SOLUTIONS

Problem 1.

Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be such that $AB^2A = AB$. Prove that:

a) $(AB)^2 = AB$.

b) $(AB - BA)^3 = O_n$.

Solution: From the hypotesis, $AB(BA - I_n) = O_n$. Based on Sylvester's inequality for ranks, it follows that

$$\operatorname{rank}(AB) + \operatorname{rank}(BA - I_n) \leqslant n + \operatorname{rank}\left(AB(BA - I_n)\right) = n. \tag{1}$$

Also, it is true in general that

$$\operatorname{rank}(AB - I_n) = \operatorname{rank}(BA - I_n), \tag{2}$$

so

$$\operatorname{rank}(AB - I_n) + \operatorname{rank}(AB) \leqslant n. \tag{3}$$

But $\operatorname{Ker}(AB - I_n) \subseteq \operatorname{Im} AB$, so

$$\operatorname{rank}(AB - I_n) + \operatorname{rank}(AB) = n + \operatorname{rank}\left((AB)^2 - AB\right) \tag{4}$$

(this is the equality case in Sylvester's inequality for the matrices $AB - I_n$ and AB). Combining (3) and (4), it follows that $(AB)^2 = AB$.
Using now the identity from the hypotesis and $(AB)^2 = AB$, we obtain

$$\begin{aligned}
(AB - BA)^2 &= (AB)^2 + (BA)^2 - AB^2A - BA^2B = (BA)^2 - BA^2B = -BA(AB - BA)\\
(AB - BA)^3 &= -BA(AB - BA)^2 = (BA)^2(AB - BA)\\
(AB - BA)^4 &= (BA)^2(AB - BA)^2 = -(BA)^3(AB - BA) = -B(AB)^2A(AB - BA)\\
&= -B(AB)A(AB - BA) = -(BA)^2(AB - BA)\\
&= -(AB - BA)^3,
\end{aligned}$$

hence $(AB - BA)^4 = -(AB - BA)^3$.
Let λ be any eigenvalue of $AB - BA$. Then the previous identity implies $\lambda^4 = -\lambda^3$, so $\lambda \in \{0, -1\}$. Since $\operatorname{Tr}(AB - BA) = 0$, it follows that all eigenvalues of $AB - BA$ must be 0. Then $(AB - BA)^n = O_n$, and hence, $(AB - BA)^3 = O_n$.

Problem 2.

Let $a, b, c \in \mathbb{R}$ be such that

$$a + b + c = a^2 + b^2 + c^2 = 1, \quad a^3 + b^3 + c^3 \neq 1.$$

We say that a function f is a *Palić function* if $f : \mathbb{R} \to \mathbb{R}$, f is continuous and satisfies

$$f(x) + f(y) + f(z) = f(ax + by + cz) + f(bx + cy + az) + f(cx + ay + bz)$$

for all $x, y, z \in \mathbb{R}$.

Prove that any Palić function is infinitely many times differentiable and find all Palić functions.

Solution: First, it is easy to show that the given conditions imply that a, b and c are nonzero. Let f be a *Palić function*. For $z = 0$ in (P), we obtain

$$f(x) + f(y) + f(0) = f(ax + by) + f(bx + cy) + f(cx + ay) \tag{1}$$

for all $x, y \in \mathbb{R}$. Since f is continuous, it follows that $F(x) = \displaystyle\int_0^x f(t)\,\mathrm{d}t$ is a primitive of f. By integrating (1) on $[0, 1]$ with respect to y, it follows that

$$f(x) + \int_0^1 f(y)\,\mathrm{d}y + f(0) = \frac{F(ax+b) - F(ax)}{b} + \frac{F(bx+c) - F(bx)}{c} + \frac{F(cx+a) - F(cx)}{a} \tag{2}$$

for all $x, y \in \mathbb{R}$. Since F is differentiable, it follows from (2) that f is also differentiable, hence F is twice differentiable. By repeating the argument (using (2)), we easily obtain that f is infinitely many times differentiable.

Next, we differentiate in (P) three times with respect to x to obtain

$$f'''(x) = a^3 f'''(ax + by + cz) + b^3 f'''(bx + cy + az) + c^3 f'''(cx + ay + bz),$$

then let $y = z = x$, hence

$$f'''(x) = (a^3 + b^3 + c^3) f'''(x)$$

for all $x \in \mathbb{R}$. Because $a^3 + b^3 + c^3 \neq 1$, it follows that $f'''(x) = 0$, so any *Palić function* is of the following type:

$$f(x) = px^2 + qx + r \quad (p, q, r \in \mathbb{R}). \tag{3}$$

Replacing the expression of f in (P) it follows that

$$\begin{aligned}
&f(ax + by + cz) + f(bx + cy + az) + f(cx + ay + bz) \\
&\quad = p\underbrace{(a^2 + b^2 + c^2)}_{1}(x^2 + y^2 + z^2) + 2p\underbrace{(ab + bc + ca)}_{0}(xy + yz + xz) + q\underbrace{(a + b + c)}_{1}(x + y + z) + 3r \\
&\quad = p(x^2 + y^2 + z^2) + q(x + y + z) + 3r = f(x) + f(y) + f(z)
\end{aligned}$$

for all $x, y, z \in \mathbb{R}$, so any function f of the form (3) is a *Palić function*.

Problem 3.

Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $A \in \mathcal{M}_n(\mathbb{C})$, $A \neq O_n$, be such that

$$A^2 + (A^*)^2 = \alpha A \cdot A^*,$$

where $A^* = \left(\overline{A}\right)^T$. Prove that $\alpha \in \mathbb{R}$, $|\alpha| \leqslant 2$, and $A \cdot A^* = A^* \cdot A$.

Solution: Let $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$. Applying the trace operator in the given identity, it follows that

$$\sum_{i,j=1}^{n} a_{ij} \cdot a_{ji} + \sum_{i,j=1}^{n} \overline{a_{ji}} \cdot \overline{a_{ij}} = \alpha \cdot \sum_{i,j=1}^{n} a_{ij} \cdot \overline{a_{ij}},$$

hence

$$2 \operatorname{Re} \sum_{i,j=1}^{n} a_{ij} \cdot a_{ji} = \alpha \underbrace{\sum_{i,j=1}^{n} |a_{ij}|^2}_{\in (0,\infty)}, \tag{1}$$

which leads to $\alpha \in \mathbb{R}$.
Since

$$|\operatorname{Re} xy| \leq |x| \cdot |y| \leq \frac{|x|^2 + |y|^2}{2} \quad \text{for all } x, y \in \mathbb{C},$$

using (1) it follows that

$$|\alpha| \cdot \sum_{i,j=1}^{n} |a_{ij}|^2 = 2 \cdot \left| \operatorname{Re} \sum_{i,j=1}^{n} a_{ij} \cdot a_{ji} \right| \leq \sum_{i,j=1}^{n} |a_{ij}|^2 + \sum_{i,j=1}^{n} |a_{ji}|^2 = 2 \underbrace{\sum_{i,j=1}^{n} |a_{ij}|^2}_{>0},$$

hence $|\alpha| \leqslant 2$.
Let $\varepsilon_1, \varepsilon_2$ be the solutions of $z^2 - \alpha z + 1 = 0$, hence $\varepsilon_1 + \varepsilon_2 = \alpha$ and $\varepsilon_1 \varepsilon_2 = 1$. Let $X = A - \varepsilon_1 A^*$ and $Y = A - \varepsilon_2 A^*$. Then

$$XY = A^2 + \underbrace{\varepsilon_1 \varepsilon_2}_{=1} (A^*)^2 - \varepsilon_1 A^* A - \varepsilon_2 A A^* = \alpha A A^* - \varepsilon_1 A^* A - \varepsilon_2 A A^* = \varepsilon_1 (A A^* - A^* A)$$

and, similarly,

$$YX = \varepsilon_2 (AA^* - A^* A).$$

Then $XY = \dfrac{\varepsilon_1}{\varepsilon_2} YX = \varepsilon_1^2 YX$, so $(XY)^2 = \varepsilon_1^4 (YX)^2$. Since $\operatorname{Tr}\left((XY)^2\right) = \operatorname{Tr}\left((YX)^2\right)$, it follows that

$$(\varepsilon_1^4 - 1) \operatorname{Tr}\left((XY)^2\right) = 0,$$

so we distinguish the following cases:

- $\varepsilon_1 \in \{-i, i\}$. Then $\alpha = 0$, which is a contradiction.
- $\varepsilon_1 \in \{-1, 1\}$. Then $\alpha \in \{-2, 2\}$, and the equality from the hypothesis becomes $(A \pm A^*)^2 = \pm (A^* A - A A^*)$. The equality of the traces gives $\operatorname{Tr}\left((A \pm A^*)^2\right) = 0$, which leads to $A \pm A^* = O_n$, and the conclusion follows.
- $\operatorname{Tr}\left((XY)^2\right) = 0$. Then $\operatorname{Tr}\left((AA^* - A^* A)^2\right) = 0$, which leads to $AA^* - A^* A = O_n$.

Problem 4.

Let $\mathcal{F}$ be the family of all nonempty finite subsets of $\mathbb{N}\cup\{0\}$. Find all positive real numbers a for which the series

$$\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} a^k}$$

is convergent.

Solution: Let $a = 2$. Any positive integer n can be uniquely represented in base 2:

$$n = 2^{k_1} + \cdots + 2^{k_s}$$

(here, $k_1, \ldots, k_s$ are distinct positive integers). Hence, there is a well-defined map $\varphi : \mathbb{N} \to \mathcal{F}$, given by

$$\varphi(n) = \{k_1, \ldots, k_s\}.$$

Clearly $\varphi(n) = \varphi(m)$ leads to $n = m$, i.e. φ is injective. Moreover

$$\varphi\left(\sum_{k\in A} 2^k\right) = A,$$

hence φ is surjective, and finally bijective. Similarly to the last, according to the definition of φ we observe

$$\sum_{k\in\varphi(n)} 2^k = n.$$

Hence, we can rewrite the series as follows

$$\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} 2^k} = \sum_{n=1}^{\infty} \frac{1}{\sum\limits_{k\in\varphi(n)} 2^k} = \sum_{n=1}^{\infty} \frac{1}{n}$$

(the series has only positive terms, so we can rearrange; also, we used that φ is bijective), which is the harmonic series, and hence divergent. Therefore, the series are divergent for all $a \leqslant 2$.

Now let $a > 2$. For any $n \geqslant 0$, let $\mathcal{F}_n$ be the subfamily of sets from $\mathcal{F}$ whose greatest element is n. Clearly, there are 2^n sets in $\mathcal{F}_n$. Observe that for every $A \in \mathcal{F}_n$ holds $\sum\limits_{k\in A} a^k \geqslant a^n$. Thus

$$\sum_{A\in\mathcal{F}_n} \frac{1}{\sum\limits_{k\in A} a^k} \leqslant \sum_{A\in\mathcal{F}_n} \frac{1}{a^n} \leqslant \frac{2^n}{a^n}.$$

Thus, for the initial series we obtain

$$\sum_{A\in\mathcal{F}} \frac{1}{\sum\limits_{k\in A} a^k} = \sum_{n=0}^{\infty} \sum_{A\in\mathcal{F}_n} \frac{1}{\sum\limits_{k\in A} a^k} \leqslant \sum_{n=0}^{\infty} \left(\frac{2}{a}\right)^n.$$

Since $a > 2$, the series is dominated by a convergent geometric series, hence it is convergent.