EXPLICIT FORM OF THE SOLUTIONS OF PELL'S EQUATION B THE DIFFERENCE EQUATIONS METHOD

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Abstract. We give one interesting method for finding explicit form of solutions of Pell's equation. This method is the method of difference equations by using Hamilton-Cayley's theorem.
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1 Introduction

The Pell's equation is one of the most interesting equations in the class of Diophantine equations and for the most part all the important results regarding its set of solutions are known. Let us remind ourselves of some of them.

Definition 1. *Diophantine equation of the form*

$$x^2 - Dy^2 = 1, \tag{1}$$

where D is a given positive nonsquare integer, is called Pell's equation.

If $D < 0$, then it is obvious that the equation has a finite number in nonnegative integers. When D is a perfect square (i.e., $D = a^2$), then Pell's equation has the form $(x - ay)(x + ay) = 1$ and has only the following solutions $(x, y) = (\pm 1, 0)$. These solutions are known as *trivial solutions*. On the other hand, if (x_0, y_0) is a solution of Pell's equation, then its solutions are $(-x_0, y_0)$, $(x_0, -y_0)$ and $(-x_0, -y_0)$, too. Therefore, in the future, we will find solutions of the Pell's equation only in positive integers. The *fundamental solution* of Pell's equation is the least solution of Pell's equation in positive integers. If D is not perfect square, then we have the next result (Andreescu, 2010).

Theorem 2. *Let (x_1, y_1) be a fundamental solution of Pell's equation (1). Then all solutions in positive integers of equation (1) are of the form*

$$x_n + y_n\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^n,\ n = 1,2,\ldots. \tag{2}$$

Especially

$$\left.\begin{aligned} x_n &= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n}{2k} x_1^{n-2k} y_1^{2k} D^k, \\ y_n &= \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \binom{n}{2k+1} x_1^{n-2k-1} y_1^{2k+1} D^k. \end{aligned}\right\} \tag{3}$$

Remark 3. *As we have seen, if* (x_n, y_n) *is a solution of equation (1), then* $(x_n, -y_n)$ *is also a solution of that equation. So,*

$$x_n - y_n\sqrt{D} = \left(x_1 - y_1\sqrt{D}\right)^n,\ n = 1,2,\dots . \qquad (4)$$

From (2) and (4) we obtain:

$$\left.\begin{aligned} x_n &= \frac{1}{2}\left(\left(x_1 + y_1\sqrt{D}\right)^n + \left(x_1 - y_1\sqrt{D}\right)^n\right) \\ y_n &= \frac{1}{2\sqrt{D}}\left(\left(x_1 + y_1\sqrt{D}\right)^n - \left(x_1 - y_1\sqrt{D}\right)^n\right) \end{aligned}\right\}, \qquad (5)$$

which are also the exact formulas for all solutions of equation (1), similar to the formulas (3).

It is well known that recursive relations exist for solutions of equation (1). Namely, the following result holds (Andreescu, 2010).

Theorem 4. *If* D *is positive integer that is not perfect square, then equation (1) has infinitely many solutions in positive integers, and general solution* (x_n, y_n), $n \geq 0$ *is given by*

$$\left.\begin{aligned} x_{n+1} &= x_1 x_n + D y_1 y_n \\ y_{n+1} &= x_1 y_n + y_1 x_n \end{aligned}\right\}, \qquad (6)$$

where $(x_0, y_0) = (1,0)$ *and* (x_1, y_1) *is fundamental solution of (1).*

In this paper, our goal is to show how from recursive formulas (6) we can yield explicit formulas (5) with one different approach. We will do this by method of difference equations (Elaydi, 2005), (Kelley & Peterson, 2001), (Mickens, 1990), (Kulenović & Merino, 2002), (Nurkanović, 2008), (Nurkanović & Nurkanović, 2016), (Spiegel, 1971) in the next section. But before that let us briefly describe how we come to a fundamental solution (x_1, y_1), without which we cannot have the formulas (5).

It is well known that all very good rational approximations of a real number can be obtained from its development into a continued fraction. The main method of determining the fundamental solution to Pell's equation (1) involves continued fractions (Andreescu, 2010), (Elaydi, 2005), (Kulenović & Merino, 2002). It is obtained by writing $\sqrt{D}$ as a simple continued fraction:

$$\sqrt{D} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where $a_0 = \lfloor\sqrt{D}\rfloor$ and $a_1, a_2, \dots$ is a periodic sequence of positive integers defined by

$$\sqrt{D} = a_0 + \frac{1}{\alpha_1}, a_1 = \lfloor\alpha_1\rfloor, \alpha_1 = a_1 + \frac{1}{\alpha_2}, \dots.$$

The continued fraction will be denoted by $[a_0; a_1, a_2, \dots]$, and the kth convergent of $[a_0; a_1, a_2, \dots]$ is the number

$$\frac{p_k}{q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}} = [a_0; a_1, \dots, a_k]$$

with p_k and q_k relatively prime.

Let us now state the algorithm for determining the numbers $a_0; a_1, \dots, a_l$, where l is period for $\sqrt{D}$:

$$a_i = \lfloor \frac{s_i+\alpha}{t_i} \rfloor, s_{i+1} = a_i t_i - s_i, t_{i+1} = \frac{D - s_{i+1}^2}{t_i}, \quad i \geq 0$$

and algorithm stops when the pair (s_k, t_k) repeats.

Theorem 5. *Suppose that l is period of $\sqrt{D}$. Then the least fundamental solution of Pell's equation is*

$$(x_1, y_1) = \begin{cases} (p_{l-1}, q_{l-1}) & \text{if } l \text{ is even} \\ (p_{2l-1}, q_{2l-1}) & \text{if } l \text{ is odd} \end{cases}$$

(Andrescu, 2010).

2 Explicit form of the solutions of Pell's equation

1) Note that recurrent relations can be considered as difference equations. So, system (6) is a homogeneous system with two first order linear difference equations and we can write it in the form

$$X_{n+1} = AX_n, (n = 0,1,2,3 \dots,) \tag{7}$$

where $X_{n+1} = \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}$, $A = \begin{bmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{bmatrix}$ and $X_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is initial condition. In this case, the solution of system (7) is

$$X_n = A^n X_0, (n = 1,2,3 \dots). \tag{8}$$

From (8) we can see that the only problem to find a solution of system (7) is to determine the nth degree of matrix A. There are several possible ways to calculate the matrix A^n. Except the induction, we can do this in one of the following ways (Elaydi, 2005), (Kelley & Peterson, 2001), (Nurkanović, 2008), (Nurkanović & Nurkanović, 2016):

- by using Hamilton-Cayley's theorem,
- by using binomial formula,
- by using the so-called Putzer's algorithm.

In this paper, we will use Hamilton-Cayley's theorem.

Theorem 6, *(Hamilton-Cayley)* *Every square matrix A satisfied its own characteristic equation, i.e.,*

$$\kappa(A) = \mathbf{0},$$

where $\mathbf{0}$ is zero matrix.

The characteristic polynomial $\kappa(\lambda)$ of the matrix A is

$$\kappa(\lambda) = det(A - \lambda I) = \begin{vmatrix} x_1 - \lambda & Dy_1 \\ y_1 & x_1 - \lambda \end{vmatrix} = \lambda^2 - 2x_1\lambda + x_1^2 - Dy_1^2 = \lambda^2 - 2x_1\lambda + 1.$$

By using Hamilton-Cayley's theorem we have

$$A^2 - 2x_1 A + I = \mathbf{0} \text{ and } A^{n+2} - 2x_1 A^{n+1} + A^n = \mathbf{0},$$

which is a second order difference equation with constant coefficients. Since the eigenvalues of the matrix A are $\lambda_{1,2} = x_1 \pm y_1\sqrt{D}$, then $A^n = C_1\lambda_1^n + C_2\lambda_2^n$,i.e.,

$$A^n = C_1(x_1 + y_1\sqrt{D})^n + C_2(x_1 - y_1\sqrt{D})^n,$$

where C_1 and C_2 constant matrices that we will determine by using the initial condition.
For $n = 0$:

$$A^0 = C_1 + C_2 = I,$$

and for $n = 1$:

$$A = C_1(x_1 + y_1\sqrt{D}) + C_2(x_1 - y_1\sqrt{D}),$$

from which is

$$C_1 = \frac{1}{2\sqrt{D}}\begin{bmatrix}\sqrt{D} & D\\ 1 & \sqrt{D}\end{bmatrix}, C_2 = \frac{1}{2\sqrt{D}}\begin{bmatrix}\sqrt{D} & -D\\ -1 & \sqrt{D}\end{bmatrix}.$$

Now,

$$A^n = \frac{1}{2\sqrt{D}}\begin{bmatrix}\sqrt{D}\left((x_1 + y_1\sqrt{D})^n + (x_1 - y_1\sqrt{D})^n\right) & D\left((x_1 + y_1\sqrt{D})^n - (x_1 - y_1\sqrt{D})^n\right)\\ (x_1 + y_1\sqrt{D})^n - (x_1 - y_1\sqrt{D})^n & \sqrt{D}\left((x_1 + y_1\sqrt{D})^n + (x_1 - y_1\sqrt{D})^n\right)\end{bmatrix}.$$

Finally, we have

$$X_n = A^n X_0 = \begin{bmatrix}\frac{1}{2}\left((x_1 + y_1\sqrt{D})^n + (x_1 - y_1\sqrt{D})^n\right)\\ \frac{1}{2\sqrt{D}}\left((x_1 + y_1\sqrt{D})^n - (x_1 - y_1\sqrt{D})^n\right)\end{bmatrix}, (n = 1,2,3,\dots). \quad (9)$$

We can always use the above procedure to solve specific Pell's equations, since it is sufficient to remember the general form of the matrix A.

Example 7. *Find all solutions of the equation*

$$x^2 - 15y^2 = 1 \quad (10)$$

in positive integers by using Hamilton-Cayley's theorem.

Solution: For equation (10), by using (6), we have the following system of difference equations

$$\left.\begin{aligned} x_{n+1} &= x_1 x_n + 15 y_1 y_n\\ y_{n+1} &= x_1 y_n + y_1 x_n \end{aligned}\right\}, \quad (11)$$

where $(x_0, y_0) = (1,0)$and(x_1, y_1) is a fundamental solution of (10). As it is first necessary to find a fundamental solution (x_1, y_1), for this purpose we will write $\sqrt{15}$ as a continuous fraction. For $D = 15$ we have that $\alpha = \sqrt{15}$ and let $s_0 = 0$ and $t_0 = 1$. Then, by using corresponding algorithm we have: $a_0 = 3$, $s_1 = 3$, $t_1 = 6$, $a_1 = 1$, $s_2 = 1$, $t_2 = 1$, $a_2 = 6$, $s_3 = 3$and $t_3 = 6$. Since $(s_3, t_3) = (s_1, t_1) = (3,6)$ we get

$$\sqrt{15} = [a_0; \overline{a_1, a_2}] = [3; \overline{1,6}],$$

and we see that the period $l = 2$ (i.e., it is even). By Theorem 5 the fundamental solution is

$$(x_1, y_1) = (p_1, q_1) = (a_0 a_1 + 1, a_1) = (4,1).$$

For the system (10) matrix A is of the form

$$A = \begin{bmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{bmatrix} = \begin{bmatrix} 4 & 15 \\ 1 & 4 \end{bmatrix}$$

with the characteristic polynomial

$$\kappa(\lambda) = det(A - \lambda I) = \begin{bmatrix} 4-\lambda & 15 \\ 1 & 4-\lambda \end{bmatrix} = \lambda^2 - 8\lambda + 1.$$

By Hamilton-Cayley's theorem we have

$$A^2 - 8A + I = 0,$$

and

$$A^{n+2} - 8A^{n+1} + A^n = 0, \tag{12}$$

which is a second order homogeneous difference equation with constant coefficients. The eigenvalues of A are $\lambda_{1,2} = 4 \pm \sqrt{15}$, so that the general solution of equation (12) is

$$A^n = C_1(4+\sqrt{15})^n + C_2(4-\sqrt{15})^n. \tag{13}$$

In doing so, C_1 and C_2 are constant matrices that we will determine from the initial conditions For $n = 0$:

$$A^0 = I = C_1 + C_2,$$

and for $n = 1$:

$$A = C_1(4+\sqrt{15}) + C_2(4-\sqrt{15}),$$

from which

$$C_1 = \frac{1}{2\sqrt{15}}\begin{bmatrix} \sqrt{15} & 15 \\ 1 & \sqrt{15} \end{bmatrix}, C_2 = \frac{1}{2\sqrt{15}}\begin{bmatrix} \sqrt{15} & -15 \\ -1 & \sqrt{15} \end{bmatrix}.$$

By substituting in (13) we obtain

$$A^n = \frac{1}{2\sqrt{15}}\begin{bmatrix} \sqrt{15}\left((4+\sqrt{15})^n + (4-\sqrt{15})^n\right) & -15\left((4+\sqrt{15})^n - (4-\sqrt{15})^n\right) \\ (4+\sqrt{15})^n - (4-\sqrt{15})^n & \sqrt{15}\left((4+\sqrt{15})^n + (4-\sqrt{15})^n\right) \end{bmatrix}.$$

Now, the solutions of equation (10) is

$$X_n = \frac{1}{2\sqrt{15}}\begin{bmatrix} \sqrt{15}\left((4+\sqrt{15})^n + (4-\sqrt{15})^n\right) & -15\left((4+\sqrt{15})^n - (4-\sqrt{15})^n\right) \\ (4+\sqrt{15})^n - (4-\sqrt{15})^n & \sqrt{15}\left((4+\sqrt{15})^n + (4-\sqrt{15})^n\right) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

i.e.,

$$X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left((4+\sqrt{15})^n + (4-\sqrt{15})^n\right) \\ \frac{1}{2\sqrt{15}}\left((4+\sqrt{15})^n - (4-\sqrt{15})^n\right) \end{bmatrix} (n = 0,1,2,\dots).$$

Remark 8. *System of difference equations (6) can be reduced to a second-order equation in* x_n *or in* y_n*. In this way, the general solution* (x_n, y_n) *of Pell's equation (1) can be expressed exactly* *only through* x_1 *or/and* y_1.

Let us show how the procedure mentioned in Remark 8 is performed.

From the first equation of System (6) by iterating, and then by using the second equation, we obtain

$$x_{n+2} = x_1 x_{n+1} + D x_1 y_1 y_n + D y_1^2 x_n. \qquad (14)$$

Also from the first equation of System (6) we have $D y_1 y_n = x_{n+1} - x_1 x_n$, and by substituting into (14), we get

$$x_{n+2} - 2x_1 x_{n+1} + (x_1^2 - D y_1^2) x_n = 0.$$

Since $x_1^2 - D y_1^2 = 1$, we finally get the second order difference equation in x_n:

$$x_{n+2} - 2x_1 x_{n+1} + x_n = 0. \qquad (15)$$

The corresponding characteristic equation is of the form $\lambda^2 - 2x_1\lambda + 1 = 0$ with the roots$\lambda_{1,2} = x_1 \pm \sqrt{x_1^2 - 1}$, so that the general solution of the difference equation (15) is

$$x_n = C_1\left(x_1 - \sqrt{x_1^2 - 1}\right)^n + C_2\left(x_1 + \sqrt{x_1^2 - 1}\right)^n.$$

By using initial values $x_0 = 1$ and x_1, we can determine the constants C_1 and C_2. Namely,

$$n = 0 \Longrightarrow x_0 = 1 = C_1 + C_2,$$
$$n = 1 \Longrightarrow x_1 = C_1\left(x_1 - \sqrt{x_1^2 - 1}\right) + C_2\left(x_1 + \sqrt{x_1^2 - 1}\right),$$

from which is $C_1 = C_2 = \frac{1}{2}$, and so on

$$x_n = \frac{1}{2}\left[\left(x_1 - \sqrt{x_1^2 - 1}\right)^n + \left(x_1 + \sqrt{x_1^2 - 1}\right)^n\right]. \qquad (16)$$

By an analogous procedure, the following second-order difference equation in y_n is obtained from System (6):

$$y_{n+2} - 2x_1 y_{n+1} + y_n = 0,$$

whose the general solution is

$$y_n = C_1\left(x_1 - \sqrt{x_1^2 - 1}\right)^n + C_2\left(x_1 + \sqrt{x_1^2 - 1}\right)^n.$$

By using the initial values $y_0 = 0$ and y_1, we can determine the constants C_1 and C_2. Namely,

$$n = 0 \Longrightarrow y_0 = 0 = C_1 + C_2,$$
$$n = 1 \Longrightarrow y_1 = C_1\left(x_1 - \sqrt{x_1^2 - 1}\right) + C_2\left(x_1 + \sqrt{x_1^2 - 1}\right),$$

from which we obtain $C_1 = -C_2 = -\frac{y_1}{2\sqrt{x_1^2 - 1}}$, and so on

$$y_n = \frac{y_1}{2\sqrt{x_1^2 - 1}}\left[\left(x_1 + \sqrt{x_1^2 - 1}\right)^n - \left(x_1 + \sqrt{x_1^2 - 1}\right)^n\right]. \qquad (17)$$

Remark 9. *Of course, if we use the equality* $x_1^2 - Dy_1^2 = 1$, *from (9) immediately the formulas (16) and (17) follow.*

2) Now, consider the equation of the form

$$ax^2 - by^2 = 1, \tag{18}$$

where a and b are natural numbers. We list the following two theorems without the proofs (Andreescu, 2010).

Theorem 10. *If* $ab = k^2$, *where* k *is a natural number greater than* 1, *then the equation* $ax^2 - by^2 = 1$ *has no solution in* $\mathbb{N}$.

The equation of the form

$$u^2 - abv^2 = 1 \tag{19}$$

is called *Pell's resolvent*.

The following theorem is a well known result (Andreescu, 2010).

Theorem 11. *Suppose that equation (18) has the solutions in set* $\mathbb{N}$ *and that* (x_0, y_0) *is the fundamental solution of (18). The general solution* $(x_n, y_n), n \geq 0$ *of equation (18) is given by the following iterations*

$$\left.\begin{aligned} x_n &= x_0 u_n + b y_0 v_n \\ y_n &= x_0 u_n + a y_0 v_n \end{aligned}\right\}, \tag{20}$$

where $(u_n, v_n), n \geq 0$ *is a solution of Pell's resolvent (19).*

Now, our goal is to find an explicit form for general solution of equation (18) by using the fundamental solution (u_1, v_1) of (19). By using Theorem 4 the solutions of (19) are given by the following recursive formulas

$$\left.\begin{aligned} u_{n+1} &= u_1 u_n + ab v_1 v_n \\ v_{n+1} &= v_1 u_n + u_1 v_n \end{aligned}\right\}, \tag{21}$$

where $(u_0, v_0) = (1,0)$ and (u_1, v_1) is the fundamental solution of (19). These recursive formulas (21) can be written in the following matrix form

$$U_{n+1} = AU_n (n = 0,1,2, \ldots), \tag{22}$$

where $U_{n+1} = \begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix}$, $A = \begin{bmatrix} u_1 & abv_1 \\ v_1 & u_1 \end{bmatrix}$ and $U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the initial condition. System of difference equations (22) is an initial value problem whose solution is given by

$$U_n = A^n U_0, (n = 1,2,3 \ldots).$$

By using Hamilton-Cayley's theorem we obtain

$$A^n = \frac{1}{2\sqrt{ab}} \begin{bmatrix} \sqrt{ab}\left((u_1 + v_1\sqrt{ab})^n + (u_1 - v_1\sqrt{ab})^n\right) & ab\left((u_1 + v_1\sqrt{ab})^n - (u_1 - v_1\sqrt{ab})^n\right) \\ (u_1 + v_1\sqrt{ab})^n - (u_1 - v_1\sqrt{ab})^n & \sqrt{ab}\left((u_1 + v_1\sqrt{ab})^n + (u_1 - v_1\sqrt{ab})^n\right) \end{bmatrix},$$

which implies that

$$U_n = \begin{bmatrix} \frac{1}{2}\left((u_1 + v_1\sqrt{ab})^n + (u_1 - v_1\sqrt{ab})^n\right) \\ \frac{1}{2\sqrt{ab}}\left((u_1 + v_1\sqrt{ab})^n - (u_1 - v_1\sqrt{ab})^n\right) \end{bmatrix}.$$

Now, since we can write system (20) in the following matrix form

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_0 & by_0 \\ x_0 & ay_0 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix},$$

we have that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{a}}{2a}\left((\sqrt{a}x_0 + \sqrt{b}y_0)(u_1 + v_1\sqrt{ab})^n + (\sqrt{a}x_0 - \sqrt{b}y_0)(u_1 - v_1\sqrt{ab})^n\right) \\ \frac{\sqrt{b}}{2b}\left((\sqrt{b}x_0 + \sqrt{a}y_0)(u_1 + v_1\sqrt{ab})^n + (\sqrt{b}x_0 + \sqrt{a}y_0)(u_1 - v_1\sqrt{ab})^n\right) \end{bmatrix}, \quad (23)$$

which is the general solution of equation (18).

The formulas (23) are complicated and in specific problems it is better to use the procedure described above.

Example 12. *Find all solutions of the equation*

$$6x^2 - 5y^2 = 1 \quad (24)$$

in the set of natural numbers.

Solution: The minimal solution in the set of natural numbers of equation (24) is $(x_0, y_0) = (1,1)$. Since $a = 6$ and $b = 5$ the corresponding Pell's resolvent is of the form

$$u^2 - 30v^2 = 1. \quad (25)$$

Equation (25) corresponds to the system of difference equations

$$\left.\begin{aligned} u_{n+1} &= u_1 u_n + 30 v_1 v_n \\ v_{n+1} &= v_1 u_n + u_1 v_n \end{aligned}\right\},$$

where $(u_0, v_0) = (1,0)$and$(u_1, v_1) = (11,2)$ is its fundamental solution. Now, we have

$$U_n = A^n U_0 (n = 0,1,2,\dots), \quad (26)$$

where $U_n = \begin{bmatrix} u_n \\ v_n \end{bmatrix}$, $A = \begin{bmatrix} 11 & 60 \\ 2 & 11 \end{bmatrix}$ and $U_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is initial value condition. The characteristic polynomial $\kappa(\lambda)$ of matrix A is

$$\kappa(\lambda) = det(A - \lambda I) = \begin{bmatrix} 11 - \lambda & 60 \\ 2 & 11 - \lambda \end{bmatrix} = \lambda^2 - 22\lambda + 1.$$

By using Hamilton-Cayley's theorem we obtain

$$A^2 - 22A + 1 = 0,$$

and

$$A^{n+2} - 22A^{n+1} + 1 = 0, \quad n = 0,1,\ldots, \tag{27}$$

from which

$$A^n = C_1\lambda_1^n + C_2\lambda_2^n = C_1(11 + 2\sqrt{30})^n + C_2(11 - 2\sqrt{30})^n,$$

where $\lambda_{1,2} = 11 \pm 2\sqrt{30}$ are the eigenvalues of the matrix A and C_1 and C_2 are constant matrices that we will determine using the initial conditions.
For $n = 0$:

$$A^0 = I = C_1 + C_2,$$

and for $n = 1$:

$$A = C_1(11 + 2\sqrt{30}) + C_2(11 - 2\sqrt{30}),$$

from which is

$$C_1 = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{30} \\ \frac{\sqrt{30}}{30} & 1 \end{bmatrix}, C_2 = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{30} \\ -\frac{\sqrt{30}}{30} & 1 \end{bmatrix}.$$

So,

$$A^n = \frac{1}{2\sqrt{30}}\begin{bmatrix} \sqrt{30}\left((11 + 2\sqrt{30})^n + (11 - 2\sqrt{30})^n\right) & 30\left((11 + 2\sqrt{30})^n - (11 - 2\sqrt{30})^n\right) \\ (11 + 2\sqrt{30})^n - (11 - 2\sqrt{30})^n & \sqrt{30}\left((11 + 2\sqrt{30})^n + (11 - 2\sqrt{30})^n\right) \end{bmatrix}.$$

By using (26) we obtain the solution of Pell's resolvent (27) in the following matrix form

$$U_n = \begin{bmatrix} \frac{1}{2}\left((11 + 2\sqrt{30})^n + (11 - 2\sqrt{30})^n\right) \\ \frac{1}{2\sqrt{30}}\left((11 + 2\sqrt{30})^n - (11 - 2\sqrt{30})^n\right) \end{bmatrix}.$$

By Theorem 11 we have that

$$\left.\begin{aligned} x_n &= u_n + 5v_n \\ y_n &= u_n + 6v_n \end{aligned}\right\},$$

that is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \frac{6+\sqrt{30}}{12}\left((11 + 2\sqrt{30})^n + \frac{6-\sqrt{30}}{12}(11 - 2\sqrt{30})^n\right) \\ \frac{5+\sqrt{30}}{10}\left((11 + 2\sqrt{30})^n + \frac{5-\sqrt{30}}{10}(11 - 2\sqrt{30})^n\right) \end{bmatrix}.$$

3)The negative Pell's equation

Now, consider so-called the negative Pell's equation (Andreescu (2010)):

$$x^2 - Dy^2 = -1. \tag{28}$$

Theorem 13. *Assume that equation (28) has the solutions in* $\mathbb{N}$ *and that* (A, B) *is the minimal solution of (28). Then the general solution* (x_n, y_n), $n \geq 0$ *of (28) has the following form*

$$\left.\begin{aligned} x_n &= Bu_n + DAv_n \\ y_n &= Au_n + Bv_n \end{aligned}\right\},$$

where $(u_n, v_n),\ n \geq 0$ *are solutions of Pell's resolvent* $u^2 - Dv^2 = 1$.

Theorem 14. *Let* p *be a prime number. Equation (28) has the solutions if and only if* $p = 2$ *or* $p \equiv 1(mod4)$.

Now, we will demonstrate the method of difference equation in the following example.

Example 15. *Find all the pairs* (k, m) $(k, m \in \mathbb{N})$ *such that* $k < m$ *and that*

$$1 + 2 + 3 + \cdots + k = (k + 1) + (k + 2) + \cdots + m. \tag{29}$$

Solution: Equation (29) is equivalent with

$$2(1 + 2 + 3 + \ldots + k) = 1 + 2 + .. + m,$$

i.e.,

$$(2m + 1)^2 - 2(2k + 1)^2 = -1.$$

If we introduce the substitutions $x = 2m + 1$ and $y = 2k + 1$, we obtain the following negative Pell's equation

$$x^2 - 2y^2 = -1, \tag{30}$$

whose the minimal solution is $(A, B) = (1,1)$. The solutions of equation (30) are of the form

$$\left.\begin{aligned} x_n &= u_n + 2v_n \\ y_n &= u_n + v_n \end{aligned}\right\},$$

where (u_n, v_n) are the solutions of Pell's resolvent $u^2 - 2v^2 = 1$. As we know, the solutions of this resolvent are given by the following system of difference equations

$$\left.\begin{aligned} u_{n+1} &= u_1 u_n + 2v_1 v_n \\ v_{n+1} &= v_1 u_n + u_1 v_n \end{aligned}\right\}$$

where $(u_0, v_0) = (1,0)$and (u_1, v_1)=(3,2) is its fundamental solution.
Now, we have that $U_n = A^n U_0 (n = 0,1,2, \ldots)$,where $U_n = \begin{bmatrix} u_n \\ v_n \end{bmatrix}$, $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $U_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the initial condition. The characteristic polynomial $\kappa(\lambda)$ of the matrix A is

$$\kappa(\lambda) = det(A - \lambda I) = \begin{bmatrix} 3 - \lambda & 4 \\ 2 & 3 - \lambda \end{bmatrix} = \lambda^2 - 6\lambda + 1.$$

By Hamilton-Cayley's theorem we obtain

$$A^2 - 6A + I = 0,$$

and

$$A^{n+2} - 6A^{n+1} + I = 0, \quad n = 0,1, \ldots,$$

from which

$$A^n = C_1 \lambda_1^n + C_2 \lambda_2^n = C_1 (3 + 2\sqrt{2})^n + C_2 (3 - 2\sqrt{2})^n,$$

where $\lambda_{1,2} = 3 \pm 2\sqrt{2}$ are the eigenvalues of the matrix A and C_1 and C_2 are constant matrices that we will determine using the initial conditions.
For $n = 0$:

$$A^0 = I = C_1 + C_2,$$

and for $n = 1$

$$A = C_1(3 + 2\sqrt{2}) + C_2(3 - 2\sqrt{2}),$$

which implies that

$$C_1 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} & \frac{1}{2} \end{bmatrix}, C_2 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{4} & \frac{1}{2} \end{bmatrix}.$$

Then, we have

$$A^n = \begin{bmatrix} \frac{1}{2}\left((3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n\right) & \frac{\sqrt{2}}{2}\left((3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n\right) \\ \frac{\sqrt{2}}{4}\left((3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n\right) & \frac{1}{2}\left((3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n\right) \end{bmatrix},$$

so it is

$$U_n = \begin{bmatrix} \frac{1}{2}\left((3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n\right) & \frac{\sqrt{2}}{2}\left((3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n\right) \\ \frac{\sqrt{2}}{4}\left((3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n\right) & \frac{1}{2}\left((3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n\right) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left((3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n\right) \\ \frac{\sqrt{2}}{4}\left((3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n\right) \end{bmatrix}.$$

Since

$$\left.\begin{aligned} x_n &= u_n + 2v_n \\ y_n &= u_n + v_n \end{aligned}\right\},$$

finally, we obtain

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{2}}{2}(3 + 2\sqrt{2})^n + \frac{1-\sqrt{2}}{2}(3 - 2\sqrt{2})^n \\ \frac{2+\sqrt{2}}{4}(3 + 2\sqrt{2})^n + \frac{2-\sqrt{2}}{4}(3 - 2\sqrt{2})^n \end{bmatrix}.$$

By using the substitution $x = 2m + 1$ and $y = 2k + 1$, we get the required solution in the following form

$$\begin{bmatrix} m_n \\ k_n \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{2}}{4}(3 + 2\sqrt{2})^n + \frac{1-\sqrt{2}}{4}(3 - 2\sqrt{2})^n - \frac{1}{2} \\ \frac{2+\sqrt{2}}{8}(3 + 2\sqrt{2})^n + \frac{2-\sqrt{2}}{8}(3 - 2\sqrt{2})^n - \frac{1}{2} \end{bmatrix}.$$

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