ИМО2021

Задача 1. Нека $n \geqslant 100$ е цел број. Иван ги запишал броевите $n, n+1, \ldots, 2n$ секој на различна карта. Потоа ги измешал овие $n+1$ карти и ги поделил во две купчиња. Докажи дека барем во едно од овие две купчиња постојат две карти такви што збирот на нивните броеви е полн квадрат.

Задача 2. Докажи дека неравенството

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

важи за секои реални броеви $x_1, \ldots, x_n$.

Задача 3. Нека D е внатрешна точка во остроаголен триагонлик ABC за кој $AB > AC$, така што $\angle DAB = \angle CAD$. За точката E на отсечката AC важи $\angle ADE = \angle BCD$, за точката F на отсечката AB важи $\angle FDA = \angle DBC$ и за точката X на правата AC важи $CX = BX$. Нека O_1 и O_2 се центрите на опишаните кружници на триаголниците ADC и EXD, соодветно. Докажи дека правите BC, EF и O_1O_2 се сечат во една точка.

Задача 4. Нека Γ е кружница со центар I и $ABCD$ е конвексен четириаголник така што секоја од отсечките AB, BC, CD и DA е тангента на Γ. Нека Ω е опишаната кружница околу триаголникот AIC. Продолжението на BA преку A ја сече Ω во точката X и продолжението на BC преку C ја сече Ω во точката Z. Продолженијата на AD и CD преку D ја сечат Ω во точките Y и T, соодветно. Докажи дека

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Задача 5. Две верверички, Бушавко и Скокалко, собрале 2021 ореви за зимата. Скокалко ги нумерирал ореввите од 1 до 2021 и ископал 2021 дупчиња наредени во кружна форма околу неговото омилено дрво. Следното утро Скокалко забележал дека Бушавко ставил по еден орев во секое дупче, но не внимавал на нумерирањето. Незадоволен од тоа, Скокалко одлучил да ги пререди оревите со помош на низа од 2021 чекори така што во k-тиот чекор, Скокалко ги заменува позициите на двата ореви соседни до оревот нумериран со k. Докажи дека постои k така што во k-тиот чекор, Скокалко заменува некои ореви нумерирани со a и b за кои $a < k < b$.

Задача 6. Нека $m \geqslant 2$ е цел број, A е конечно множество од цели броеви (кои не мора да бидат позитивни) и $B_1, B_2, B_3, \ldots, B_m$ се подмножества од A. Да претпоставиме дека за секој $k = 1, 2, \ldots, m$ збирот на елементите од B_k е m^k. Докажи дека A содржи барем $m/2$ елементи.

1 Let $n \geqslant 100$ be an integer. The numbers $n, n+1, \ldots, 2n$ are written on $n+1$ cards, one number per card. The cards are shuffled and divided into two piles. Prove that one of the piles contains two cards such that the sum of their numbers is a perfect square.

Solution. To solve the problem it suffices to find three squares and three cards with numbers a, b, c on them such that pairwise sums $a+b, b+c, a+c$ are equal to the chosen squares. By choosing the three consecutive squares $(2k-1)^2$, $(2k)^2$, $(2k+1)^2$ we arrive at the triple

$$(a, b, c) = \left(2k^2 - 4k, \quad 2k^2 + 1, \quad 2k^2 + 4k\right).$$

We need a value for k such that

$$n \leqslant 2k^2 - 4k, \quad \text{and} \quad 2k^2 + 4k \leqslant 2n.$$

A concrete k is suitable for all n with

$$n \in \left[k^2 + 2k, 2k^2 - 4k + 1\right] =: I_k.$$

For $k \geqslant 9$ the intervals I_k and I_{k+1} overlap because

$$(k+1)^2 + 2(k+1) \leqslant 2k^2 - 4k + 1.$$

Hence $I_9 \cup I_{10} \cup \ldots = [99, \infty)$, which proves the statement for $n \geqslant 99$.

Comment 1. There exist approaches which only work for sufficiently large n.

One possible approach is to consider three cards with numbers $70k^2, 99k^2, 126k^2$ on them. Then their pairwise sums are perfect squares and so it suffices to find k such that $70k^2 \geqslant n$ and $126k^2 \leqslant 2n$ which exists for sufficiently large n.

Another approach is to prove, arguing by contradiction, that a and $a-2$ are in the same pile provided that n is large enough and a is sufficiently close to n. For that purpose, note that every pair of neighbouring numbers in the sequence $a, x^2 - a, a + (2x+1), x^2 + 2x + 3 - a, a - 2$ adds up to a perfect square for any x; so by choosing $x = \lfloor\sqrt{2a}\rfloor + 1$ and assuming that n is large enough we conclude that a and $a-2$ are in the same pile for any $a \in [n+2, 3n/2]$. This gives a contradiction since it is easy to find two numbers from $[n+2, 3n/2]$ of the same parity which sum to a square.

It then remains to separately cover the cases of small n which appears to be quite technical.

Comment 2. An alternative formulation for this problem could ask for a proof of the statement for all $n > 10^6$. An advantage of this formulation is that some solutions, e.g. those mentioned in Comment 1 need not contain a technical part which deals with the cases of small n. However, the original formulation seems to be better because the bound it gives for n is almost sharp, see the next comment for details.

Comment 3. The statement of the problem is false for $n = 98$. As a counterexample, the first pile may contain the even numbers from 98 to 126, the odd numbers from 129 to 161, and the even numbers from 162 to 196.

2. Show that for all real numbers $x_1, \dots, x_n$ the following inequality holds:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}.$$

Solution 1. If we add t to all the variables then the left-hand side remains constant and the right-hand side becomes

$$H(t) := \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j + 2t|}.$$

Let T be large enough such that both $H(-T)$ and $H(T)$ are larger than the value L of the left-hand side of the inequality we want to prove. Not necessarily distinct points $p_{i,j} := -(x_i + x_j)/2$ together with T and $-T$ split the real line into segments and two rays such that on each of these segments and rays the function $H(t)$ is concave since $f(t) := \sqrt{|\ell + 2t|}$ is concave on both intervals $(-\infty, -\ell/2]$ and $[-\ell/2, +\infty)$. Let $[a, b]$ be the segment containing zero. Then concavity implies $H(0) \geqslant \min\{H(a), H(b)\}$ and, since $H(\pm T) > L$, it suffices to prove the inequalities $H(-(x_i + x_j)/2) \geqslant L$, that is to prove the original inequality in the case when all numbers are shifted in such a way that two variables x_i and x_j add up to zero. In the following we denote the shifted variables still by x_i.

If $i = j$, i.e. $x_i = 0$ for some index i, then we can remove x_i which will decrease both sides by $2\sum_k \sqrt{|x_k|}$. Similarly, if $x_i + x_j = 0$ for distinct i and j we can remove both x_i and x_j which decreases both sides by

$$2\sqrt{2|x_i|} + 2 \cdot \sum_{k \neq i,j}\left(\sqrt{|x_k + x_i|} + \sqrt{|x_k + x_j|}\right).$$

In either case we reduced our inequality to the case of smaller n. It remains to note that for $n = 0$ and $n = 1$ the inequality is trivial.

Solution 2. For real p consider the integral

$$I(p) = \int_0^\infty \frac{1 - \cos(px)}{x\sqrt{x}}\,dx,$$

which clearly converges to a strictly positive number. By changing the variable $y = |p|x$ one notices that $I(p) = \sqrt{|p|}I(1)$. Hence, by using the trigonometric formula $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$ we obtain

$$\sqrt{|a+b|} - \sqrt{|a-b|} = \frac{1}{I(1)}\int_0^\infty \frac{\cos((a-b)x) - \cos((a+b)x)}{x\sqrt{x}}\,dx = \frac{1}{I(1)}\int_0^\infty \frac{2\sin(ax)\sin(bx)}{x\sqrt{x}}\,dx,$$

from which our inequality immediately follows:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|} - \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} = \frac{2}{I(1)}\int_0^\infty \frac{\left(\sum_{i=1}^{n}\sin(x_i x)\right)^2}{x\sqrt{x}}\,dx \geqslant 0.$$

Comment 1. A more general inequality

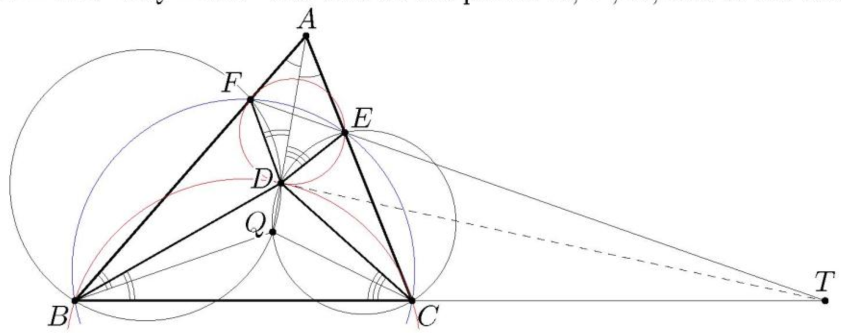
$$\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|^r \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}|x_i + x_j|^r$$

holds for any $r \in [0, 2]$. The first solution can be repeated verbatim for any $r \in [0, 1]$ but not for $r > 1$. In the second solution, by putting x^{r+1} in the denominator in place of $x\sqrt{x}$ we can prove the inequality for any $r \in (0, 2)$ and the cases $r = 0, 2$ are easy to check by hand.

Comment 2. In fact, the integral from Solution 2 can be computed explicitly, we have $I(1) = \sqrt{2\pi}$.

3 A point D is chosen inside an acute-angled triangle ABC with $AB > AC$ so that $\angle BAD = \angle DAC$. A point E is constructed on the segment AC so that $\angle ADE = \angle DCB$. Similarly, a point F is constructed on the segment AB so that $\angle ADF = \angle DBC$. A point X is chosen on the line AC so that $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and DXE. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Common remarks. Let Q be the isogonal conjugate of D with respect to the triangle ABC. Since $\angle BAD = \angle DAC$, the point Q lies on AD. Then $\angle QBA = \angle DBC = \angle FDA$, so the points Q, D, F, and B are concyclic. Analogously, the points Q, D, E, and C are concyclic. Thus $AF \cdot AB = AD \cdot AQ = AE \cdot AC$ and so the points B, F, E, and C are also concyclic.



Let T be the intersection of BC and FE.

Claim. $TD^2 = TB \cdot TC = TF \cdot TE$.

Proof. We will prove that the circles (DEF) and (BDC) are tangent to each other. Indeed, using the above arguments, we get

$$\angle BDF = \angle AFD - \angle ABD = (180^\circ - \angle FAD - \angle FDA) - (\angle ABC - \angle DBC)$$
$$= 180^\circ - \angle FAD - \angle ABC = 180^\circ - \angle DAE - \angle FEA = \angle FED + \angle ADE = \angle FED + \angle DCB,$$

which implies the desired tangency.

Since the points B, C, E, and F are concyclic, the powers of the point T with respect to the circles (BDC) and (EDF) are equal. So their radical axis, which coincides with the common tangent at D, passes through T, and hence $TD^2 = TE \cdot TF = TB \cdot TC$. $\square$

Solution 1. Let TA intersect the circle (ABC) again at M. Due to the circles $(BCEF)$ and $(AMCB)$, and using the above Claim, we get $TM \cdot TA = TF \cdot TE = TB \cdot TC = TD^2$; in particular, the points A, M, E, and F are concyclic.

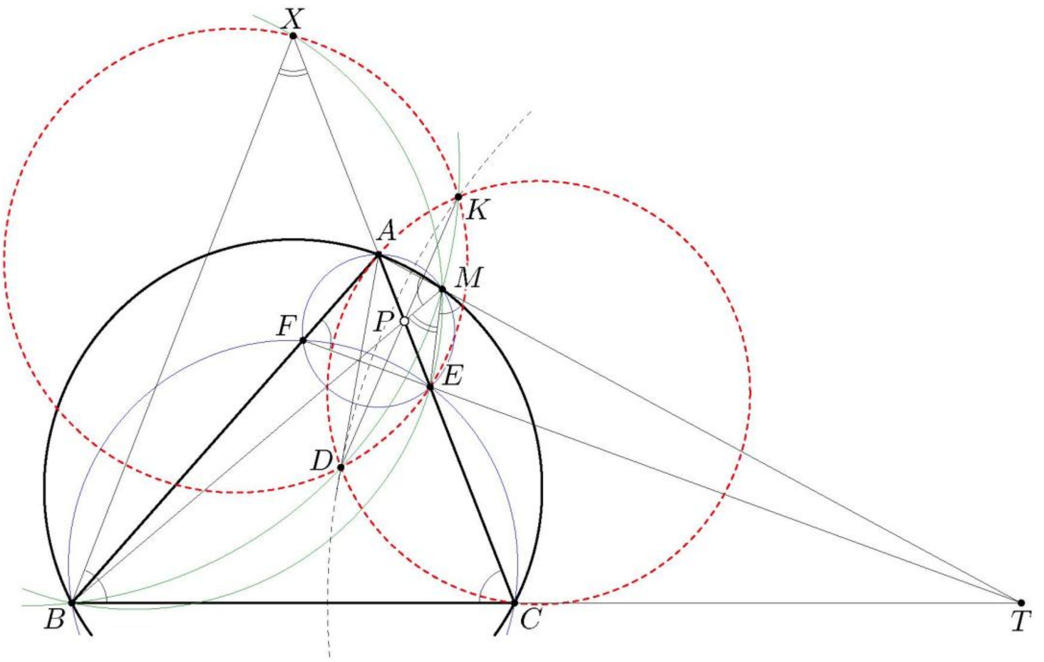
Under the inversion with centre T and radius TD, the point M maps to A, and B maps to C, which implies that the circle (MBD) maps to the circle (ADC). Their common point D lies on the circle of the inversion, so the second intersection point K also lies on that circle, which means $TK = TD$. It follows that the point T and the centres of the circles (KDE) and (ADC) lie on the perpendicular bisector of KD.

Since the center of (ADC) is O_1, it suffices to show now that the points D, K, E, and X are concyclic (the center of the corresponding circle will be O_2).

The lines BM, DK, and AC are the pairwise radical axes of the circles $(ABCM)$, $(ACDK)$ and $(BMDK)$, so they are concurrent at some point P. Also, M lies on the circle (AEF), thus

$$\measuredangle(EX, XB) = \measuredangle(CX, XB) = \measuredangle(XC, BC) + \measuredangle(BC, BX) = 2\measuredangle(AC, CB)$$
$$= \measuredangle(AC, CB) + \measuredangle(EF, FA) = \measuredangle(AM, BM) + \measuredangle(EM, MA) = \measuredangle(EM, BM),$$

so the points M, E, X, and B are concyclic. Therefore, $PE \cdot PX = PM \cdot PB = PK \cdot PD$, so the points E, K, D, and X are concyclic, as desired.



Comment 1. We present here a different solution which uses similar ideas.

Perform the inversion ι with centre T and radius TD. It swaps B with C and E with F; the point D maps to itself. Let $X' = \iota(X)$. Observe that the points E, F, X, and X' are concyclic, as well as the points B, C, X, and X'. Then

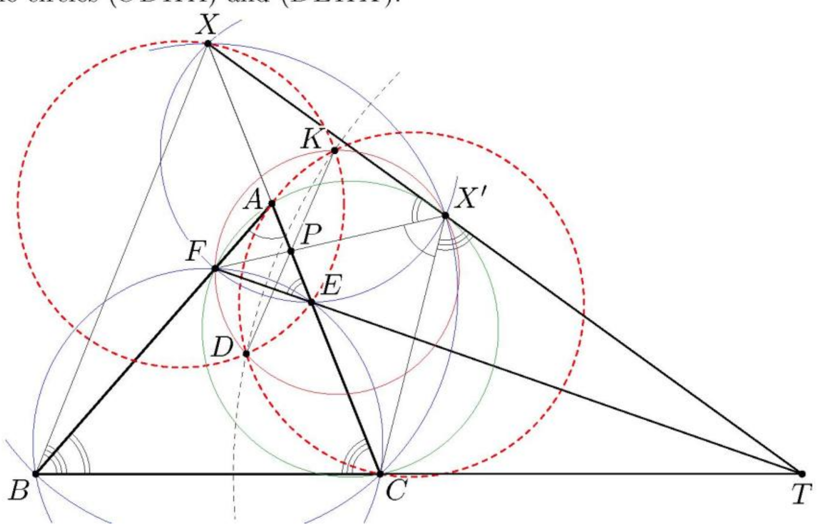
$$\begin{aligned}\measuredangle(CX', X'F) &= \measuredangle(CX', X'X) + \measuredangle(X'X, X'F) = \measuredangle(CB, BX) + \measuredangle(EX, EF) \\ &= \measuredangle(XC, CB) + \measuredangle(EC, EF) = \measuredangle(CA, CB) + \measuredangle(BC, BF) = \measuredangle(CA, AF),\end{aligned}$$

therefore the points C, X', A, and F are concyclic.

Let $X'F$ intersect AC at P, and let K be the second common point of DP and the circle (ACD). Then

$$PK \cdot PD = PA \cdot PC = PX' \cdot PF = PE \cdot PX;$$

hence, the points K, X, D, and E lie on some circle ω_1, while the points K, X', D, and F lie on some circle ω_2. (These circles are distinct since $\angle EXF + \angle EDF < \angle EAF + \angle DCB + \angle DBC < 180^\circ$). The inversion ι swaps ω_1 with ω_2 and fixes their common point D, so it fixes their second common point K. Thus $TD = TK$ and the perpendicular bisector of DK passes through T, as well as through the centres of the circles $(CDKA)$ and $(DEKX)$.



Solution 2. We use only the first part of the Common remarks, namely, the facts that the tuples (C, D, Q, E) and (B, C, E, F) are both concyclic. We also introduce the point $T = BC \cap EF$. Let the circle (CDE) meet BC again at E_1. Since $\angle E_1CQ = \angle DCE$, the arcs DE and QE_1 of the circle (CDQ) are equal, so $DQ \parallel EE_1$.

Since $BFEC$ is cyclic, the line AD forms equal angles with BC and EF, hence so does EE_1. Therefore, the triangle EE_1T is isosceles, $TE = TE_1$, and T lies on the common perpendicular bisector of EE_1 and DQ.

Let U and V be the centres of circles (ADE) and $(CDQE)$, respectively. Then UO_1 is the perpendicular bisector of AD. Moreover, the points U, V, and O_2 belong to the perpendicular bisector of DE. Since $UO_1 \parallel VT$, in order to show that O_1O_2 passes through T, it suffices to show that

$$\frac{O_2U}{O_2V} = \frac{O_1U}{TV}. \tag{1}$$

Denote angles A, B, and C of the triangle ABC by α, β, and γ, respectively. Projecting onto AC we obtain

$$\frac{O_2U}{O_2V} = \frac{(XE - AE)/2}{(XE + EC)/2} = \frac{AX}{CX} = \frac{AX}{BX} = \frac{\sin(\gamma - \beta)}{\sin \alpha} \tag{2}$$

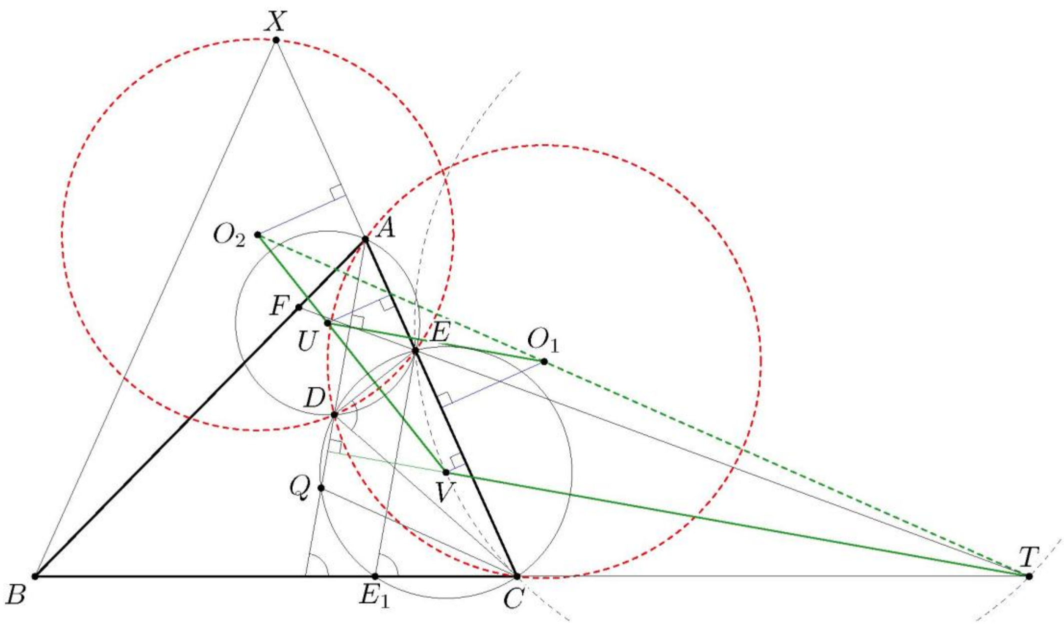
The projection of O_1U onto AC is $(AC - AE)/2 = CE/2$; the angle between O_1U and AC is $90^\circ - \alpha/2$, so

$$\frac{O_1U}{EC} = \frac{1}{2\sin(\alpha/2)} \tag{3}$$

Next, we claim that E, V, C, and T are concyclic. Indeed, the point V lies on the perpendicular bisector of CE, as well as on the internal angle bisector of $\angle CTF$. Therefore, V coincides with the midpoint of the arc CE of the circle (TCE).

Now we have $\angle EVC = 2\angle EE_1C = 180^\circ - (\gamma - \beta)$ and $\angle VET = \angle VE_1T = 90^\circ - \angle E_1EC = 90^\circ - \alpha/2$. Therefore,

$$\frac{EC}{TV} = \frac{\sin \angle ETC}{\sin \angle VET} = \frac{\sin(\gamma - \beta)}{\cos(\alpha/2)}. \tag{4}$$



Recalling (2) and multiplying (3) and (4) we establish (1):

$$\frac{O_2U}{O_2V} = \frac{\sin(\gamma-\beta)}{\sin\alpha} = \frac{1}{2\sin(\alpha/2)}\cdot\frac{\sin(\gamma-\beta)}{\cos(\alpha/2)} = \frac{O_1U}{EC}\cdot\frac{EC}{TV} = \frac{O_1U}{TV}.$$

Solution 3. Notice that $\angle AQE = \angle QCB$ and $\angle AQF = \angle QBC$; so, if we replace the point D with Q in the problem set up, the points E, F, and T remain the same. So, by the Claim, we have $TQ^2 = TB\cdot TC = TD^2$.

Thus, there exists a circle Γ centred at T and passing through D and Q. We denote the second meeting point of the circles Γ and (ADC) by K. Let the line AC meet the circle (DEK) again at Y; we intend to prove that $Y = X$. As in Solution 1, this will yield that the point T, as well as the centres O_1 and O_2, all lie on the perpendicular bisector of DK.

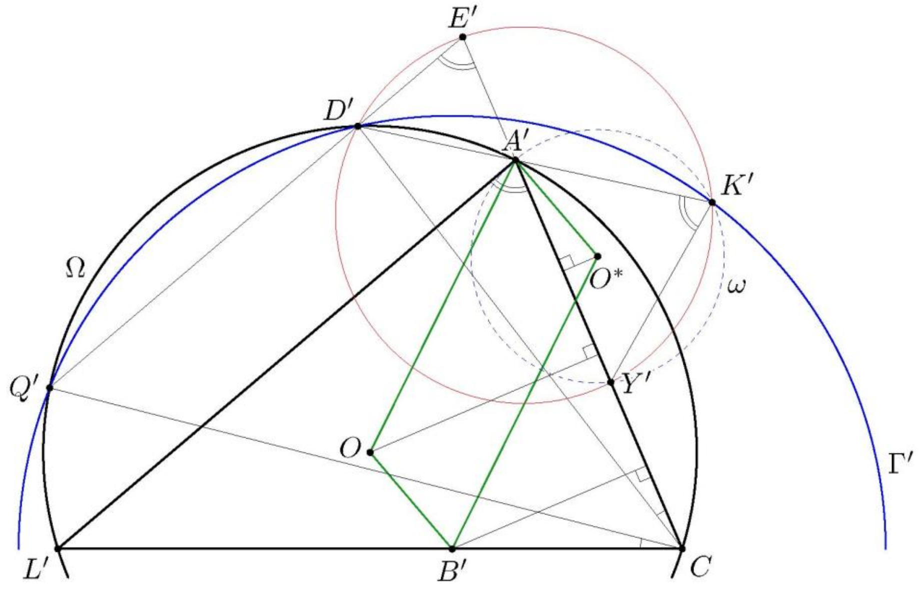
Let $L = AD \cap BC$. We perform an inversion centred at C; the images of the points will be denoted by primes, e.g., A' is the image of A. We obtain the following configuration, constructed in a triangle $A'CL'$.

The points D' and Q' are chosen on the circumcircle Ω of $A'L'C$ such that $\measuredangle(L'C, D'C) = \measuredangle(Q'C, A'C)$, which means that $A'L' \parallel D'Q'$. The lines $D'Q'$ and $A'C$ meet at E'.

A circle Γ' centred on CL' passes through D' and Q'. Notice here that B' lies on the segment CL', and that $\angle A'B'C = \angle BAC = 2\angle LAC = 2\angle A'L'C$, so that $B'L' = B'A'$, and B' lies on the perpendicular bisector of $A'L'$ (which coincides with that of $D'Q'$). All this means that B' is the centre of Γ'.

Finally, K' is the second meeting point of $A'D'$ and Γ', and Y' is the second meeting point of the circle $(D'K'E')$ and the line $A'E'$, We have $\measuredangle(Y'K', K'A') = \measuredangle(Y'E', E'D') = \measuredangle(Y'A', A'L')$, so $A'L'$ is tangent to the circumcircle ω of the triangle $Y'A'K'$.

Let O and O^* be the centres of Ω and ω, respectively. Then $O^*A' \perp A'L' \perp B'O$. The projections of vectors $\overrightarrow{O^*A'}$ and $\overrightarrow{B'O}$ onto $K'D'$ are equal to $\overrightarrow{K'A'}/2 = \overrightarrow{K'D'}/2 - \overrightarrow{A'D'}/2$. So $\overrightarrow{O^*A'} = \overrightarrow{B'O}$, or equivalently $\overrightarrow{A'O} = \overrightarrow{O^*B'}$. Projecting this equality onto $A'C$, we see that the projection of $\overrightarrow{O^*B'}$ equals $\overrightarrow{A'C}/2$. Since O^* is projected to the midpoint of $A'Y'$, this yields that B' is projected to the midpoint of CY', i.e., $B'Y' = B'C$ and $\angle B'Y'C = \angle B'CY'$. In the original figure, this rewrites as $\angle CBY = \angle BCY$, so Y lies on the perpendicular bisector of BC, as desired.



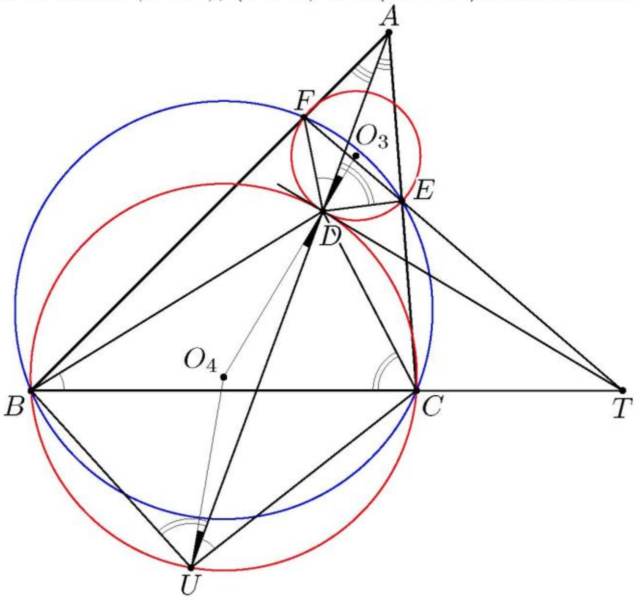
Comment 2. The point K appears to be the same in Solutions 1 and 3 (and Comment 1 as well). One can also show that K lies on the circle passing through A, X, and the midpoint of the arc BAC.

Comment 3. There are different proofs of the facts from the Common remarks, namely, the cyclicity of B, C, E, and F, and the Claim. We present one such alternative proof here.

We perform the composition ϕ of a homothety with centre A and the reflection in AD, which maps E to B. Let $U = \phi(D)$. Then $\measuredangle(BC, CD) = \measuredangle(AD, DE) = \measuredangle(BU, UD)$, so the points B, U, C, and D are concyclic. Therefore, $\measuredangle(CU, UD) = \measuredangle(CB, BD) = \measuredangle(AD, DF)$, so $\phi(F) = C$. Then the coefficient of the homothety is $AC/AF = AB/AE$, and thus points C, E, F, and B are concyclic.

Denote the centres of the circles (EDF) and $(BUCD)$ by O_3 and O_4, respectively. Then $\phi(O_3) = O_4$, hence $\measuredangle(O_3D, DA) = -\measuredangle(O_4U, UA) = \measuredangle(O_4D, DA)$, whence the circle (BDC) is tangent to the circle (EDF).

Now, the radical axes of circles (DEF), (BDC) and $(BCEF)$ intersect at T, and the claim follows.

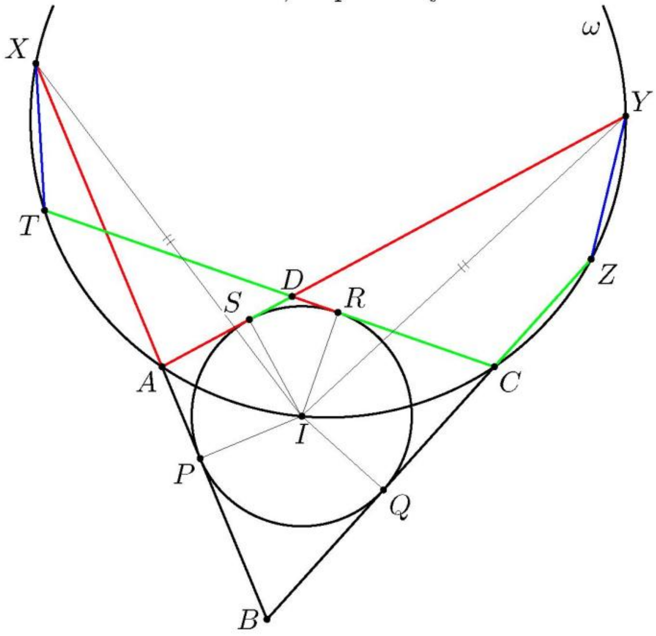


This suffices for Solution 1 to work. However, Solutions 2 and 3 need properties of point Q, established in Common remarks before Solution 1.

Comment 4. In the original problem proposal, the point X was hidden. Instead, a circle γ was constructed such that D and E lie on γ, and its center is collinear with O_1 and T. The problem requested to prove that, in a fixed triangle ABC, independently from the choice of D on the bisector of $\angle BAC$, all circles γ pass through a fixed point.

4 Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Prove that the perimeters of the (possibly self-intersecting) quadrilaterals $ADTX$ and $CDYZ$ are equal.

Solution. The point I is the intersection of the external bisector of the angle TCZ with the circumcircle ω of the triangle TCZ, so I is the midpoint of the arc TCZ and $IT = IZ$. Similarly, I is the midpoint of the arc YAX and $IX = IY$. Let O be the centre of ω. Then X and T are the reflections of Y and Z in IO, respectively. So $XT = YZ$.



Let the incircle of $ABCD$ touch AB, BC, CD, and DA at points P, Q, R, and S, respectively.

The right triangles IXP and IYS are congruent, since $IP = IS$ and $IX = IY$. Similarly, the right triangles IRT and IQZ are congruent. Therefore, $XP = YS$ and $RT = QZ$.

Denote the perimeters of $ADTX$ and $CDYZ$ by P_{ADTX} and P_{CDYZ} respectively. Since $AS = AP$, $CQ = RC$, and $SD = DR$, we obtain

$$\begin{aligned} P_{ADTX} &= XT + XA + AS + SD + DT = XT + XP + RT \\ &= YZ + YS + QZ = YZ + YD + DR + RC + CZ = P_{CDYZ}, \end{aligned}$$

as required.

Comment 1. After proving that X and T are the reflections of Y and Z in IO, respectively, one can finish the solution as follows. Since $XT = YZ$, the problem statement is equivalent to

$$XA + AD + DT = YD + DC + CZ. \tag{1}$$

Since $ABCD$ is circumscribed, $AB - AD = BC - CD$. Adding this to (1), we come to an equivalent equality $XA + AB + DT = YD + BC + CZ$, or

$$XB + DT = YD + BZ. \tag{2}$$

Let $\lambda = \frac{XZ}{AC} = \frac{TY}{AC}$. Since $XACZ$ is cyclic, the triangles ZBX and ABC are similar, hence

$$\frac{XB}{BC} = \frac{BZ}{AB} = \frac{XZ}{AC} = \lambda.$$

It follows that $XB = \lambda BC$ and $BZ = \lambda AB$. Likewise, the triangles TDY and ADC are similar, hence

$$\frac{DT}{AD} = \frac{DY}{CD} = \frac{TY}{AC} = \lambda.$$

Therefore, (2) rewrites as $\lambda BC + \lambda AD = \lambda CD + \lambda AB$.

This is equivalent to $BC + AD = CD + AB$ which is true as $ABCD$ is circumscribed.

Comment 2. Here is a more difficult modification of the original problem, found by the PSC.

Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Let $U = BC \cap AD$ and $V = BA \cap CD$. Let I_U be the incentre of UYZ and let J_V be the V-excentre of VXT. Then $I_U J_V \perp BD$.

5 A thimblerigger has 2021 thimbles numbered from 1 through 2021. The thimbles are arranged in a circle in arbitrary order. The thimblerigger performs a sequence of 2021 moves; in the k^{th} move, he swaps the positions of the two thimbles adjacent to thimble k.

Prove that there exists a value of k such that, in the k^{th} move, the thimblerigger swaps some thimbles a and b such that $a < k < b$.

Solution. Assume the contrary. Say that the k^{th} thimble is the *central thimble* of the k^{th} move, and its position on that move is the *central position* of the move.

Step 1: Black and white colouring.

Before the moves start, let us paint all thimbles in white. Then, after each move, we repaint its central thimble in black. This way, at the end of the process all thimbles have become black.

By our assumption, in every move k, the two swapped thimbles have the same colour (as their numbers are either both larger or both smaller than k). At every moment, assign the colours of the thimbles to their current positions; then the only position which changes its colour in a move is its central position. In particular, each position is central for exactly one move (when it is being repainted to black).

Step 2: Red and green colouring.

Now we introduce a colouring of the *positions*. If in the k^{th} move, the numbers of the two swapped thimbles are both less than k, then we paint the central position of the move in red; otherwise we paint that position in green. This way, each position has been painted in red or green exactly once. We claim that among any two adjacent positions, one becomes green and the other one becomes red; this will provide the desired contradiction since 2021 is odd.

Consider two adjacent positions A and B, which are central in the a^{th} and in the b^{th} moves, respectively, with $a < b$. Then in the a^{th} move the thimble at position B is white, and therefore has a number greater than a. After the a^{th} move, position A is green and the thimble at position A is black. By the arguments from Step 1, position A contains only black thimbles after the a^{th} step. Therefore, on the b^{th} move, position A contains a black thimble whose number is therefore less than b, while thimble b is at position B. So position B becomes red, and hence A and B have different colours.

Comment 1. Essentially, Step 1 provides the proof of the following two assertions (under the indirect assumption):

(1) Each position P becomes central in exactly one move (denote that move's number by k); and

(2) Before the k^{th} move, position P always contains a thimble whose number is larger than the number of the current move, while after the k^{th} move the position always contains a thimble whose number is smaller than the number of the current move.

Both (1) and (2) can be proved without introduction of colours, yet the colours help to visualise the argument.

After these two assertions have been proved, Step 2 can be performed in various ways, e.g., as follows.

At any moment in the process, the black positions are split into several groups consisting of one or more contiguous black positions each; different groups are separated by white positions. Now one can prove by induction on k that, after the k^{th} move, all groups have odd sizes. Indeed, in every move, the new black position either forms a separate group, or merges two groups (say, of lengths a and b) into a single group of length $a + b + 1$.

However, after the 2020^{th} move the black positions should form one group of length 2020. This is a contradiction.

This argument has several variations; e.g., one can check in a similar way that, after the process starts, at least one among the groups of *white* positions has an even size.

6 Let A be a finite set of (not necessarily positive) integers, and let $m \geqslant 2$ be an integer. Assume that there exist non-empty subsets $B_1, B_2, B_3, \ldots, B_m$ of A whose elements add up to the sums $m^1, m^2, m^3, \ldots, m^m$, respectively. Prove that A contains at least $m/2$ elements.

Solution. Let $A = \{a_1, \ldots, a_k\}$. Assume that, on the contrary, $k = |A| < m/2$. Let

$$s_i := \sum_{j: a_j \in B_i} a_j$$

be the sum of elements of B_i. We are given that $s_i = m^i$ for $i = 1, \ldots, m$.

Now consider all m^m expressions of the form

$$f(c_1, \ldots, c_m) := c_1 s_1 + c_2 s_2 + \ldots + c_m s_m, \ c_i \in \{0, 1, \ldots, m-1\} \text{ for all } i = 1, 2, \ldots, m.$$

Note that every number $f(c_1, \ldots, c_m)$ has the form

$$\alpha_1 a_1 + \ldots + \alpha_k a_k, \ \alpha_i \in \{0, 1, \ldots, m(m-1)\}.$$

Hence, there are at most $(m(m-1)+1)^k < m^{2k} < m^m$ distinct values of our expressions; therefore, at least two of them coincide.

Since $s_i = m^i$, this contradicts the uniqueness of representation of positive integers in the base-m system.

Comment 1. For other rapidly increasing sequences of sums of B_i's the similar argument also provides lower estimates on $k = |A|$. For example, if the sums of B_i are equal to $1!, 2!, 3!, \ldots, m!$, then for any fixed $\varepsilon > 0$ and large enough m we get $k \geqslant (1/2 - \varepsilon)m$. The proof uses the fact that the combinations $\sum c_i i!$ with $c_i \in \{0, 1, \ldots, i\}$ are all distinct.

Comment 2. The problem statement holds also if A is a set of real numbers (not necessarily integers), the above proofs work in the real case.