Mathematical Society
of South Eastern Europe

Technical University
"Gheorghe Asachi" of Iasi

Romanian
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South Eastern European Mathematical Olympiad for University Students

Iaşi, Romania - April 11, 2024

Problem 1. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1 \in (0,1)$ and $x_{n+1} = x_n - \dfrac{x_n^2}{\sqrt{n}}$ for all $n \geq 1$.

Find the values of $\alpha \in \mathbb{R}$ for which the series $\displaystyle\sum_{n=1}^{\infty} x_n^{\alpha}$ is convergent.

Problem 2. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ two real, symmetric matrices with nonnegative eigenvalues. Prove that $A^3 + B^3 = (A+B)^3$ if and only if $AB = \mathcal{O}_n$.

Problem 3. For every $n \geq 1$ define x_n by

$$x_n = \int_0^1 \ln(1 + x + x^2 + \ldots + x^n) \cdot \ln \frac{1}{1-x}\, \mathrm{d}x, \quad n \geq 1.$$

a) Show that x_n is finite for every $n \geq 1$ and $\lim\limits_{n\to\infty} x_n = 2$.

b) Calculate $\lim\limits_{n\to\infty} \dfrac{n}{\ln n}(2 - x_n)$.

Problem 4. Let $n \in \mathbb{N}$, $n \geq 2$. Find all the values $k \in \mathbb{N}$, $k \geq 1$, for which the following statement holds:

"If $A \in \mathcal{M}_n(\mathbb{C})$ is such that $A^k A^* = A$, then $A = A^*$."

(here, $A^* = \overline{A}^t$ denotes the transpose conjugate of A).

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Solutions and marking scheme

Problem 1. Let $(x_n)_{n\geq 1}$ be the sequence defined by $x_1 \in (0,1)$ and $x_{n+1} = x_n - \dfrac{x_n^2}{\sqrt{n}}$ for all $n \geq 1$. Find the values of $\alpha \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} x_n^\alpha$ is convergent.

Solution:

By induction we deduce that $x_n \in (0,1)$ for all $n \geq 1$. Let $n \geq 1$. From $x_n - x_{n+1} = \dfrac{x_n^2}{\sqrt{n}}$ for all $n \geq 1$ we deduce that $1 - \dfrac{x_{n+1}}{x_n} = \dfrac{x_n}{\sqrt{n}}$ and since $0 < \dfrac{x_n}{\sqrt{n}} < \dfrac{1}{\sqrt{n}}$, $\forall n \geq 1$ we deduce that $\lim\limits_{n\to\infty} \dfrac{x_n}{\sqrt{n}} = 0$ and hence $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = 1$. Now let $n \geq 1$. By the recurence relation we have $\dfrac{1}{x_{n+1}} - \dfrac{1}{x_n} = \dfrac{x_n - x_{n+1}}{x_n x_{n+1}} = \dfrac{x_n}{x_{n+1}} \cdot \dfrac{1}{\sqrt{n}}$ which implies that

$$\lim_{n\to\infty} \frac{\frac{1}{x_{n+1}} - \frac{1}{x_n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = 1.$$

Since $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n-1}}\right) = \infty$ by the Stolz-Cesaro lemma it follows that

$$\lim_{n\to\infty} \frac{\frac{1}{x_n}}{1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}}} = 1.$$

Now if we use that, again by the Stolz-Cesaro lemma

$$\lim_{n\to\infty} \frac{\sqrt{n}}{1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}}} = \lim_{n\to\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{2}$$

we get $\lim\limits_{n\to\infty} \dfrac{\frac{1}{x_n}}{\sqrt{n}} = 2$ and hence $\lim\limits_{n\to\infty} \dfrac{x_n^\alpha}{\frac{1}{n^{\frac{\alpha}{2}}}} = 2^{-\alpha}$. By the comparison criterion for the positive series it follows that the series $\sum\limits_{n=1}^{\infty} x_n^\alpha$ is convergent if and only if the $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{\alpha}{2}}}$ is convergent that is if and only if $\frac{\alpha}{2} > 1$, $\alpha > 2$.

Marking scheme:

I) Proving that $(x_n)_{n\geq 1}$ is convergent. **1p**

II) Proving that $x_n \to 0$. **2p**

III) Proving that there exists a constant $c_1 > 0$ such that $x_n \leq \frac{c_1}{\sqrt{n}}$. **3p**

IV) Proving that there exists a constant $c_2 > 0$ such that $x_n \geq \frac{c_2}{\sqrt{n}}$. **3p**

V) Conclusion. **1p**

First remark: Using the Stolz-Cesaro lemma to prove that $x_n \sim \frac{1}{\sqrt{n}}$ generates **6p**, since it replaces parts III and IV from the previous mentioned mark scheme.
Second remark: Using the Stolz-Cesaro lemma without arguing that the denominator is increasing and unbounded generates only **5p**.
Third remark: Claiming that $x_n \sim \frac{1}{\sqrt{n}}$ without a proof will only generate **1p**, which is **not** additive with V.

Problem 2. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ two real, symmetric matrices with nonnegative eigenvalues. Prove that $A^3 + B^3 = (A+B)^3$ if and only if $AB = \mathcal{O}_n$.

Solution (Author): If $AB = \mathcal{O}_n$ then

$$AB = \mathcal{O}_n = (AB)^T = B^T A^T = BA$$

therefore A and B commute and

$$(A+B)^3 = A^3 + B^3 + 3AB(A+B) = A^3 + B^3.$$

Assume now that $A^3 + B^3 = (A+B)^3$. Since the trace operator is linear and invariant under cyclic permutations it follows that

$$\operatorname{Tr}(ABA) + \operatorname{Tr}(BAB) = 0. \tag{1}$$

We recall that a real, symmetric matrix M has nonnegative eigenvalues $\lambda_1, ..., \lambda_n$ i.e. M is positive semidefinite if and only if M can be decomposed as a product $M = Q^T Q$ for some real matrix Q. Moreover, if for such a matrix $\operatorname{Tr} M = 0$ then $M = \mathcal{O}_n$. Let $U, V \in \mathcal{M}_n(\mathbb{R})$ such that $A = U^T U$ and $B = V^T V$. Then, using the symmetry of A and B we get

$$ABA = AV^T VA = (VA)^T (VA) \qquad BAB = BU^T UB = (UB)^T (UB)$$

so $\operatorname{Tr}(ABA) \geq 0$ and $\operatorname{Tr}(BAB) \geq 0$. From (1) it follows that we must have $\operatorname{Tr}(ABA) = \operatorname{Tr}(BAB) = 0$ and therefore $ABA = BAB = \mathcal{O}_n$.

In particular, for every $x \in \mathbb{R}^n$ we have

$$\|VAx\|^2 = x^T (VA)^T (VA) x = x^T ABAx = 0$$

so $VA = \mathcal{O}_n$. Again, for every $x \in \mathbb{R}^n$

$$\|ABx\|^2 = x^T (AB)^T (AB) x = x^T V^T (VA) ABx = 0$$

and, finally, we find $AB = \mathcal{O}_n$.

Alternative solution (2). For every $x \in \mathbb{R}^n$ we have, on account of B being positive semidefinite $< Bx, x > \geq 0$ and equality holds only for $x \in \ker B$. But then $(ABA)^T = ABA$ and

$$< ABAx, x > = < BAx, Ax > \geq 0$$

so ABA is positive semidefinite and $\operatorname{Tr}(ABA) \geq 0$. In the same manner we get BAB as positive semidefinite and $\operatorname{Tr}(ABA) \geq 0$ which leads to $\operatorname{Tr}(ABA) = \operatorname{Tr}(BAB) = 0$ and, next, to $ABA = BAB = \mathcal{O}_n$. Finally, for every $x \in \mathbb{R}^n$ we have

$$0 = < BABx, x > = < ABx, Bx >$$

which implies $Bx \in \ker A$, $\forall x \in \mathbb{R}^n$, which concludes the proof.

Marking scheme:

I) $AB = O_n \Rightarrow (A+B)^3 = A^3 + B^3$ **2p**

II) $\operatorname{Tr}(ABA) + \operatorname{Tr}(BAB) = 0$ **2p**

III) $ABA = BAB = O_n$ **4p**

IV) Conclusion $AB = O_n$. **2p**

Problem 3. For every $n \geq 1$ define x_n by

$$x_n = \int_0^1 \ln(1 + x + x^2 + \ldots + x^n) \cdot \ln \frac{1}{1-x} \, \mathrm{d}x, \quad n \geq 1.$$

a) Show that x_n is finite for every $n \geq 1$ and $\lim\limits_{n \to \infty} x_n = 2$.

b) Calculate $\lim\limits_{n \to \infty} \dfrac{n}{\ln n}(2 - x_n)$.

Solution. a) For all $n \geq 1$ and $x \in [0, 1)$,

$$\frac{1}{1-x} \geq 1 \quad \text{and} \quad 0 \leq \ln(1 + x + x^2 + \ldots + x^n) \cdot \ln \frac{1}{1-x} \leq \ln n \cdot \ln \frac{1}{1-x}.$$

Since $\displaystyle\int_0^1 \ln \frac{1}{1-x} \, \mathrm{d}x$ is convergent (to 1, by a direct computation), it follows that the sequence is well-defined.

Next, the sequence of functions $f_n(x) = \ln(1 + x + x^2 + \ldots + x^n) \cdot \ln \dfrac{1}{1-x}$ satisfies:

$$0 \leq f_n(x) \leq f_{n+1}(x), \quad \text{for all } x \in [0,1) \text{ and } n \geq 1,$$

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \left(\ln \frac{1 - x^{n+1}}{1-x} \cdot \ln \frac{1}{1-x} \right) = \ln^2 \frac{1}{1-x}, \quad \text{for all } x \in [0,1).$$

It follows by *the Lebesgue-Beppo-Levi theorem* (of *monotone convergence*) that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \int_0^1 f_n(x) \, \mathrm{d}x = \int_0^1 \ln^2 \frac{1}{1-x} \, \mathrm{d}x = 2$$

(the last equality follows by an elementary computation).

b) From (a),

$$2 - x_n = \int_0^1 \left(\ln^2 \frac{1}{1-x} - \ln \frac{1 - x^{n+1}}{1-x} \cdot \ln \frac{1}{1-x} \right) \mathrm{d}x = \int_0^1 \ln(1 - x^{n+1}) \cdot \ln(1-x) \, \mathrm{d}x$$

and with the change of variable $y = x^{n+1}$, it follows that

$$2 - x_n = \frac{1}{n+1} \int_0^1 \ln(1-y) \cdot \ln\left(1 - y^{\frac{1}{n+1}}\right) \cdot y^{\frac{1}{n+1}-1}\, \mathrm{d}y.$$

By shifting the index, for convenience, it follows that

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{\ln n}(2 - x_n) &= \lim_{n\to\infty} \frac{n-1}{\ln(n-1)}(2 - x_{n-1}) \\
&= \lim_{n\to\infty} \frac{n-1}{n} \cdot \lim_{n\to\infty} \frac{\ln n}{\ln(n-1)} \cdot \lim_{n\to\infty} \frac{1}{\ln n} \int_0^1 \ln(1-y) \cdot \ln\left(1 - y^{\frac{1}{n}}\right) \cdot y^{\frac{1}{n}-1}\, \mathrm{d}y \\
&= \lim_{n\to\infty} \int_0^1 \frac{\ln(1-y)}{y} \cdot \frac{y^{\frac{1}{n}} \ln(1 - y^{\frac{1}{n}})}{\ln n}\, \mathrm{d}y.
\end{aligned}$$

We want to verify the conditions in the *Lebesgue dominated convergence theorem*, so consider

$$g_n(y) = \frac{\ln(1-y)}{y} \cdot \frac{y^{\frac{1}{n}} \ln(1 - y^{\frac{1}{n}})}{\ln n}, \text{ for } y \in (0,1), \text{ and } n \geq 2.$$

The pointwise convergence follows in a standard manner: we start from

$$\lim_{n\to\infty} \frac{y^{\frac{1}{n}} - 1}{\frac{1}{n}} = \ln y, \quad \text{hence} \quad \lim_{n\to\infty} n\left(1 - y^{\frac{1}{n}}\right) = \ln\frac{1}{y} > 0,$$

which leads to

$$\lim_{n\to\infty} \left(\ln\left(1 - y^{\frac{1}{n}}\right) + \ln n\right) = \ln\left(\ln\frac{1}{y}\right).$$

Then

$$\begin{aligned}
\lim_{n\to\infty} g_n(y) &= \frac{\ln(1-y)}{y} \cdot \lim_{n\to\infty} y^{\frac{1}{n}} \cdot \lim_{n\to\infty} \frac{\ln\left(1 - y^{\frac{1}{n}}\right)}{\ln n} \\
&= \frac{\ln(1-y)}{y} \cdot \lim_{n\to\infty} \left(\frac{\ln\left(1 - y^{\frac{1}{n}}\right) + \ln n}{\ln n} - 1\right) \\
&= \frac{\ln(1-y)}{y} \left(\ln\left(\ln\frac{1}{y}\right) \cdot \frac{1}{\infty} - 1\right) = -\frac{\ln(1-y)}{y}, \quad \text{for all } y \in (0,1).
\end{aligned}$$

To check *the domination condition*, let $g(t) = -\ln(1-t) = \ln\dfrac{1}{1-t}$, for $t \in [0,1)$. Note that g is positive. Since $0 \leq y^{\frac{1}{n}} \leq 1$, it follows that

$$0 \leq g_n(y) \leq \frac{\ln(1-y)}{y} \cdot \frac{\ln\left(1 - y^{\frac{1}{n}}\right)}{\ln n} = \frac{g(y)}{y} \cdot \frac{g\left(y^{\frac{1}{n}}\right)}{\ln n}, \quad \text{for all } n \geq 2 \text{ and } y \in (0,1). \tag{1}$$

From

$$g(t) - g(t^n) = \ln\frac{1-t^n}{1-t} = \ln(1 + t + \ldots + t^{n-1}) \leq \ln n, \quad \text{for all } t \in (0,1) \text{ and } n \geq 1,$$

it follows that $g\left(y^{\frac{1}{n}}\right) - g(y) \leq \ln n$, hence

$$\frac{g\left(y^{\frac{1}{n}}\right)}{\ln n} \leq 1 + \frac{g(y)}{\ln n} \leq 1 + g(y), \quad \text{for all } n \geq 3. \tag{2}$$

Combining (1) and (2) and replacing g, we finally obtain

$$0 \leq g_n(y) \leq \frac{\ln^2(1-y) - \ln(1-y)}{y}, \quad \text{for all } n \geq 3 \text{ and } y \in (0,1).$$

It is an elementary exercise to check that $\int_0^1 \frac{\ln^2(1-y) - \ln(1-y)}{y}\, \mathrm{d}y$ is convergent, which concludes the proof of the domination condition and establishes that

$$L = \lim_{n\to\infty} \frac{n}{\ln n}(2 - x_n) = -\int_0^1 \frac{\ln(1-y)}{y}\, \mathrm{d}y = \frac{\pi^2}{6},$$

where the last equality is a well know result, that can be obtained by integrating the Maclaurin series of $-\frac{\ln(1-y)}{y}$ and then using Euler's identity $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Marking scheme:

a)

• The convergence of the integral defining x_n .... **1** p

• Apply a convergence theorem (e.g., *Beppo–Levi monotone convergence*) for the sequence of functions

$$f_n(x) = \ln(1 + x + x^2 + \ldots + x^n) \cdot \ln \frac{1}{1-x}, \quad x \in [0, 1) \text{ and } n \geq 1$$

to obtain that $\lim_{n\to\infty} x_n = \int_0^1 \lim_{n\to\infty} f_n \, \mathrm{d}x = \int_0^1 \ln^2 \frac{1}{1-x}\, \mathrm{d}x$ .... **1** p

• Compute $\int_0^1 \ln^2 \frac{1}{1-x}\, \mathrm{d}x = 2$ .... **1** p

b)

• Obtain $2 - x_n = \int_0^1 \ln(1 - x^{n+1}) \cdot \ln(1-x)\, \mathrm{d}x$ .... **1** p

• Use the change of variable $y = x^{n+1}$, and rewrite

$$\frac{n}{\ln n}(2 - x_n) = \frac{n}{n+1} \cdot \frac{\ln(n+1)}{\ln n} \int_0^1 \frac{\ln(1-y)}{y} \cdot \frac{y^{\frac{1}{n+1}} \ln(1 - y^{\frac{1}{n+1}})}{\ln(n+1)}\, \mathrm{d}y$$ .... **1** p

• For the sequence of functions

$$g_n(y) = \frac{\ln(1-y)}{y} \cdot \frac{y^{\frac{1}{n}} \ln(1 - y^{\frac{1}{n}})}{\ln n}, \quad \text{for } y \in (0,1) \text{ and } n \geq 2$$

compute $\lim_{n\to\infty} g_n(y) = -\frac{\ln(1-y)}{y}$, for all $y \in (0, 1)$ .... **2** p

• Apply a convergence theorem to obtain that

$$\lim_{n\to\infty} \frac{n}{\ln n}(2 - x_n) = \int_0^1 \lim_{n\to\infty} g_n(y)\, \mathrm{d}y = -\int_0^1 \frac{\ln(1-y)}{y}\, \mathrm{d}y$$ .... **2** p*

(***0.5** p for choosing a convergence theorem and stating the conditions that need to be verified, without completing the corresponding computations)

E.g., use the *Lebesgue dominated convergence* with the domination

$$0 \leq g_n(y) \leq \frac{\ln^2(1-y) - \ln(1-y)}{y}, \quad \text{for all } n \geq 3 \text{ and } y \in (0, 1)$$

and check that $\int_0^1 \frac{\ln^2(1-y) - \ln(1-y)}{y}\, \mathrm{d}y$ is convergent.

• Concluding, $\lim_{n\to\infty} \frac{n}{\ln n}(2 - x_n) = -\int_0^1 \frac{\ln(1-y)}{y}\, \mathrm{d}y = \frac{\pi^2}{6}$ .... **1** p*

(***0.5** p for the value of the integral, without proof)

Problem 4. Let $n \in \mathbb{N}$, $n \geq 2$. Find all the values $k \in \mathbb{N}$, $k \geq 1$, for which the following statement holds:

$$\text{“If } A \in \mathcal{M}_n(\mathbb{C}) \text{ is such that } A^k A^* = A, \text{ then } A = A^*.\text{”} \tag{*}$$

(here, $A^* = \overline{A}^t$ denotes the transpose conjugate of A).

Solution (Author). First, we limit the range of the possible values for k, by choosing $A = \varepsilon I_n$, with suitable $\varepsilon \in \mathbb{C}$, $|\varepsilon| = 1$, such that the implication in (*) is false, so we ask that $A^k A^* = A$, but $A \neq A^*$. Then $\varepsilon I_n = A = A^k A^* = \varepsilon^k \overline{\varepsilon} I_n = \varepsilon^{k-1} I_n$ and $\varepsilon I_n = A \neq A^* = \overline{\varepsilon} I_n$, which are equivalent to $\varepsilon^{k-2} = 1$ and $\varepsilon \notin \mathbb{R}$. In consequence,

- if $k = 2$, then let $\varepsilon = i$.
- if $k \geq 5$, then let $\varepsilon = \cos \dfrac{2\pi}{k-2} + i \sin \dfrac{2\pi}{k-2} \notin \mathbb{R}$ (since $\dfrac{2\pi}{k-2} \in (0, \pi)$).

This means that $k \in \{1, 3, 4\}$. We prove next that the statement (*) is true for these values of k.
For $k = 1$, if $A \cdot A^* = A$, then $A^* = (A \cdot A^*)^* = (A^*)^* \cdot A^* = A \cdot A^* = A$, so (*) is true.
For $k \in \{3, 4\}$, we provide two methods.

First method.
$A^k A^* = A$ implies that $\operatorname{rank} A = \operatorname{rank}\left(A^k A^*\right) \leq \operatorname{rank} A^k \leq \operatorname{rank} A$, so $\operatorname{rank} A^k = \operatorname{rank} A = \operatorname{rank} A^*$. By the rank–nullity theorem, it follows that $\dim\ker A^k = \dim\ker A = \dim\ker A^*$. Since $\operatorname{Ker} A^* \subseteq \operatorname{Ker} A$ (by $A^k A^* = A$) and $\operatorname{Ker} A \subseteq \operatorname{Ker} A^k$, we obtain

$$\operatorname{Ker} A^* = \operatorname{Ker} A^k = \operatorname{Ker} A. \tag{1}$$

Next, $A^k A^* A^{k-1} = A \cdot A^{k-1} = A^k$, so $A^k \left(A^* A^{k-1} - I_n\right) = O_n$, then $A^* \left(A^* A^{k-1} - I_n\right) = O_n$, by (1), hence

$$\left(A^*\right)^2 A^{k-1} = A^*. \tag{2}$$

For $k = 3$, (2) becomes $\left(A^*\right)^2 A^2 = A^*$, so $A = \left(\left(A^*\right)^2 A^2\right)^* = \left(A^*\right)^2 A^2 = A^*$, which means that the statement (*) is true.

For $k = 4$, (2) becomes $\left(A^*\right)^2 A^3 = A^*$, so $\left(A^*\right)^2 A^4 A^* = \left(A^*\right)^2 A^3 \cdot AA^* = A^* AA^*$. At the same time, $\left(A^*\right)^2 A^4 A^* = \left(A^*\right)^2 A$, so $\left(A^*\right)^2 A = A^* AA^*$, which leads to $\left(A^*\right)^2 A^2 = \left(A^* A\right)^2$. With $B = A^* A - AA^*$, we have $B^* = B$ and

$$\operatorname{Tr} BB^* = \operatorname{Tr} B^2 = \operatorname{Tr}\left(A^* A - AA^*\right)^2 = 2\left(\operatorname{Tr}\left(A^* A\right)^2 - \operatorname{Tr}\left(\left(A^*\right)^2 A^2\right)\right) = 0,$$

hence $B = O_n$. This proves that $A^* A = AA^*$ (i.e., A is normal), so A is unitarily diagonalizable, $A = U^* D U$, $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, $U \in \mathcal{M}_n(\mathbb{C})$ with $U^{-1} = U^*$.Then $A^* = U^* \overline{D} U$, and $A^4 A^* = A$ becomes $D^4 \overline{D} = D$, which means that $\lambda_i^4 \overline{\lambda_i} = \lambda_i$, for all $i = 1, 2, \ldots, n$. It follows that $\lambda_i \in \{-1, 0, 1\}$, for all $i = 1, 2, \ldots, n$, so $\overline{D} = D$, therefore $A^* = A$, which means that the statement (*) is true.

Second method. We continue from relation (1) (from the first method).
It is true in general, for any matrix $A \in \mathcal{M}_n(\mathbb{C})$, that $\operatorname{Ker} A^* \perp \operatorname{Im} A$ [indeed, if $Y \in \operatorname{Ker} A^*$ and $Z = AX \in \operatorname{Im} A$, then $\langle Z, Y \rangle = \langle AX, Y \rangle = \langle X, A^* Y \rangle = \langle X, O \rangle = 0$].

Next, by (1), it follows that $\operatorname{Ker} A \perp \operatorname{Im} A$, so $\mathbb{C}^n = \operatorname{Ker} A \oplus \operatorname{Im} A$.

Consider an orthonormal basis in $\operatorname{Ker} A$ and an orthonormal basis in $\operatorname{Im} A$, which together give an orthonormal basis in $\mathbb{C}^n$ such that $A = U^* A_1 U$, where $A_1 = \begin{bmatrix} B & O \\ O & O \end{bmatrix}$ with $B \in \mathcal{M}_m(\mathbb{C})$ invertible, and $U \in \mathcal{M}_n(\mathbb{C})$ with $U^{-1} = U^*$. Then the relation $A^k A^* = A$ becomes $B^k B^* = B$, hence $B^* = (B^{-1})^{k-1}$. From the Cayley-Hamilton theorem, it follows that $B^{-1} = f(B)$ for some polynomial f of degree at most $n-1$, so $B^* = (f(B))^{k-1}$, which leads to $B^* B = B B^*$ (B is normal). Just like in the previous approach, B is unitarily diagonalizable, $B = V^* D V$, $\quad D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\lambda_1, \lambda_2, \ldots, \lambda_m \neq 0$, $\quad V \in \mathcal{M}_m(\mathbb{C})$ with $V^{-1} = V^*$.Then $B^* = V^* \overline{D} V$, and the relation $B^k B^* = B$ becomes $D^k \overline{D} = D$, which leads to $\lambda_i^{k-1} \overline{\lambda_i} = 1$, for all i. It follows that $|\lambda_i| = 1$ and $\lambda_i^{k-2} = 1$, for all i. When $k = 3$ or $k = 4$, then $\lambda_i \in \{-1, 1\}$ for all i, so $\overline{D} = D$, therefore $B^* = B$, then $A^* = A$, which means that the statement (*) is true.

Conclusion: $k \in \{1, 3, 4\}$.

Marking scheme:

1. Solve case $k = 1$ 1p
2. Eliminate $k = 2$ and $k \geq 5$ 3p
3. Find the relation $\operatorname{Ker} A^* = \operatorname{Ker} A^k = \operatorname{Ker} A$ 2p

Now, we solve cases $k \in \{3, 4\}$ with two methods.

First method

1. Find the relation $(A^*)^2 A^{k-1} = A^*$. 1p
2. Solve case $k = 3$ 1p
3. Solve case $k = 4$ 2p

Second method

1. $\mathbb{C}^n = \operatorname{Ker} A \oplus \operatorname{Im} A$ 1p
2. Consider an orthonormal basis in $\operatorname{Ker} A$ and an orthonormal basis in $\operatorname{Im} A$ using these basis to write $A = U^* A_1 U$, where $A_1 = \begin{bmatrix} B & O \\ O & O \end{bmatrix}$ with $B \in \mathcal{M}_m(\mathbb{C})$ invertible
   .... 1p
3. Find B is normal and therefore $B^* = B$ from which the conclusion folows for $k \in \{3, 4\}$
   .... 2p

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