX = \lambda AX \quad \Rightarrow \quad \lambda X = \lambda^2 X$$

$$(\lambda - \lambda^2)X = 0_{M_{n\times 1}(\mathbb{C})}, \quad X \neq 0_{M_{n\times 1}(\mathbb{C})} \quad \Rightarrow \quad \lambda - \lambda^2 = 0 \quad \Rightarrow \quad \lambda = 0 \text{ or } \lambda = 1.$$

In the following, we will prove that the matrix *A* is diagonalisable.

If we suppose the contrary, the matrix *A* has a Jordan canonical form having at least one Jordan block of order *k*, for instance:

$$J_k = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} = \lambda I_k + E_k.$$

Therefore,

$$J_k^2 = (\lambda I_k + E_k)^2 = \lambda^2 I_k + 2\lambda E_k + E_k^2 = \begin{pmatrix} \lambda^2 & 2\lambda & 1 & \dots & 0 \\ 0 & \lambda^2 & 2\lambda & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2\lambda \\ 0 & 0 & 0 & \dots & \lambda^2 \end{pmatrix}.$$

Taking into account the condition from hypothesis, we obtain:

$$A^2 = A, A = PJP^{-1}, A^2 = PJ^2P^{-1} \ \Rightarrow \ J^2 = J.$$

On the other hand, $J_k^2 \neq J_k$, for all eigenvalues of the matrix *A*. So, our supposition was wrong and consequently, the matrices *A* and *B* are diagonalisable.

The diagonal forms of the matrices *A* and *B* have on the diagonal only 0 and 1. Taking into account the condition $\operatorname{rank} A = \operatorname{rank} B$ it follows that the diagonal forms of *A* and *B* contain the same number of entries 1 on the diagonal and 0 in rest.

So, we can choose the same diagonal matrix for both matrices *A* and *B:*

$$D = \operatorname{diag}(\underbrace{1,1,\dots,1}_{p},\underbrace{0,\dots,0}_{n-p}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

where $p = \operatorname{rank} A = \operatorname{rank} B$.

Of course, we have the same diagonal matrix but it is obtained from *A* and *B* using different modal matrices $P, S \in \mathcal{M}_n(\mathbb{R}), \det P \neq 0$ and $\det S \neq 0$:

$$A = PDP^{-1}, \quad B = SDS^{-1}.$$

Thus,

$$\begin{aligned} D = P^{-1}AP,\, D = S^{-1}BS \quad &\Rightarrow \quad P^{-1}AP = S^{-1}BS \\ &\Rightarrow \quad A = PS^{-1}BSP^{-1} = (SP^{-1})^{-1}B(SP^{-1}). \end{aligned}$$

Denoting

$$C = SP^{-1} \in M_n(\mathbb{R}),\ \det C = \det S \cdot \det P^{-1} \neq 0$$

we obtain the conclusion.

Problem 8. a) Let A, B be two $m \times n$ matrices over $\mathbb{R}$. Show that if A and B have the same image, then there is an invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$ such that $A = BP$. (The *image* of an $m \times n$ matrix A is the subset $\{Ax \mid x \in \mathbb{R}^n\}$ of $\mathbb{R}^m$.)

b) Let

$$A = \begin{pmatrix} 0 & 0 \\ X^2 & 2(X+1) \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 2 & X^2 \end{pmatrix}$$

be two 2×2 matrices over the polynomial ring $R = \mathbb{Z}[X]$. Show that A and B have the same image, but there is no invertible matrix $P \in \mathcal{M}_2(R)$ such that $A = BP$.

Solution. a) Set $M = \operatorname{Im} A = \operatorname{Im} B$. This is a subspace of $\mathbb{R}^m$, and let $r = \dim M$. We then have $M \simeq \mathbb{R}^r$ hence we may assume $\operatorname{Im} A = \operatorname{Im} B = \mathbb{R}^r$.

Since $A: \mathbb{R}^n \to \mathbb{R}^r$ is surjective there is $A' \in \mathcal{M}_{n \times r}(\mathbb{R})$ such that $AA' = I_r$. Then $\mathbb{R}^n = \ker A \oplus \operatorname{Im} A'$, and since $\dim \ker A = n - r$ there is a matrix $X' \in \mathcal{M}_{n \times (n-r)}(\mathbb{R})$ whose columns form a basis for $\ker A$. Thus the matrix $(A' \mid X') \in \mathcal{M}_n(\mathbb{R})$ has the property that its columns form a basis for $\mathbb{R}^n$, hence it is invertible and its inverse has the form $\begin{pmatrix} A \\ X \end{pmatrix}$.

Similarly, we get a matrix $(B' \mid Y')$ whose columns form a basis for $\mathbb{R}^n$ and its inverse has the form $\begin{pmatrix} B \\ Y \end{pmatrix}$.

Since the matrices $\begin{pmatrix} A \\ X \end{pmatrix}$ and $\begin{pmatrix} B \\ Y \end{pmatrix}$ provide bases for $\mathbb{R}^n$ they differ by an invertible matrix, and we are done.

b) To show that $\operatorname{Im} A = \operatorname{Im} B$ is straightforward.

Suppose there is an invertible matrix $P \in \mathcal{M}_2(R)$ such that $A = BP$. Set

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From $A = BP$ we get

$$P = \begin{pmatrix} X^2 f & (X+1) - X^2 g \\ 1 - 2f & 2g \end{pmatrix}.$$

Since P is invertible we have $\det P = \pm 1$. But $\det P = (X+1)(2f-1) + X^2 g$. If $(X+1)(2f-1) + X^2 g = -1$ we send X to 0 and get $f(0) = 0$, hence $f = Xf_1$. Plugging this into the previous equation one gets $(X+1)(2Xf_1 - 1) + X^2 g = -1$, equivalents

$$2X^2 f_1 - X + 2Xf_1 + X^2 g = 0.$$

This gives $2Xf_1 - 1 + 2f_1 + Xg = 0$, and sending again X to 0 one obtains $2f_1(0) = 1$, a contradiction.

If $(X+1)(2f-1) + X^2 g = 1$ we send X to 0 and get $f(0) = 1$, hence $f = Xf_1 + 1$. Plugging this into the previous equation one gets $(X+1)(2Xf_1 + 1) + X^2 g = 1$, equivalents $2X^2 f_1 + X + 2Xf_1 + X^2 g = 0$. This gives $2Xf_1 + 1 + 2f_1 + Xg = 0$ and sending again X to 0 one obtains $2f_1(0) = -1$, a contradiction.

Problem 9. Let A and B be two 3×3 complex matrices such that

$$2Tr((A+B)^3)+(Tr(A+B))^3 \neq 3Tr(A+B)Tr((A+B)^2)\ ,$$

where Tr denotes the trace of the matrix in cause. Prove that $A+B$ is invertible, and

$$A((A+B)^{-1}B)^n(A(A+B)^{-1})^m B - B((A+B)^{-1}A)^m(B(A+B)^{-1})^n A$$

is the null matrix for all $m,n\in\mathbb{N}$.

Solution. Taking into attention the eigenvalues of of the matrix $A+B$, denoted here by λ_1,λ_2 and λ_3, the following equalities may be used :

$$Tr(A+B)=\lambda_1+\lambda_2+\lambda_3,\ Tr((A+B)^2)=\lambda_1^2+\lambda_2^2+\lambda_3^2,\ \ Tr((A+B)^3)=\lambda_1^3+\lambda_2^3+\lambda_3^3.$$

In this way, remarking that

$$6\lambda_1\lambda_2\lambda_3=(\lambda_1+\lambda_2+\lambda_3)^3-3(\lambda_1+\lambda_2+\lambda_3)(\lambda_1^2+\lambda_2^2+\lambda_3^2)+2(\lambda_1^3+\lambda_2^3+\lambda_3^3)\ ,$$

the relationship

$$2Tr((A+B)^3)+(Tr(A+B))^3 \neq 3Tr(A+B)Tr((A+B)^2)$$

offered by hypothesis, is easily transposed into the following one: $\lambda_1\lambda_2\lambda_3\neq 0$.

This means just that $\det(A+B)\neq 0$, which ensures that $A+B$ is invertible. In other words, it can be next counted on the matrix $(A+B)^{-1}$.

It is noticeable now that $A(A+B)^{-1}B=B(A+B)^{-1}A$, by virtue of the following sequence of equalities:

$$\begin{aligned}A(A+B)^{-1}B&=(A+B-B)(A+B)^{-1}B=B-B(A+B)^{-1}B\\&=B(A+B)^{-1}(A+B)-B(A+B)^{-1}B=B(A+B)^{-1}A.\end{aligned}$$

On such a basis, it is inductively inferred the fact:

$$A((A+B)^{-1}B)^n=(B(A+B)^{-1})^n A,\ \text{for all } n\in\mathbb{N}.$$

At the same time, one can similarly be seen the fact:

$$(A(A+B)^{-1})^m B=B((A+B)^{-1}A)^m,\ \text{for all } m\in\mathbb{N}.$$

Finally, the nullity of the matrix specified in this problem derives by the substraction of suitable products of terms in latter two equalities.

Problem 10. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a diagonalisable matrix, and $V \in \mathcal{M}_n(\mathbb{R})$ be another matrix, such that $V^2 = I_n$.

a) Prove that, for any $\varepsilon > 0$ sufficiently small, the matrix equation $AX + \varepsilon X = V$ has an unique solution $X \in \mathcal{M}_n(\mathbb{R})$, denoted by $X(\varepsilon)$.

b) Prove that

$$\lim_{\varepsilon \to 0_+} \varepsilon \cdot \mathrm{Tr}(VX(\varepsilon)) = \mathrm{null}\, A.$$

Solution. a) Remark that the matrix $A + \varepsilon I_n$ has the eigenvalues $\lambda_1 + \varepsilon, ..., \lambda_n + \varepsilon$, where by $\lambda_1, ..., \lambda_n$ we have denoted the eigenvalues of A. If all λ_i are nonzero, for any $\varepsilon > 0$ sufficiently small, $\lambda_1 + \varepsilon, ..., \lambda_n + \varepsilon$ are nonzero, hence the matrix $A + \varepsilon I_n$ is nonsingular. If 0 is eigenvalue for A, again, the matrix $A + \varepsilon I_n$ has as eigenvalues ε or $\lambda_i + \varepsilon$, with $\lambda_i \neq 0$, which are nonzero for any ε sufficiently small, therefore $A + \varepsilon I_n$ is nonsingular, too.

b) Since A is diagonalisable, $A + \varepsilon I_n$ is diagonalisable. There is an nonsingular matrix $P \in \mathcal{M}_n(\mathbb{R})$ such that $A + \varepsilon I_n = PD_\varepsilon P^{-1}$, where

$$D_\varepsilon = \begin{pmatrix} \lambda_1 + \varepsilon & 0 & ... & 0 \\ 0 & \lambda_2 + \varepsilon & ... & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & \lambda_n + \varepsilon \end{pmatrix}.$$

For every $\varepsilon > 0$ sufficiently small, $A + \varepsilon I_n$ is invertible, and its inverse can be written as

$$(A + \varepsilon I_n)^{-1} = P \cdot \begin{pmatrix} \frac{1}{\lambda_1+\varepsilon} & 0 & ... & 0 \\ 0 & \frac{1}{\lambda_2+\varepsilon} & ... & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & \frac{1}{\lambda_n+\varepsilon} \end{pmatrix} \cdot P^{-1}.$$

It follows that

$$\varepsilon X(\varepsilon) = \varepsilon (A + \varepsilon I_n)^{-1} V = P \cdot \begin{pmatrix} \frac{\varepsilon}{\lambda_1+\varepsilon} & 0 & ... & 0 \\ 0 & \frac{\varepsilon}{\lambda_2+\varepsilon} & ... & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & \frac{\varepsilon}{\lambda_n+\varepsilon} \end{pmatrix} \cdot P^{-1} V,$$

and, furthermore, using that $V^2 = I_n$, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, and the fact that the traces of similar matrices are equal, we obtain

$$\varepsilon \mathrm{Tr}(VX(\varepsilon)) = \mathrm{Tr}(\varepsilon X(\varepsilon) V) = \mathrm{Tr}(\varepsilon (A + \varepsilon I_n)^{-1} V^2) = \tfrac{\varepsilon}{\lambda_1+\varepsilon} + ... + \tfrac{\varepsilon}{\lambda_n+\varepsilon}.$$

Then $\lim\limits_{\varepsilon \to 0_+} \varepsilon \cdot \mathrm{Tr}(\varepsilon X(\varepsilon)) = k$, where k is the number of zero eigenvalues of the matrix A, i.e., $\mathrm{null}\, A$.

Problem 11. Given a triangle $A_0B_0C_0$. On its sides are built squares outside the triangle to form a new triangle $A_1B_1C_1$ with vertices the centers of these squares. Continuing in the same way an infinite sequence of triangles is obtained and $S_0, S_1, \ldots, S_n, \ldots$ are the areas of them.

a) Let $\mathcal{A}: \mathbb{C}^3 \to \mathbb{C}^3$ be linear operator defined by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and a_0, b_0, c_0 are the complex numbers corresponding to the vertices A_0, B_0, C_0 of the triangle $A_0B_0C_0$, and vector $\mathbf{z_0} = (a_0, b_0, c_0)$. Prove that

$$S_0 = \tfrac{1}{2}\operatorname{Im}(\mathcal{A}\mathbf{z_0}, \mathbf{z_0}),$$

where $(\bullet,\bullet)$ is the standard inner product in $\mathbb{C}^3$.

b) Prove that

$$S_n = 2S_{n-1} - \tfrac{1}{4}S_{n-2},\ n \ge 2$$

and

$$S_n = \tfrac{1}{\sqrt{3}}((S_1 - q_2 S_0)q_1^n - (S_1 - q_1 S_0)q_2^n)$$

where $q_1 = \frac{2+\sqrt{3}}{2}, q_2 = \frac{2-\sqrt{3}}{2}$.

Solution. a) We will prove more: if $A_1, A_2, \ldots, A_m$ are vertices of polygon and $a_1, a_2, \ldots a_m$ are the their corresponding complex numbers, and $\mathbf{w} = (a_1, a_2, \ldots, a_m)$ then his area is

$$S = \tfrac{1}{2}\operatorname{Im}(\mathcal{A}\mathbf{w}, \mathbf{w}),$$

where $\mathcal{A}\mathbf{w} = (a_2, a_3, \ldots, a_m, a_1)$.

Really, if $a_k = r_k(\cos\varphi_k + i\sin\varphi_k)$, then $\operatorname{Im}(\mathcal{A}\mathbf{w}, \mathbf{w}) = \sum_{k=1}^{m} r_k r_{k+1}\sin(\varphi_{k+1} - \varphi_k)$ $(a_{m+1} = a_1)$. Let us denote that $\operatorname{Im}(\mathcal{A}\mathbf{w}, \mathbf{w})$ doesn't change in translation (i. e. in addition to $\mathbf{w}$ of a vector $\mathbf{v} = k(1,1,\ldots,1), k \in \mathbb{C}$).

b) First note that $\mathcal{A}$ is unitary operator and $\mathcal{A}^3 = \mathrm{Id}$. Put $\mathbf{z}_k = (a_k, b_k, c_k)$ where a_k, b_k, c_k are the corresponding complex numbers to the vertices A_k, B_k, C_k. Let us define the operator $\mathcal{B} = \alpha\mathcal{A} + \overline{\alpha}\mathcal{A}^2$, where $\alpha = \frac{1}{2}(1+i)$. It is not difficult to see that $\mathbf{z}_k = \mathcal{B}\mathbf{z}_{k-1}, k \ge 1$, and $S_k = \frac{1}{2}\operatorname{Im}(\mathcal{A}\mathbf{z}_k, \mathbf{z}_k)$, $k \ge 0$.

It is sufficient to prove requested equality for $n = 2$.

By using the properties of the unitary operators, we obtain consecutively

$$(\mathcal{A}\mathbf{z}_1, \mathbf{z}_1) = (\mathcal{A}\mathbf{z}_0, \mathbf{z}_0) + \tfrac{i}{2}(\mathbf{z}_0, \mathbf{z}_0) - \tfrac{i}{2}(\mathcal{A}^2\mathbf{z}_0, \mathbf{z}_0)$$

whence

$$2S_1 = \operatorname{Im}(\mathcal{A}\mathbf{z}_1, \mathbf{z}_1) = 2S_0 - \operatorname{Im}\tfrac{i}{2}(\mathcal{A}^2\mathbf{z}_0, \mathbf{z}_0) + \tfrac{1}{2}(\mathbf{z}_0, \mathbf{z}_0).$$

Similarly,

$$2S_2 = \operatorname{Im}(\mathcal{A}\mathbf{z}_2, \mathbf{z}_2) = \tfrac{7}{2}S_0 + (\mathbf{z}_0, \mathbf{z}_0) - \operatorname{Im} i(\mathcal{A}^2\mathbf{z}_0, \mathbf{z}_0).$$

From the last two equalities follows requested result.

Finally, characteristic equation of sequence S_n is $q^2 - 2q + \frac{1}{4} = 0$ with roots $q_1 = \frac{2+\sqrt{3}}{2}, q_2 = \frac{2-\sqrt{3}}{2}$. Then S_n has the form $S_n = C_1 q_1^n + C_2 q_2^n$ where the constants C_1 and C_2 are obtained from the system

$$\begin{cases} S_0 = C_1 + C_2 \\ S_1 = C_1 q_1 + C_2 q_2 \end{cases}$$

<u>Problem 12.</u> Let $A \in \mathcal{M}_n(\mathbb{R})$ be a fixed non-zero matrix. Define the function

$$f_A : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R}),$$
$$f_A(X) = AX - XA, \ \forall X \in \mathcal{M}_n(\mathbb{R}).$$

a) Show that $f_A = \theta$ if and only if $A = \lambda I_n$, where I_n is the identity matrix.

b) Show that $f_A \circ f_B = f_B \circ f_A$ if and only if $AB = BA$.

c) If A is a matrix with n distinct real eigenvalues, find the dimension of $\ker(f_A)$.

*<u>Solution</u>***.** a) Since $f_A = \theta$, it follows that $AX - XA = 0_n$, $\forall X \in \mathcal{M}_n(\mathbb{R})$.

Denote by E_{ij} the matrix with 1 on the (i, j) position and 0 elsewhere. For $X = E_{ij}$, we have $AE_{ij} = E_{ij}A$.

AE_{ij} is the matrix whose j-th column is the i-th column of A and 0 elsewhere, and $E_{ij}A$ is the matrix whose i-th column is the j-th column of A and 0 elsewhere. For $i \neq j$, one has $a_{ii} = a_{jj}$, $i = 1,..,n$, $j = 1,..,n$, and for $i = j$, one has $a_{ik} = 0$, $a_{ki} = 0$, $k \neq i$, $k = 1,...,n$. It follows that A is a diagonal matrix, $A = \lambda I_n$.

b) One has

$$\begin{aligned} & f_A \circ f_B = f_B \circ f_A \\ \Leftrightarrow\ & (f_A \circ f_B)(X) = (f_B \circ f_A)(X), \ \forall X \in \mathcal{M}_n(\mathbb{R}) \\ \Leftrightarrow\ & ABX + XBA = BAX + XAB, \ \forall X \in \mathcal{M}_n(\mathbb{R}) \\ \Leftrightarrow\ & (AB - BA)X = X(AB - BA), \ \forall X \in \mathcal{M}_n(\mathbb{R}). \end{aligned}$$

If $AB = BA$, then $f_A \circ f_B = f_B \circ f_A$.

If $f_A \circ f_B = f_B \circ f_A$ then

$$(AB - BA)X - X(AB - BA) = 0_n, \ \forall X \in \mathcal{M}_n(\mathbb{R}),$$

hence $f_{AB-BA}(X) = 0_n, \forall X \in \mathcal{M}_n(\mathbb{R})$. Using a), it follows that

$$AB - BA = \lambda I_n.$$

Since $\mathrm{Tr}(AB - BA) = \mathrm{Tr}(\lambda I_n)$, it follows $0 = n\lambda$, hence $\lambda = 0$, and, finally, $AB = BA$.

c) We have

$$\ker(f_A) = \{X \in \mathcal{M}_n(\mathbb{R}) : \ AX = XA\}.$$

We prove that all the matrices $X \in \ker(f_A)$ have the same eigenvectors, due the condition that A is a matrix with n distinct real eigenvalues.

Indeed, take an eigenvector v corresponding to the eigenvalue λ for A. Then

$$AXv = XAv = X\lambda v = \lambda Xv,$$

hence Xv is an eigenvector for A corresponding to λ. But since the eigenspace corresponding to λ has dimension 1, it follows that there is α such that $Xv = \alpha v$, hence v is an eigenvector for X. Since X can have at most n eigenvectors which are linearly independent, and all the eigenvectors for A are eigenvectors for X , it follows that X has the same eigenvectors as A.

Observe that every X is diagonalisable, since its eigenvectors are linearly independent. We want to prove that there exist a polynomial f of degree at most $n-1$ such that

$$X = f(A).$$

Denote by $\alpha_1,...,\alpha_n$ the eigenvalues of X, and by $\lambda_1,...,\lambda_n$ the eigenvalues of A. Then $X = PD_X P^{-1}$, and $A = PD_A P^{-1}$, where by P we have denoted the matrix whose columns are the eigenvectors of A and X (which are the same), and the relation $X = f(A)$ reduces to $D_X = f(D_A)$, or

$$\begin{pmatrix} \alpha_1 & 0 & ... & 0 \\ 0 & \alpha_2 & ... & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & \alpha_n \end{pmatrix} = \begin{pmatrix} f(\lambda_1) & 0 & ... & 0 \\ 0 & f(\lambda_2) & ... & 0 \\ .. & .. & .. & .. \\ 0 & 0 & .. & f(\lambda_n) \end{pmatrix}.$$

Since $\lambda_1,...,\lambda_n$ are distinct, the Lagrange interpolation polynomial, of degree $n-1$ in our case, satisfies $f(\lambda_i)=\alpha_i$, $\forall i=1,...,n.$ It follows that $\{I_n, A, A^2,..., A^{n-1}\}$ generates $\ker(f_A)$. Moreover, the linear independence of $\{I_n, A, A^2,..., A^{n-1}\}$ reduces to the fact that the Vandermonde determinant

$$\begin{vmatrix} 1 & \lambda_1 & ... & \lambda_1^{n-1} \\ 1 & \lambda_2 & ... & \lambda_2^{n-1} \\ .. & .. & .. & .. \\ 1 & \lambda_n & .. & \lambda_n^{n-1} \end{vmatrix}$$

is nonzero, which is true due to the fact that λ_i are all distinct. Then, $\{I_n, A, A^2,..., A^{n-1}\}$ form a basis in $\ker(f_A)$, so the dimension of $\ker(f_A)$ is n.

ANALYSIS

Problem 1. Let f be continuous on [0,1] and differentiable on (0,1). Suppose that $f(0) = f(1) = 0$ and that there is $x_0 \in (0,1)$ such that $f(x_0) = 1$. Prove that $|f'(c)| > 2$ for some $c \in (0,1)$.

Solution. Suppose first that $x_0 \neq \frac{1}{2}$. Then either $[0, x_0]$ or $[x_0, 1]$ has length less than 1. Suppose, for example, that this is $[x_0, 1]$. By the mean value theorem,

$$\frac{-1}{1-x_0} = \frac{f(1) - f(x_0)}{1-x_0} = f'(c)$$

and consequently, $|f'(c)| > 2$.

Suppose now that $x_0 = \frac{1}{2}$ and that f is linear on $[0, \frac{1}{2}]$. Then $f(x) = 2x$ for $x \in [0, \frac{1}{2}]$. Since $f'(\frac{1}{2}) = 2$, there is $x_1 > \frac{1}{2}$ such that $f(x_1) > 1$. In this case, the assertion follows from the mean value theorem applied to f on $[x_1, 1]$. Finally, suppose that f is not linear on $[0, \frac{1}{2}]$. If there is $x_2 \in (0, \frac{1}{2})$ such that $f(x_2) > 2x_2$, then to get the desired result it is enough to apply the mean value theorem on $[0, x_2]$. $f(x_2) < 2x_2$, then one can apply the mean value theorem on $[x_0, \frac{1}{2}]$.

Problem 2. Draw a tangent line of parabola $y=x^2$ at the point $A(1,1)$. Suppose the line intersects the *x*-axis and *y*-axis at *D* and *B* respectively. Let point *C* be on the parabola and point *E* on *AC* such that $\frac{AE}{EC}=k_1$. Let point *F* be on *BC* such that $\frac{BF}{FC}=k_2$ and $k_1+k_2=1$. Assume that *CD* intersects *EF* at point *P*. When point C moves along the parabola, find the equation of the trail of *P*.

Solution. The slope of the tangent line passing through *A* is $y'(1)=2$. So the equation of the tangent line *AB* is $y=2x-1$. Hence coordinates of *B* and *D* are $B(0;-1), D(\frac{1}{2},0)$. Thus *D* is midpoint of the line segment *AB*.

Consider $P(x;y), C(x_0,x_0^2), E(x_1,y_1), F(x_2,y_2)$. Then by $\frac{AE}{EC}=k_1$, we get $x_1=\frac{1+k_1x_0}{1+k_1}$, $y_1=\frac{1+k_1x_0^2}{1+k_1}$. From $\frac{BF}{FC}=k_2$, we get $x_2=\frac{k_2x_0}{1+k_2}, y_2=\frac{-1+k_2x_0^2}{1+k_2}$. Therefore the equation of line *EF* is

$$\frac{y-\frac{1+k_1x_0^2}{1+k_1}}{\frac{-1+k_2x_0^2}{1+k_2}-\frac{1+k_1x_0^2}{1+k_1}}=\frac{x-\frac{1+k_1x_0}{1+k_1}}{\frac{k_2x_0}{1+k_2}-\frac{1+k_1x_0}{1+k_1}}.$$

Simplifying it, we get

$$[(k_2-k_1)x_0-(1+k_2)]y=[(k_2-k_1)x_0^2-3]x+1-x_0-k_2x_0^2. \qquad (1)$$

When $x_0\neq\frac{1}{2}$ the equation of line *CD* is

$$y=\frac{2x_0^2x-x_0^2}{2x_0-1}. \qquad (2)$$

From (1) and (2), we get $x=\frac{x_0+1}{3}, y=\frac{x_0}{3}$. By elimination of x_0, we get the equation of the trail of point *P* as

$$y=\frac{1}{3}(3x-1)^2.$$

When $x_0=\frac{1}{2}$ the equation of *EF* is $-\frac{3}{2}y=(\frac{1}{4}k_2-\frac{1}{4}k_1-3)x+\frac{3}{2}-\frac{1}{4}k_2$, the equation *CD* is $x=\frac{1}{2}$. Combining then, we conclude that $(x,y)=(\frac{1}{2},\frac{1}{12})$ is on the trail of *P*. Since *C* and *A* cannot be congruent, $x_0\neq 1, x\neq\frac{2}{3}$

Therefore the equation of trail is $y=\frac{1}{3}(3x-1)^2$, $x\neq\frac{2}{3}$.

Problem 3. The function $f(x)$ has a derivative of order two in the interval $[a,b]$ with a length 2, $f(a)=f(b)=0$ and there is a point $x\in(a,b)$ such that $f(x)>0$. Prove that

$$\inf_{x\in[a,b]} f''(x)+\max_{x\in[a,b]} f(x)\le 0.$$

<u>*Solution.*</u> Let $x_0\in(a,b)$ be such that $\max\limits_{x\in[a,b]} f(x)=f(x_0)$ and $m=\inf\limits_{x\in[a,b]} f''(x)$. By the Taylor's Theorem we have

$$0=f(a)=f(x_0)+f'(x_0)(a-x_0)+\tfrac{1}{2}f''(\xi_1)(a-x_0)^2$$

$$0=f(b)=f(x_0)+f'(x_0)(b-x_0)+\tfrac{1}{2}f''(\xi_2)(b-x_0)^2$$

for some $\xi_1,\, a<\xi_1<x_0$ and $\xi_2,\, x_0<\xi_2<b$.

Because $x_0\in(a,b)$, $f'(x_0)=0$ and we obtain

$$f(x_0)+\tfrac{1}{4}(f''(\xi_1)(a-x_0)^2+f''(\xi_2)(b-x_0)^2)=0,$$

and

$$f(x_0)+\tfrac{m}{4}((a-x_0)^2+(b-x_0)^2)\le 0$$

(obviously $m<0$).

The maximal value of the function $y(x)=(a-x)^2+(b-x)^2$ in $[a,b]$ is equal to 4, which proves the requested inequality.

Problem 4. Prove the inequality

$$\int_{-1}^{1} \frac{3^x}{(1+3^x)^2(1+x^2)^2}\,\mathrm{d}x \geq \frac{2+\pi}{16\ln 3} .$$

Solution. First stage. Let us denote $f(x)=\dfrac{3^x}{(1+3^x)^2}$, $g(x)=\dfrac{1}{(1+x^2)^2}$. The function $f(x)$ is even since $f(x)-f(-x)=0$. That is why the product $f(x)g(x)$ is also an even function. Therefore

$$\int_{-1}^{1} \frac{3^x}{(1+3^x)^2(1+x^2)^2}\,\mathrm{d}x = 2\int_{0}^{1} \frac{3^x}{(1+3^x)^2(1+x^2)^2}\,\mathrm{d}x .$$

Computing the derivatives:

$$f'(x)=-\frac{3^x(-1+3^x)\ln 3}{(1+3^x)^3} \text{ and } g'(x)=-\frac{4x}{(1+x^2)^3}$$

we conclude that both functions are decreasing in $[0,1]$. Additionally, $f(x)$ and $g(x)$ are positive.

Second stage. The above properties of the functions $f(x)$ and $g(x)$ make valid Chebishev's inequality

$$2\int_{0}^{1} \frac{3^x}{(1+3^x)^2(1+x^2)^2}\,\mathrm{d}x \geq 2\int_{0}^{1} \frac{3^x}{(1+3^x)^2}\,\mathrm{d}x \int_{0}^{1} \frac{1}{(1+x^2)^2}\,\mathrm{d}x . \qquad (1)$$

We continue with the antiderivatives of $f(x)$ and $g(x)$:

$$F(x)=-\frac{1}{(1+3^x)\ln 3},\ G(x)=\frac{1}{2}\left(\frac{x}{1+x^2}+\arctan x\right).$$

Therefore:

$$\int_{0}^{1} \frac{3^x}{(1+3^x)^2}\,\mathrm{d}x = \frac{1}{4\ln 3} \text{ and } \int_{0}^{1} \frac{1}{(1+x^2)^2}\,\mathrm{d}x = \frac{2+\pi}{8} .$$

It remains just to replace the latter results in (1) to complete the proof.

Problem 5. Let f be a nontrivial function, of class C^2, continuous, such that $f:[1,2]\to[0,R],\ 0<R<\infty$ and $f(1)=f(2)=0$. Prove that

$$\int_1^2 \left|\frac{f''(t)}{f(t)}\right| dt > 2x \text{, for } x\in[1,2].$$

Solution. It is enough to prove the following inequality

$$\int_1^2 \left|\frac{f''(t)}{f(t)}\right| dt > 4 .$$

Let $M=\max\limits_{x\in[1,2]}|f(x)|>0$. The

$$I=\int_1^2 \left|\frac{f''(t)}{f(t)}\right| dt > \frac{1}{M}\int_1^2 |f''(x)|\,dx . \tag{1}$$

There is a $x_0\in(1,2)$ such that $M=f(x_0)$. It' s clear that $x_0\neq 1,2$. From mean - value theorem:

- There is a $\xi_1\in(1,x_0)$ such that $f'(\xi_1)=\frac{M}{x-1}$,
- There is a $\xi_2\in(x_0,2)$ such that $f'(\xi_2)=\frac{-M}{2-x}$.

From (1) and Cauchy-Schwarz inequality in Engel form, since $x_0\in(1,2)$, we have

$$I>\frac{1}{M}\int_1^2 |f''(x)|\,dx \geq \frac{1}{M}\int_{\xi_1}^{\xi_2} |f''(x)|\,dx \geq \frac{1}{M}\left|\int_{\xi_1}^{\xi_2} f''(x)dx\right|$$

$$=\frac{1}{M}|f'(\xi_2)-f'(\xi_1)|=\frac{1}{M}\left|-\frac{M}{2-x}-\frac{M}{x-1}\right|=\left|\frac{1}{2-x}+\frac{1}{x-1}\right|$$

$$=\frac{1}{2-x}+\frac{1}{x-1}\geq \frac{(1+1)^2}{(2-x)+(x-1)}=4.$$

Problem 6. a) Calculate the limit

$$\lim_{n\to\infty} \frac{\int_n^{n+1} \frac{dx}{\log x} - \frac{1}{\log n}}{\frac{1}{\log(n+1)} - \frac{1}{\log n}} .$$

b) Let $f:(a,\infty)\to\mathbb{R}$ ($a>0$) be differentiable such that f' is monotone, has no roots and $\lim_{n\to\infty} \frac{f'(n)}{f'(n+1)} = 1$. Prove that

$$\lim_{n\to\infty} \frac{\int_n^{n+1} f(x)dx - f(n)}{f(n+1) - f(n)} = \frac{1}{2} .$$

Solution. b) Using Taylor's formula with the Lagrange remainder for the function

$$F(t) = \int_0^t f(x)dx,\ t>a$$

we obtain that for every $n>a$, there exists some $x_n \in (n,n+1)$ such that

$$F(n+1) = F(n) + F'(n) + \frac{1}{2}F''(x_n) ,$$

hence

$$\int_n^{n+1} f(x)dx - f(n) = \frac{1}{2} f'(x_n) .$$

Similarly, for every $n>a$, there exists some $y_n \in (n,n+1)$ such that

$$f(n+1) = f(n) + f'(y_n) ,$$

hence

$$\frac{\int_n^{n+1} f(x)dx - f(n)}{f(n+1) - f(n)} = \frac{1}{2} \cdot \frac{f'(x_n)}{f'(y_n)} .$$

Note that f' has the Darboux (i.e. intermediate value) property, while being nonzero on (a,∞), which leads to f' having constant sign (without any loss of generality, we may assume that f' is positive on (a,∞)). Next, using the monotonicity of f', it follows that

$$\frac{f'(n)}{f'(n+1)} \le \frac{f'(x_n)}{f'(y_n)} \le \frac{f'(n+1)}{f'(n)}$$

hence

$$\frac{f'(x_n)}{f'(y_n)} \to 1, \text{ as } n\to\infty$$

which leads to the conclusion.

a) We apply B) for $f(x) = \frac{1}{\log x}$.

Problem 7. Let $f \in C^1(\mathbb{R})$ is a positive valued function. Prove that

$$|\int_0^2 (f(x))^3 dx - (f(0))^2 \int_0^2 f(x)dx| \le \max_{0\le x\le 2} |f'(x)| (\int_0^2 f(t)dt)^2 .$$

Solution. Let $M = \max_{0\le t\le 2} |f'(t)|$. We have

$$-Mf(t) \le f'(t)f(t) \le Mf(t), \quad \forall t \in [0,2].$$

Integrating on $[0,x]$:

$$-M\int_0^x f(t)dt \le \int_0^x f(t)f'(t)dt \le M\int_0^x f(t)dt$$

$$-M\int_0^x f(t)dt \le \tfrac{1}{2}((f(x))^2 - (f(0))^2) \le M\int_0^x f(t)dt.$$

Multiply the last inequalities by $f(x)$:

$$-Mf(x)\int_0^x f(t)dt \le \tfrac{1}{2}((f(x))^3 - (f(0))^2 f(x)) \le Mf(x)\int_0^x f(t)dt$$

$$-M(\int_0^x f(t)dt)' \cdot \int_0^x f(t)dt \le \tfrac{1}{2}((f(x))^3 - (f(0))^2 f(x)) \le M(\int_0^x f(t)dt)' \cdot \int_0^x f(t)dt$$

$$-\tfrac{M}{2}((\int_0^x f(t)dt)^2)' \le \tfrac{1}{2}((f(x))^3 - (f(0))^2 f(x)) \le \tfrac{M}{2}((\int_0^x f(t)dt)^2)'.$$

Intergrating on [0,2]:

$$-\tfrac{M}{2}\int_0^2((\int_0^x f(t)dt)^2)'dx \le \tfrac{1}{2}\int_0^2 (f(x))^3 dx - \tfrac{1}{2}(f(0))^2\int_0^2 f(x)dx \le \tfrac{M}{2}\int_0^2((\int_0^x f(t)dt)^2)'dt$$

$$-\tfrac{M}{2}(\int_0^2 f(t)dt)^2 \le \tfrac{1}{2}\int_0^2 (f(x))^3 dx - \tfrac{1}{2}(f(0))^2\int_0^2 f(x)dx \le \tfrac{M}{2}(\int_0^2 f(t)dt)^2$$

$$|\int_0^2 (f(x))^3 dx - (f(0))^2 \int_0^2 f(x)dx| \le \max_{0\le x\le 2} |f'(x)| (\int_0^2 f(t)dt)^2.$$

Problem 8. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Prove that

$$\int_0^4 f(x(x-3)^2)dx = 2\int_1^3 f(x(x-3)^2)dx .$$

Solution. Let $g:[0,4]\to\mathbb{R}$ defined by $g(x)=x(x-3)^2$. Then $g'(x)=3(x-1)(x-3)$ and the behaviour of function g is given in the following table:

x	0		1		3		4
$g'(x)$	+	+	0	–	0	+	+
$g(x)$	0	↗	4	↘	0	↗	4

Let g_1, g_2, g_3 be the restrictions of g over $(0,1), (1,3)$ and $(3,4)$, respectively, and let h_1, h_2, h_3 be their inverses:

$$h_1:(0,4)\to(0,1),\ h_2:(0,4)\to(1,3),\ h_3:(0,4)\to(3,4)$$

where, for every $t\in(0,4)$,

$x_1 = h_1(t)$ is the solution of $x(x-3)^2 = t$ in $(0,1)$,

$x_2 = h_2(t)$ is the solution of $x(x-3)^2 = t$ in $(1,3)$,

$x_3 = h_3(t)$ is the solution of $x(x-3)^2 = t$ in $(3,4)$.

Using the changes of variable $x = h_i(t)$ $(i=1,2,3)$, we have that

$$\begin{aligned}\int_0^4 f(x(x-3)^2)dx - 2\int_1^3 f(x(x-3)^2)dx &= \int_0^1 f(g(x))dx - \int_1^3 f(g(x))dx + \int_3^4 f(g(x))dx \\ &= \int_0^4 f(t)\cdot h_1'(t)dt - \int_4^0 f(t)\cdot h_2'(t)dt + \int_0^4 f(t)\cdot h_3'(t)dt \\ &= \int_0^4 f(t)\cdot (h_1'(t)+h_2'(t)+h_3'(t))dt.\end{aligned}$$

Since the sum of the roots of the polynomial equation $x(x-3)^2 = t$ is 6, it follows that

$$h_1(t)+h_2(t)+h_3(t) = 6 \text{ for every } t\in(0,4),$$

hence

$$h_1'(t)+h_2'(t)+h_3'(t) = 0 \text{ for every } t\in(0,4),$$

which concludes the proof.

Remark. Since $g'(1)=g'(3)=0$, it follows that $h_1'(4), h_2'(0), h_2'(4)$ and $h_3'(0)$ are infinite, hence the integrals $\int_0^4 f(t)\cdot|h_1'(t)|\,dt$, $\int_0^4 f(t)\cdot|h_2'(t)|\,dt$ and $\int_0^4 f(t)\cdot|h_3'(t)|\,dt$ are improper, yet convergent, because they where obtained from proper integrals by a change of variable.

Problem 9. Let $f:[0,\infty)\to\mathbb{R}$ be the function denned by $f(x)=\log_3(3^x-x)$, $\forall x\in[0,\infty)$.

a) Considering the sequence $\{x_n\}_{n\in\mathbb{N}}$, where $x_0=\frac{1}{2}$ and $x_{n+1}=f(x_n)$ for all $n\in\mathbb{N}$, evaluate $\sum\limits_{n=0}^{\infty} x_n$.

b) Calculate

$$\lim_{x\to 0}\left(x^{2017}\left[(x-f(x))\ln 3-\sum_{k=1}^{2016} k^{-1}x^k 3^{-kx}\right]^{-1}\right).$$

Solution. a) It is useful to be observed that $3^x>e^x>x+1$, $\forall x\in[0,\infty)$. Thus it is guaranteed that $f(x)>0$, $\forall x\in[0,\infty)$. Consequently, $x_n>0$ for all $n\in\mathbb{N}$. At the same time, seeing that $3^{x_n}-3^{x_{n+1}}=x_n>0$, $\forall n\in\mathbb{N}$, one can deduce that $x_n>x_{n+1}$, $\forall n\in\mathbb{N}$. These mean that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is decreasing and bounded below. So it is convergent. By taking $l=\lim\limits_{n\to\infty} x_n$ and relying on the equality $3^{x_n}-3^{x_{n+1}}=x_n$, it appears that $l=3^l-3^l=0$. Therefore, the necessary criterion for convergence of the series $\sum\limits_{n=0}^{\infty} x_n$ is accomplished. More than that, inasmuch as

$$\sum_{k=0}^{n} x_k=\sum_{k=0}^{n}(3^{x_k}-3^{x_{k+1}})=3^{x_0}-3^{x_{n+1}} \text{ and } \lim_{n\to\infty}(3^{x_0}-3^{x_{n+1}})=\sqrt{3}-1$$

we may conclude that $\sum\limits_{n=0}^{\infty} x_n$ is a convergent series and its sum is $\sqrt{3}-1$.

b) Taking into account that $f(x)=\log_3(3^x-x)=x-\log_3(1-\frac{x}{3^x})=x-\frac{1}{\ln 3}\ln(1-\frac{x}{3^x})$, $\forall x\in[0,\infty)$ and $\ln(1-t)=-\sum\limits_{k=1}^{\infty}\frac{t^k}{k}, \forall t\in(-1,1)$, it is obvious that $(x-f(x))\ln 3=\sum\limits_{k=1}^{\infty}\frac{x^k}{k3^{kx}}$, because $\frac{x}{3^x}<1$, $\forall x\in[0,\infty)$. Then it follows that

$$(x-f(x))\ln 3-\sum_{k=1}^{2016} k^{-1}x^k 3^{-kx}=\sum_{k=2017}^{\infty}\frac{x^k}{k3^{kx}},\ \forall x\in[0,\infty)$$

and so

$$x^{2017}\left[(x-f(x))\ln 3-\sum_{k=1}^{2016} k^{-1}x^k 3^{-kx}\right]^{-1}=\frac{1}{\sum\limits_{k=2017}^{\infty}\frac{x^{k-2017}}{k3^{kx}}},\ \forall x\in[0,\infty).$$

Accordingly, we obtain:

$$\lim_{x\to 0}\left(x^{2017}\left[(x-f(x))\ln 3-\sum_{k=1}^{2016} k^{-1}x^k 3^{-kx}\right]^{-1}\right)=2017.$$

<u>Problem 10.</u> Given is the function $f \in C^2(\mathbb{R} \setminus \{0\})$ for which $\lim_{x\to 0} f(x) = \infty$, $\lim_{x\to 0} f'(x) = \infty$ and

$$\lim_{x\to 0} \frac{f''(x)}{f'^2(x)} = 0. \qquad (*)$$

We define the function

$$g(x) = \begin{cases} \sin f(x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

a) Prove that $g(x)$ has a primitive on $\mathbb{R}$, (i.e. $\exists G(x): \mathbb{R} \to \mathbb{R}$ so that $G'(x) = g(x)$). Is this true if the condition (*) is not satisfied?

b) Let $G(x)$ be a primitive of $g(x)$. Prove that exists a function $\xi(x)$, satisfying the condition $G(x) - G(0) = xg(\xi(x))$ where $\xi(x)$ is between 0 and x, and this function has points of discontinuity randomly near zero.

<u>Solution.</u> a) Let us define $G(x) = \int_0^x g(t)\mathrm{d}t$ for $x \neq 0$ and $G(0) = 0$. If $x \neq 0$ by the Newton-Leibniz Theorem we have $G'(x) = g(x)$. It remains to prove that there exists $G'(0)$ and $G'(0) = 0$. By definition

$$G'(0) = \lim_{x\to 0} \frac{G(x)}{x} = \lim_{x\to 0} \frac{1}{x}\int_0^x g(t)\mathrm{d}t.$$

We consecutively obtain (for example for $x > 0$)

$$\int_0^x g(t)\mathrm{d}t = \lim_{\varepsilon\to 0+} \int_\varepsilon^x \frac{\sin f(t)}{f'(t)}\mathrm{d}f(t).$$

After integration by parts we have

$$\int_0^x g(t)\mathrm{d}t = \lim_{\varepsilon\to 0+} \left(-\frac{\cos f(t)}{f'(t)}\Big|_\varepsilon^x - \int_\varepsilon^x \frac{f''(t)\cos f(t)}{f'^2(t)}\mathrm{d}t\right) = -\frac{\cos f(x)}{f'(x)} - \int_0^x \frac{f''(t)\cos f(t)}{f'^2(t)}\mathrm{d}t.$$

Now,

$$G'(0) = \lim_{x\to 0}\left(-\frac{\cos f(x)}{xf'(x)} - \frac{1}{x}\int_0^x \frac{f''(t)\cos f(t)}{f'^2(t)}\mathrm{d}t\right) = 0$$

by the L'Hospital rule and the condition (*).

Note that the statement is not true if the condition (*) is not satisfied, for example $f(x) = \ln|x|$.

b) Such a function $\xi(x)$ exists (for example for $x > 0$) by Lagrange Theorem. Let us assume that there exists $\varepsilon > 0$ such that $\xi(x)$ is continuous in $(0, \varepsilon)$. Then the function $f(\xi(x))$ transforms the interval $(0, \varepsilon)$ onto infinite interval and $\lim_{x\to 0} \sin f(\xi(x))$ does not exist contrary to the fact

$$\lim_{x\to 0} g(\xi(x)) = \lim_{x\to 0} \frac{G(x) - G(0)}{x} = G'(0) = 0.$$

Problem 11. Assume that g is a continuous function from $\mathbb{R}\setminus\{0,1\}$ to $\mathbb{R}$, such that $g(x)+g(1-\frac{1}{x})$ denoted by $h(x)$, is admitted to be known for every $x\in\mathbb{R}\setminus\{0,1\}$.

1) Find $\int_0^1 g(x)dx$, when $h(x)=\ln^2|x|$.

2) If h is so that $\int_0^1 (h(x)+h(\frac{1}{1-x}))dx+\frac{\pi^2}{3}=\int_0^1 h(1-\frac{1}{x})dx+4$, prove the existence of a number $r\in(0,1)$ such that $g(r)=\ln r\ln(1-r)$.

Solution. First of all, it is important to realize that the following relation is in effect:

$$2g(x)=h(x)+h(\tfrac{1}{1-x})-h(1-\tfrac{1}{x}),\ \forall x\in\mathbb{R}\setminus\{0,1\}. \tag{1}$$

Based on this, noting that, at 1),

$$h(x)+h(\tfrac{1}{1-x})-h(1-\tfrac{1}{x})=2\ln|x|\ln|x-1|,\ \forall x\in\mathbb{R}\setminus\{0,1\},$$

one may be found that $g(x)=\ln x\ln(1-x)$ for all x in $(0,1)$. Therefore, in this case, we have:

$$\int_0^1 g(x)dx=\int_0^1 \ln x\ln(1-x)dx. \tag{2}$$

Inasmuch as the improper integral $\int_0^1 \ln x\ln(1-x)dx$ is convergent, and its value can be calculated on the path of the following sequence of equalities

$$\int_0^1 \ln x\ln(1-x)dx=-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^k\ln x dx=\sum_{k=1}^{\infty}\frac{1}{k(k+1)^2}=\sum_{k=1}^{\infty}\frac{1}{k(k+1)}-\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}=1-(\frac{\pi^2}{6}-1), \tag{3}$$

we obtain:

$$\int_0^1 g(x)dx=2-\frac{\pi^2}{6}. \tag{4}$$

In the situation of 2), taking into account the assumption in effect and once again the relation (1) we deduce that, in fact, the equality (4) occurs. Having in mind (3), this means (2) , that is:

$$\int_0^1 (g(x)-\ln x\ln(1-x))dx=0.$$

From here, applying the mean value theorem for integrals, the desired conclusion is finally achieved.

Problem 12. Find all functions $f:[0,\frac{2}{3}]\to(0,\infty)$ of class C^1 satisfying the following conditions:

$$\int_0^{\frac{2}{3}}[f'(x)]^2dx+\int_0^{\frac{2}{3}}\frac{1}{f(x)}dx\le 4,\ \text{ and}$$

$$\sqrt{f(\tfrac{2}{3})}=1+\sqrt{f(0)}.$$

*Solution***.** We have

$$0\le\int_0^{\frac{2}{3}}[f'(x)-\frac{1}{\sqrt{f(x)}}]^2dx\le\int_0^{\frac{2}{3}}[f'(x)]^2dx+\int_0^{\frac{2}{3}}\frac{1}{f(x)}dx-2\int_0^{\frac{2}{3}}\frac{f'(x)}{\sqrt{f(x)}}dx$$

$$\le 4-4\sqrt{f(x)}\Big|_0^{\frac{2}{3}}=4-4(\sqrt{f(\tfrac{2}{3})}-\sqrt{f(0)})=0.$$

Thus,

$$\int_0^{\frac{2}{3}}[f'(x)-\frac{1}{\sqrt{f(x)}}]^2dx=0,$$

and since f is C^1, it follows that

$$f'(x)=\frac{1}{\sqrt{f(x)}},\quad \forall x\in[0,\tfrac{2}{3}].$$

We obtain

$$[f^{\frac{3}{2}}(x)]'=\frac{3}{2}\Leftrightarrow\sqrt{f^3(x)}=\frac{3}{2}(x+C)\Leftrightarrow f(x)=\sqrt[3]{\frac{9}{4}(x+C)^2}.$$

The condition $\sqrt{f(\frac{2}{3})}=1+\sqrt{f(0)}$ leads us to $C=0$ and $C=-\frac{2}{3}.$ Consequently, we obtain two functions satisfying the hypotesis:

$$f(x)=\sqrt[3]{\frac{9}{4}x^2}\quad\text{and}\quad f(x)=\sqrt[3]{\frac{9}{4}x^2-3x+1}.$$

Problem 13. Let $p>1$ be a real number, and let $C[0,1]$ denote the set of all continuous functions $f:[0,1]\to\mathbb{R}$. Find $\max\limits_{f\in C[0,1]} I(f)$ where

$$I(f)=\int_0^1 x^p\,|f(x)|\,dx-\int_0^1 x\,|f(x)|^p\,dx\,.$$

Solution. Answer: $\max\limits_{f\in C[0,1]} I(f)=\frac{1}{p+2}(p^{\frac{1}{1-p}}-p^{\frac{p}{1-p}})$.

Let $q=\frac{p}{p-1}$ be the conjugate of p (i.e. $\frac{1}{p}+\frac{1}{q}=1$). By virtue of the Hölder inequality we have

$$\int_0^1 x^p\,|f(x)|\,dx=\int_0^1 x^{p-\frac{1}{p}}x^{\frac{1}{p}}\,|f(x)|\,dx\le\Big(\int_0^1 x^{q(p-\frac{1}{p})}dx\Big)^{\frac{1}{q}}\cdot\Big(\int_0^1 x\,|f(x)|^p\,dx\Big)^{\frac{1}{p}}.$$

Since

$$\int_0^1 x^{q(p-\frac{1}{p})}dx=\int_0^1 x^{p+1}dx=\frac{1}{p+2}$$

it follows that

$$\int_0^1 x^p\,|f(x)|\,dx\le\frac{1}{(p+2)^{\frac{p-1}{p}}}\Big(\int_0^1 x\,|f(x)|^p\,dx\Big)^{\frac{1}{p}}.$$

Letting $\alpha_p=\frac{1}{(p+2)^{\frac{p-1}{p}}}$ and $A=\int_0^1 x\,|f(x)|^p\,dx$ we deduce that

$$I(f)\le\alpha_p A^{\frac{1}{p}}-A\,.$$

An elementary computation shows that the function $g:[0,\infty)\to\mathbb{R}$ defined by $g(y)=\alpha_p y^{\frac{1}{p}}-y$, has a unique critical point, namely $y_0=(\frac{\alpha_p}{p})^{\frac{p}{p-1}}$. In addition, g is increasing on $[0,y_0]$ and decreasing on $[y_0,\infty)$. Consequently, we have

$$I(f)\le g(A)\le g(y_0)=\alpha_p(\frac{\alpha_p}{p})^{\frac{1}{p-1}}-(\frac{\alpha_p}{p})^{\frac{p}{p-1}},$$

whence

$$I(f)\le\alpha_p^{\frac{p}{p-1}}\Big(\frac{1}{p^{\frac{1}{p-1}}}-\frac{1}{p^{\frac{p}{p-1}}}\Big)=\frac{1}{p+2}(p^{\frac{1}{1-p}}-p^{\frac{p}{1-p}})\,.$$

Equality holds, for instance, in the case of the function $f(x)=p^{\frac{1}{1-p}}x$ for all $x\in[0,1]$.

Remark. This problem is a generalization of problem B5 in the 2006 William Lowell Putnam mathematical competition.

Problem 14. Consider the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ and the sequence $x_n = 1-\frac{1}{n},\ n\geq 1$. Define the function

$$f:[0,1]\to\mathbb{R},\qquad f(x)=\begin{cases}\sum\limits_{x\leq x_n}\frac{(-1)^{n+1}}{n}, & x\in[0,1)\\ 0, & x=1.\end{cases}$$

a) Study the continuity of f.

b) Prove that f is Riemann integrable on $[0,1]$ and compute $\int_0^1 f(x)dx$.

Solution. a) Denote by $a_n=\frac{(-1)^{n+1}}{n}, n\geq 1$. It is clear that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is convergent, the sum of the series is $S=\ln 2$, and that the sequence x_n is increasing to 1 and all its terms lie in $[0,1]$. Given $x\in[0,1)$, because $x_n\to 1$, there is some $n_x\in\mathbb{N}$ such that $x_n>x$ for all $n\geq n_x$. Taking n_x the smallest one with this property, and taking into account that (x_n) is increasing, then

$$f(x)=\sum_{n=n_x}^{\infty}a_n=S-S_{n_x-1},$$

which is finite. Here, S_n denotes the partial sum sequence associated with the given series. In this way, one can write

$$f(x)=\begin{cases}S, & x=0\\ S-S_1, & x\in(0,\frac{1}{2}]\\ S-S_2, & x\in(\frac{1}{2},\frac{2}{3}]\\ \vdots & \\ S-S_n, & x\in(\frac{n-1}{n},\frac{n}{n+1}]\\ \vdots & \end{cases}$$

It follows that f is continuous at every $x\in(\frac{n-1}{n},\frac{n}{n+1})$ for every $n\geq 1$. Furthermore,

$$f(x_n+0)=S-S_n, \text{ and}$$
$$f(x_n-0)=f(x_n)=S-S_{n-1}.$$

Thus, f is continuous at every $x\in[0,1]\setminus\{x_n \mid n\geq 1\}$, is continuous from the left at every x_n, and is not continuous from the right at any x_n, $n\geq 1$.

b) Observe that f is bounded, since (S_n) is convergent (hence bounded). Then, f is Riemann integrable over $[0,1]$, since is bounded and its discontinuity set is at most countable, and

$$\int_0^1 f(x)dx=\sum_{n=1}^{\infty}\int_{x_n}^{x_{n+1}}f(x)dx=\sum_{n=1}^{\infty}(S-S_n)(\tfrac{n}{n+1}-\tfrac{n-1}{n})=\sum_{n=1}^{\infty}(S-S_n)(\tfrac{1}{n}-\tfrac{1}{n+1}).$$

Denote by T_n the partial sum of the series $\sum_{n=1}^{\infty}(S-S_n)(\frac{1}{n}-\frac{1}{n+1})$. Then

$$\begin{aligned}T_n &= (S-S_1)(\tfrac{1}{1}-\tfrac{1}{2})+(S-S_2)(\tfrac{1}{2}-\tfrac{1}{3})+\ldots+(S-S_n)(\tfrac{1}{n}-\tfrac{1}{n+1})\\ &= S(1-\tfrac{1}{n+1})-\Big[S_1(\tfrac{1}{1}-\tfrac{1}{2})+S_2(\tfrac{1}{2}-\tfrac{1}{3})+\ldots+S_n(\tfrac{1}{n}-\tfrac{1}{n+1})\Big]\\ &= S(1-\tfrac{1}{n+1})-\Big[\tfrac{1}{1}S_1+\tfrac{1}{2}(S_2-S_1)+\tfrac{1}{3}(S_3-S_2)+\ldots+\tfrac{1}{n}(S_n-S_{n-1})\Big]+\tfrac{S_n}{n+1}\\ &= S(1-\tfrac{1}{n+1})-(\tfrac{a_1}{1}+\tfrac{a_2}{2}+\ldots+\tfrac{a_n}{n})+\tfrac{S_n}{n+1}.\end{aligned}$$

Consider now the series $\sum\limits_{n=1}^{\infty}\frac{a_n}{n}=\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. This is absolutely convergent so its associated partial sums sequence is also convergent. Moreover, because (S_n) is bounded, it follows that $\frac{S_n}{n+1}\to 0$ as $n\to\infty$. Thus, (T_n) is convergent and

$$\int_0^1 f(x)dx = S-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.$$

Because $S=\ln 2$ and $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$, it follows that

$$\int_0^1 f(x)dx=\ln 2-\frac{\pi^2}{12}.$$

Problem 15. a) Let $n \ge 0$ be an integer. Calculate $\int_0^1 (1-t)^n e^t dt$.

b) Let $k \ge 0$ be a fixed integer and let $(x_n)_{n\ge k}$ be the sequence defined by

$$x_n = \sum_{i=k}^{n} \binom{i}{k}\left(e-1-\frac{1}{1!}-\frac{1}{2!}-\dots-\frac{1}{i!}\right).$$

Prove that the sequence converges and find its limit.

<u>*Solution.*</u> a) Let $I_n = \int_0^1 (1-t)^n e^t dt$, $n \ge 0$. We integrate by parts and we get that $I_n = -1 + nI_{n-1}$, $n \ge 1$ which implies that $\frac{I_n}{n!} = -\frac{1}{n!} + \frac{I_{n-1}}{(n-1)!}$. It follows that

$$\frac{I_n}{n!} = I_0 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{n!} = e - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{n!}.$$

Thus,

$$I_n = n!\left(e-1-\frac{1}{1!}-\frac{1}{2!}-\dots-\frac{1}{n!}\right), n \ge 0.$$

b) We have

$$x_{n+1} - x_n = \binom{n+1}{k}\left(e-1-\frac{1}{1!}-\frac{1}{2!}-\dots-\frac{1}{(n+1)!}\right) > 0$$

hence the sequence is strictly increasing.

On the other hand, we have based on Taylor's formula, that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{e^\theta}{(n+1)!}$$

for some $\theta \in (0,1)$. It follows that

$$0 < e - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{n!} < \frac{e}{(n+1)!}.$$

Therefore

$$x_n \le \sum_{i=k}^{n} \binom{i}{k} \frac{e}{(i+1)!} \le \frac{e}{k!} \sum_{i=k}^{n} \frac{1}{(i-k)!} = \frac{e}{k!}\left(\frac{1}{0!} + \frac{1}{1!} + \dots + \frac{1}{(n-k)!}\right) \le \frac{e^2}{k!}$$

which implies the sequence is bounded. Since the sequence is bounded and increasing it converges.

To find $\lim_{n\to\infty} x_n$ we apply part a) of the problem and we have, since

$$e - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{i!} = \frac{1}{i!}\int_0^1 (1-t)^i e^t dt$$

that

$$x_n = \sum_{i=k}^{n} \binom{i}{k} \frac{1}{i!} \int_0^1 (1-t)^i e^t dt = \frac{1}{k!}\int_0^1 (1-t)^k e^t \left(\sum_{i=k}^{n} \frac{(1-t)^{i-k}}{(i-k)!}\right) dt.$$

Since $\lim_{n\to\infty} \sum_{i=k}^{n} \frac{(1-t)^{i-k}}{(i-k)!} = e^{1-t}$ and $\sum_{i=k}^{n} \frac{(1-t)^{i-k}}{(i-k)!} < e^{1-t}$, we get based on Lebesgue Dominated Convergence Theorem

$$\lim_{n\to\infty} x_n = \frac{1}{k!}\int_0^1 (1-t)^k e^t e^{1-t} dt = \frac{e}{(k+1)!}.$$

<u>**Remark.**</u> Part b) of the problem has an equivalent formulation

$$\sum_{i=k}^{\infty} \binom{i}{k}\left(e-1-\frac{1}{1!}-\frac{1}{2!}-\dots-\frac{1}{i!}\right) = \frac{e}{(k+1)!}.$$

Problem 16. Let C be the set of all real numbers x for which the series

$$\sum_{n=1}^{\infty} \sin^2(2\pi n! x) \qquad (1)$$

converges. Prove that:

a) $\mathbb{Q} \subseteq C$, but $C \neq \mathbb{Q}$.

b) There exists a dense subset A of $\mathbb{R}$ such that $A \subseteq \mathbb{R} \setminus C$.

Solution. a) If $x = \frac{p}{q}$ is an arbitrary rational number, with $p, q \in \mathbb{Z}$, $q \neq 0$, then for every $n \geq |q|$ we have $\sin^2(2\pi n! x) = 0$, hence the series (1) converges. Therefore, we have $\mathbb{Q} \subseteq C$. In order to prove that $C \neq \mathbb{Q}$, we show that $e \in C$. It is well-known that for each $n \geq 1$ there exists some $\theta_n \in (0,1)$ such that $e = 1 + \frac{1}{1!} + ... + \frac{1}{n!} + \frac{\theta_n}{n \cdot n!}$, whence

$$2\pi n! e = 2\pi n! (1 + \frac{1}{1!} + ... + \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{\theta_{n+1}}{(n+1)\cdot(n+1)!}) = 2\pi k_n + \frac{2\pi}{n+1} + \frac{2\pi\theta_{n+1}}{(n+1)^2},$$

where $k_n = n!(1 + \frac{1}{1!} + ... + \frac{1}{n!})$ is a positive integer and $\theta_{n+1} \in (0,1)$. Taking into account that $\sin^2 x = O(x^2)$ as $x \to 0$, it follows that

$$\sin^2(2\pi n! e) = \sin^2(\frac{2\pi}{n+1} + \frac{2\pi\theta_{n+1}}{(n+1)^2}) = O((\frac{2\pi}{n+1} + \frac{2\pi\theta_{n+1}}{(n+1)^2})^2) = O(\frac{1}{n^2}), \text{ as } n \to \infty.$$

Consequently, the series $\sum_{n=1}^{\infty} \sin^2(2\pi n! e)$ has the same nature as the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent. This shows that $e \in C$, as claimed.

b) We prove first that $\frac{e}{3} \notin C$. Indeed, for each positive integer $n > 3$ there exists some $\theta_n \in (0,1)$ such that

$$2\pi n! \frac{e}{3} = 2\pi \frac{n!}{3}(1 + \frac{1}{1!} + ... + \frac{1}{(n-3)!} + \frac{1}{(n-2)!} + \frac{1}{(n-1)!} + \frac{1}{n!} + \frac{\theta_n}{n \cdot n!}) = 2\pi m_n + \frac{2\pi}{3}(n(n-1) + n + 1) + \frac{2\pi\theta_n}{3n},$$

where

$$m_n = \frac{n!}{3}(1 + \frac{1}{1!} + ... + \frac{1}{(n-3)!}) = \frac{n(n-1)(n-2)}{3}(n-3)!(1 + \frac{1}{1!} + ... + \frac{1}{(n-3)!})$$

is a positive integer. Therefore, we have

$$\sin^2(2\pi n! \frac{e}{3}) = \sin^2(\frac{2\pi}{3}(n^2 + 1) + \frac{2\pi\theta_n}{3n}).$$

If $n \equiv 0 (\mathrm{mod}\, 3)$, then $n^2 + 1 \equiv 1 (\mathrm{mod}\, 3)$, whence

$$\lim_{\substack{n \to \infty \\ n \equiv 0 (\mathrm{mod}\, 3)}} \sin^2(2\pi n! \frac{e}{3}) = \lim_{\substack{n \to \infty \\ n \equiv 0 (\mathrm{mod}\, 3)}} \sin^2(\frac{2\pi}{3}(n^2 + 1) + \frac{2\pi\theta_n}{3n}) = \sin^2 \frac{2\pi}{3} = \frac{3}{4}.$$

Consequently, the series $\sum_{n=1}^{\infty} \sin^2(2\pi n! \frac{e}{3})$ diverges, showing that $\frac{e}{3} \notin C$, as claimed.

Now set $A = \{\frac{e}{3}\} + \mathbb{Q} = \{\frac{e}{3} + x \mid x \in \mathbb{Q}\}$. If $x = \frac{p}{q}$ is an arbitrary rational number, with $p, q \in \mathbb{Z}$, $q \neq 0$, then for every $n \geq |q|$ we have $\sin^2(2\pi n!(\frac{e}{3} + x)) = \sin^2(2\pi n! \frac{e}{3})$, whence $\frac{e}{3} + x \notin C$. It follows that $A \subseteq \mathbb{R} \setminus C$ and A is obviously a dense subset of $\mathbb{R}$.

Problem 17. Consider the function $f:[0,\infty)\to\mathbb{R}$ given by $f(x)=xe^{-x}$.

1) Prove that, for every $\varepsilon\in(0,1)$, there exists $\delta\in(0,1)$ such that, for every $x\in(1-\delta,1+\delta)$, one has:

$$e^{-1-\frac{(x-1)^2}{2}(1+\varepsilon)}\le f(x)\le e^{-1-\frac{(x-1)^2}{2}(1-\varepsilon)}.$$

2) Prove that:

$$\lim_{n\to\infty}\frac{1}{n!}\cdot\int_0^n f(t)\cdot t^{n-1}dt=\frac{1}{2}.$$

3) Compute the limit:

$$\lim_{n\to\infty}e^{-n}\cdot\left(\sum_{k=0}^{n}\frac{n^k}{k!}\right).$$

Solution. 1) Observe that f achives a strict maximum at 1 and that $f(0)=0$ and $\lim\limits_{x\to\infty}f(x)=0$. Consider $g:(0,\infty)\to\mathbb{R}$ given as $g(x)=\ln(f(x))$. Remark that $g(1)=-1$, $g'(1)=0$ and $g''(1)=-1$. Then, by the Taylor formula, there exists a function α such that $\lim\limits_{x\to1}\alpha(x)=0$ such that

$$g(x)=-1-\frac{(x-1)^2}{2}(1+\alpha(x)).$$

Using $\lim\limits_{x\to1}\alpha(x)=0$ it follows that for every $\varepsilon\in(0,1)$, there exists $\delta\in(0,1)$ such that, for every $x\in(1-\delta,1+\delta)$, one has $|\alpha(x)|<\varepsilon$. The conclusion follows.

2) Denote by

$$\ell=\lim_{n\to\infty}\frac{1}{n!}\cdot\int_0^n f(t)\cdot t^{n-1}dt.$$

By the change of the variable $nt=x$, one has

$$\ell=\lim_{n\to\infty}\frac{n^{n+1}}{n!}\cdot\int_0^1(xe^{-x})^n\,dx.$$

Denote

$$I_n=\int_0^1(f(x))^n\,dx=\int_0^1(xe^{-x})^n\,dx.$$

Observe that, in view of 1), that for every $\varepsilon\in(0,1)$, there exists $\delta\in(0,1)$ such that

$$I_n\ge\int_{1-\delta}^{1}(xe^{-x})^n\,dx\ge\int_{1-\delta}^{1}e^{-n-\frac{(x-1)^2}{2}(1+\varepsilon)n}dx,$$

hence, using also the Stirling formula, there exists $\theta_n\to1$ such that

$$\frac{n^{n+1}}{n!}I_n=\frac{n^{n+1}}{\sqrt{2n\pi}(\frac{n}{e})^n\theta_n}I_n\ge\frac{n^{n+1}}{\sqrt{2n\pi}(\frac{n}{e})^n\theta_n}\cdot\frac{1}{e^n}\cdot\int_{1-\delta}^{1}e^{-\frac{(x-1)^2}{2}(1+\varepsilon)n}dx.$$

Now, observe that, by the change of variable $\sqrt{(1+\varepsilon)n}(x-1)=-y$, the integral

$\int_{1-\delta}^{1} e^{-\frac{(x-1)^2}{2}(1+\varepsilon)n} dx$ becomes $\frac{1}{\sqrt{(1+\varepsilon)n}} \int_0^{\delta\sqrt{(1+\varepsilon)n}} e^{-\frac{y^2}{2}} dy.$

It follows that

$$\frac{n^{n+1}}{n!} I_n \geq \frac{1}{\theta_n \sqrt{2\pi(1+\varepsilon)}} \int_0^{\delta\sqrt{(1+\varepsilon)n}} e^{-\frac{y^2}{2}} dy.$$

Passing to the limit for $n \to \infty$ and using the fact that $\int_0^{\infty} e^{-\frac{y^2}{2}} dy = \frac{\sqrt{2\pi}}{2}$, we obtain that $\ell \geq \frac{1}{2\sqrt{(1+\varepsilon)}}$.

For the upper bound for I_n, observe that

$$I_n - \int_{1-\delta}^{1} (xe^{-x})^n dx = \int_0^{1-\delta} (xe^{-x})^n dx \leq (\max_{x \in [0,1-\delta]} f(x))^{n-1} \cdot I_1.$$

Since $m_\delta := \max_{x \in [0,1-\delta]} f(x) < f(1) = \frac{1}{e}$, it follows that

$$I_n \leq \int_{1-\delta}^{1} (xe^{-x})^n dx + m_\delta^{n-1} \cdot I_1 = (m_\delta e)^n \cdot \frac{1}{e^n} \cdot \frac{I_1}{m_\delta},$$

hence for every $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$, $A := m_\delta e \in (0,1)$ and $k := \frac{I_1}{m_\delta} > 0$, such that

$$I_n \leq \int_{1-\delta}^{1} (xe^{-x})^n dx + k \cdot (\tfrac{A}{e})^n.$$

Then, reasoning as above, it follows that

$$\frac{n^{n+1}}{n!} I_n = \frac{n^{n+1}}{\sqrt{2n\pi}(\frac{n}{e})^n \theta_n} I_n \leq \frac{n^{n+1}}{\sqrt{2n\pi}(\frac{n}{e})^n \theta_n} \cdot [\tfrac{1}{e^n} \int_{1-\delta}^{1} e^{-\frac{(x-1)^2}{2}(1-\varepsilon)n} dx + k \cdot (\tfrac{A}{e})^n]$$

$$= \frac{1}{\theta_n \sqrt{2\pi(1-\varepsilon)}} \int_0^{\delta\sqrt{(1-\varepsilon)n}} e^{-\frac{y^2}{2}} dy + \frac{k}{\theta_n \sqrt{2\pi}} \cdot \sqrt{n} \cdot A^n.$$

Passing to the limit for $n \to \infty$, it follows that $\ell \leq \frac{1}{2\sqrt{(1-\varepsilon)}}$.

In conclusion,

$$\frac{1}{2\sqrt{(1+\varepsilon)}} \leq \ell \leq \frac{1}{2\sqrt{(1-\varepsilon)}},$$

for every $\varepsilon \in (0,1)$, hence $\ell = \frac{1}{2}$.

3) Observe, integrating by parts, that

$$\int_0^n e^{-t} t^n dt = n! - e^{-n}(n^n + n \cdot n^{n-1} + n(n-1) \cdot n^{n-2} + ... + n!).$$

Using 2), it follows that

$$\frac{1}{2} = \lim_{n\to\infty} \frac{1}{n!} \int_0^n e^{-t} t^n dt = 1 - \lim_{n\to\infty} e^{-n} \cdot (\sum_{k=0}^{n} \frac{n^k}{k!}).$$

Hence, the desired limit equals $\frac{1}{2}$. ?

DISCRETE MATHEMATICS

Problem 1. Let $n > 1$ be an integer which not divisible by 2017. Consider two sequences

$$a_i = i + \frac{ni}{2017}, \ (i = 1, 2, 3, \ldots, 2016)$$

$$b_j = j + \frac{2017j}{n}, \ \ (j = 1, 2, 3, \ldots, n-1) \ .$$

Writing all members of these two sequences in the increasing order, we get the sequence

$$c_1 \le c_2 \le c_3 \le \ldots \le c_{n+2015} \, .$$

Prove that

$$c_{k+1} - c_k \le 2, \text{ for all } k = 1, 2, 3, \ldots, n+2014 \, .$$

Solution. Replace 2017 by the number *m* which not divide *n*. Let

$$a_i = i + \frac{ni}{m}, \ (i = 0, 1, 2, \ldots, m)$$

$$b_j = j + \frac{mj}{n}, \ (j = 0, 1, 2, \ldots, n) \, .$$

We show that these sequences have the same property. We have

$$a_0 = 0 < a_1 < a_2 < \ldots < a_{m-1} < a_m = m + n \, ,$$

$$b_0 = 0 < b_1 < b_2 < \ldots < b_{n-1} < b_n = m + n \, .$$

We may assume that $n < m$. Then

$$a_{i+1} - a_i = 1 + \frac{n}{m} < 2 \, .$$

For each $k = 1, \ldots, m+n-2$ there is unique j such that

$$a_j \le c_k < a_{j+1}, \ (0 \le j \le n-1).$$

Then $c_{k+1} = a_{j+1}$ and

$$c_{k+1} - c_k \le a_{j+1} - a_j < 2 \, .$$

Problem 2. Let $(T_n)_{n>0}$ be the sequence of polynomials defined by

$$T_0 : \mathbb{R} \to \mathbb{R}, \quad T_0(x) = 1, \ (x \in \mathbb{R})$$
$$T_1 : \mathbb{R} \to \mathbb{R}, \quad T_1(x) = x, \ (x \in \mathbb{R})$$
$$T_n : \mathbb{R} \to \mathbb{R}, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \ (x \in \mathbb{R}, n \geq 1)$$

and $(F_n)_{n\geq 0}$ the sequence of numbers defined by

$$F_0 = 0, \ F_1 = 1,$$
$$F_{n+1} = F_n + F_{n-1}, \ (n \geq 1).$$

Prove that:

a) $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, for every $n \geq 1$,

b) $T_n(-\frac{3}{2}) = 1 + (-1)^n \cdot \frac{5}{2} F_n^2$ for every $n \in \mathbb{N}$.

Solution. a) Proof by induction, or by other methods - this is a well-known identity of the Fibonacci sequence (any method is accepted).

b) We give a proof by induction. It is easy to check that the conclusion takes place for $n = 0,1$. We assume the relation to be true for $n = 0,1,...,k$ $(k \geq 1)$ and prove it for $k+1$. Since

$$T_{k-1}(-\tfrac{3}{2}) = 1 + (-1)^{k-1} \cdot \tfrac{5}{2} F_{k-1}^2$$
$$T_k(-\tfrac{3}{2}) = 1 + (-1)^k \cdot \tfrac{5}{2} F_k^2,$$

by using the recurrence that defines T_n, we have that

$$T_{k+1}(-\tfrac{3}{2}) = -3T_k(-\tfrac{3}{2}) - T_{k-1}(-\tfrac{3}{2}) = -4 + (-1)^{k+1} \cdot \tfrac{5}{2}[3F_k^2 - F_{k-1}^2]$$

so we need to prove the identity:

$$1 + (-1)^{k+1} \cdot \tfrac{5}{2} F_{k+1}^2 = -4 + (-1)^{k+1} \cdot \tfrac{5}{2}[3F_k^2 - F_{k-1}^2]$$

which is equivalent to

$$F_{k+1}^2 - 3F_k^2 + F_{k-1}^2 = 2(-1)^k .$$

We have

$$\begin{aligned} F_{k+1}^2 - 3F_k^2 + F_{k-1}^2 &= (F_k + F_{k-1})^2 - 3F_k^2 + F_{k-1}^2 \\ &= 2(F_{k-1}^2 - F_k^2 + F_k F_{k-1}) \\ &= 2[F_{k-1}(F_k + F_{k-1}) - F_k^2] \\ &= 2(F_{k-1}F_{k+1}) - F_k^2) \\ &= 2(-1)^k \end{aligned}$$

by a), which concludes the proof.

Remark: T_n are the Chebyshev polynomials, defined for $x \in [-1,1]$ by

$$T_n(x) = \cos(n \arccos x) .$$

<u>Problem 3.</u> Given a positive integer n, let T_n denote the set of all permutations of $\{1,2,...,n\}$ without fixed points. Find $\sum_{\sigma\in T_n} \varepsilon(\sigma)$ where $\varepsilon(\sigma)$ denotes the sign of the permutation σ.

<u>Solution.</u> We have

$$\sum_{\sigma\in T_n} \varepsilon(\sigma) = \begin{vmatrix} 0 & 1 & 1 & ... & 1 & 1 & 1 \\ 1 & 0 & 1 & ... & 1 & 1 & 1 \\ 1 & 1 & 0 & ... & 1 & 1 & 1 \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 0 & 1 & 1 \\ 1 & 1 & 1 & ... & 1 & 0 & 1 \\ 1 & 1 & 1 & ... & 1 & 1 & 0 \end{vmatrix}.$$

By adding all the lines $2,3,...,n$ to the first line we get

$$\sum_{\sigma\in T_n} \varepsilon(\sigma) = (n-1)\begin{vmatrix} 1 & 1 & 1 & ... & 1 & 1 & 1 \\ 1 & 0 & 1 & ... & 1 & 1 & 1 \\ 1 & 1 & 0 & ... & 1 & 1 & 1 \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & 1 & 1 & ... & 0 & 1 & 1 \\ 1 & 1 & 1 & ... & 1 & 0 & 1 \\ 1 & 1 & 1 & ... & 1 & 1 & 0 \end{vmatrix}.$$

By subtracting the first line from each of the lines $2,3,...,n$ we obtain

$$\sum_{\sigma\in T_n} \varepsilon(\sigma) = (n-1)\begin{vmatrix} 1 & 1 & 1 & ... & 1 & 0 & 0 \\ 0 & -1 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & -1 & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & -1 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & -1 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & -1 \end{vmatrix}$$

Whence

$$\sum_{\sigma\in T_n} \varepsilon(\sigma) = (-1)^{n-1}(n-1).$$

Problem 4. A group of students is arranged on a circle around their professor. The professor gave each student a positive number of coins. The game begins when student gives the "extra-half" of his coins (it means that if the number of his coins is even, hi gives the half of his number and if not, he takes one more coin from the professor and gives the half) to the friend standing on his right side. Then this one, after receiving the coins from the former, gives the "extra-half" of all his coins to the friend standing on his right side, and so on. Prove that we shall arrive at situation where if a student, at his turn, gives the "extra-half" of his coins not his friend, but to the professor, then each student will have the same number of coins.

Solution I. Consider two students *A* and *B* standing next to the each other, in the order of passing coins. At the moment *n* when *A* is giving x_n coins to *B*, suppose *A* has a_n coins and *B* has b_n coins (x_n coins not belong to both *A* and *B* at this moment). Let M_n, P_n be the maximum and the minimum number of coins of all students at the moment *n* (not including x_n).

At the moment $n+1$ when *B* is giving x_{n+1} coins to the next student, the number of coins that *B* has is

$$b_{n+1} = x_{n+1} = \begin{cases} \dfrac{b_n + x_n}{2}, & if\ b_n + x_n \text{ s even} \\ \dfrac{b_n + x_n + 1}{2}, & \text{if } b_n + x_n \text{ is odd} \end{cases}$$

At this time, the student next to *B* has not received any coin, and the number of coins of each student except B is unchanged in comparison with the moment *n*.

Consider following cases:

1) If $a_n = x_n = b_n$, then $b_{n+1} = b_n = a_n$, thus $M_{n+1} = M_n$ and $P_{n+1} = P_n$.
2) If $a_n = x_n \neq b_n$
   a) Consider M_{n+1}:
   $$b_{n+1} \leq \frac{b_n + x_n + 1}{2} \leq \frac{M_n + M_n + 1}{2} = M_n + \frac{1}{2}$$
   Since M_n and b_{n+1} are integers, we have $b_{n+1} \leq M_n$, therefore $M_{n+1} \leq M_n$
   From 1) and a), we conclude that $\{M_n\}$ is increasing integer sequence.
   b) Consider P_{n+1}: if $x_n < b_n$ then $b_n \geq x_n + 1 = a_n + 1 \geq P_n + 1$. If $x_n > b_n$ then $x_n \geq b_n + 1 \geq P_n + 1$. In both cases, we always have
   $$b_{n+1} \geq \frac{b_n + x_n}{2} \geq \frac{P_n + P_n + 1}{2} = P_n + \frac{1}{2}$$
   Since both P_n and b_{n+1} are integers, $b_{n+1} \geq P_n + 1$. Therefore, either $P_{n+1} > P_n$ if at the moment *n*, there is exactly one $b_n = P_n$, or $P_{n+1} = P_n$ if there is least one student different from *B* having P_n coins.

Generally $\{P_n\}$ is non-decreasing integer sequence. Moreover, when $b_n = P_n < a_n$ then at the moment $n+1$, we will have $b_{n+1} \geq P_n + 1$, hence $\{P_n\}$ strictly increasing sometimes.

As $\{M_n\}$ is non-increasing integer sequence and $\{P_n\}$ is non-decreasing integer sequence and strictly increases sometimes, there must exist a moment *k* such that $M_k = P_k$. At that moment, all students have (including coins in giving process) equal number of coins.

Solution II. Let us consider two students A and B next to each other. When the student A possess $2x_i$ coins and he should give x_i coins to the student B, we correspond the following sequence of coins

$$x_1, x_2, \ldots, x_i, x_i, x_{i+1}, x_{i+2}, \ldots, x_n \quad (n \text{ is the number of students}).$$

The the next sequence will be

$$x_1, x_2, \ldots, x_i, \tfrac{x_i+x_{i+1}}{2}, \tfrac{x_i+x_{i+1}}{2}, x_{i+2}, \ldots, x_n \quad (\text{if } x_i + x_{i+1} \text{ is even number})$$

or

$$x_1, x_2, \ldots, x_i, \tfrac{x_i+x_{i+1}+1}{2}, \tfrac{x_i+x_{i+1}+1}{2}, x_{i+2}, \ldots, x_n \quad (\text{if } x_i + x_{i+1} \text{ is odd number}).$$

We should prove that after finite number of steps all terms of the sequence will be mutually equal.

Let M be a sufficiently large number, such that each student has less than M coins at the initial moment. Then for each sequence of $n+1$ terms $y_1, \ldots, y_{n+1}$ we map into the following positive integer $S = (M - y_1)^2 + \ldots + (M - y_{n+1})^2$. Hence the obtained sequence of numbers $S_1, S_2, \ldots$ is non-increasing. Indeed,

$$\begin{aligned} S_k &= (M - x_1)^2 + \ldots + (M - x_i)^2 + (M - x_i)^2 + (M - x_{i+1})^2 + \ldots + (M - x_n)^2 \\ &\geq (M - x_1)^2 + \ldots + (M - x_i)^2 + (M - \tfrac{x_i+x_{i+1}}{2})^2 + (M - \tfrac{x_i+x_{i+1}}{2})^2 + \ldots + (M - x_n)^2 = S_{k+1} \end{aligned}$$

and also

$$\begin{aligned} S_k &= (M - x_1)^2 + \ldots + (M - x_i)^2 + (M - x_i)^2 + (M - x_{i+1})^2 + \ldots + (M - x_n)^2 \\ &\geq (M - x_1)^2 + \ldots + (M - x_i)^2 + (M - \tfrac{x_i+x_{i+1}}{2})^2 + (M - \tfrac{x_i+x_{i+1}}{2})^2 + \ldots + (M - x_n)^2 \\ &\geq (M - x_1)^2 + \ldots + (M - x_i)^2 + (M - \tfrac{x_i+x_{i+1}+1}{2})^2 + (M - \tfrac{x_i+x_{i+1}+1}{2})^2 + \ldots + (M - x_n)^2 = S_{k+1} \end{aligned}$$

Hence, $S_{k+1} \leq S_k$ for each k. Since $S_1, S_2, \ldots$ is a non-increasing sequence of positive integer numbers, there exists a positive integer m, such that $S_m = S_{m+1} = S_{m+2} = \ldots$ and it is easy to see that this is satisfied only for constant sequence $y_1, \ldots, y_{n+1}$, i.e. only for $y_1 = \ldots = y_{n+1}$.

Problem 5. Let $(x_n)_{n\ge 0}$ be the sequence defined by

$$x_0 = 0, x_1 = 1, x_2 = 1 \text{ and } x_{n+3} = x_{n+2} + x_{n+1} + x_n + n, \ \forall n \ge 0.$$

Prove the series $\sum_{n=1}^{\infty} \frac{x_n}{2^n}$ converges and find its sum.

Solution. Let $y_n = x_n + \frac{n}{2}$. It follows that the sequence $(y_n)_{n\ge 0}$ verifies the recurrence formula $y_{n+3} = y_{n+2} + y_{n+1} + y_n$, $\forall n \ge 0$. The characteristic equation of this recurrence relation is $t^3 - t^2 - t - 1 = 0$. We have, based on the study of the graph of the function $f:\mathbb{R}\to\mathbb{R},\ f(t) = t^3 - t^2 - t - 1$ that the equation $f(t) = 0$ has a real root $t_1 \in (1,2)$ and two complex conjugate roots,

$$t_2 = \rho(\cos\theta + i\sin\theta) \text{ and } t_3 = \rho(\cos\theta - i\sin\theta).$$

We have, based on Viete's formula, that $t_1 t_2 t_3 = 1$ which implies $t_1\rho^2 = 1$. Thus, $\rho = \frac{1}{\sqrt{t_1}} < 1$. It follows that

$$y_n = A\rho^n \cos n\theta + B\rho^n \sin n\theta + Ct_1^n,$$

for some constants $A, B, C \in \mathbb{R}$. This implies that

$$x_n = y_n - \frac{n}{2} = A\rho^n \cos n\theta + B\rho^n \sin n\theta + Ct_1^n - \frac{n}{2}.$$

Since $t_1 \in (1,2)$ and $\rho = \frac{1}{\sqrt{t_1}} \in (0,1)$ we have that

$$\sum_{n=1}^{\infty} \frac{x_n}{2^n} = \sum_{n=1}^{\infty} \frac{|x_n|}{2^n} \le (|A| + |B|)\sum_{n=1}^{\infty} \left(\frac{\rho}{2}\right)^n + |C| \sum_{n=1}^{\infty} \left(\frac{t_1}{2}\right)^n + \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{2^n} < \infty,$$

and this implies the series $\sum_{n=1}^{\infty} \frac{x_n}{2^n}$ converges.

Let $S = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$. We prove that $S = 6$. We have

$$\begin{aligned} S &= \frac{1}{2} + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{x_n}{2^n} = \frac{3}{4} + \sum_{n=3}^{\infty} \frac{x_{n-1}+x_{n-2}+x_{n-3}+n-3}{2^n} \\ &= \frac{3}{4} + \frac{1}{2}\sum_{n=3}^{\infty} \frac{x_{n-1}}{2^{n-1}} + \frac{1}{2^2}\sum_{n=3}^{\infty} \frac{x_{n-2}}{2^{n-2}} + \frac{1}{2^3}\sum_{n=3}^{\infty} \frac{x_{n-3}}{2^{n-3}} + \sum_{n=4}^{\infty} \frac{n-3}{2^n} \\ &= 1 + \frac{1}{2}\left(S - \frac{1}{2}\right) + \frac{S}{4} + \frac{S}{8}, \end{aligned}$$

since $\sum_{n=4}^{\infty} \frac{n-3}{2^n} = \frac{1}{4}$ (This follows from the geometric series). This implies that $S = 6$ and the problem is solved.