XVI Asian Pacific Mathematics Olympiad
March 2004

Time allowed: 4 hours
No calculators are to be used
Each question is worth 7 points

Problem 1.

Determine all finite nonempty sets S of positive integers satisfying

$$\frac{i+j}{(i,j)} \qquad \text{is an element of } S \text{ for all } i,j \text{ in } S,$$

where (i,j) is the greatest common divisor of i and j.

Problem 2.

Let O be the circumcentre and H the orthocentre of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

Problem 3.

Let a set S of 2004 points in the plane be given, no three of which are collinear. Let $\mathcal{L}$ denote the set of all lines (extended indefinitely in both directions) determined by pairs of points from the set. Show that it is possible to colour the points of S with at most two colours, such that for any points p, q of S, the number of lines in $\mathcal{L}$ which separate p from q is odd if and only if p and q have the same colour.

Note: A line ℓ *separates* two points p and q if p and q lie on opposite sides of ℓ with neither point on ℓ.

Problem 4.

For a real number x, let $\lfloor x \rfloor$ stand for the largest integer that is less than or equal to x. Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integer n.

Problem 5.

Prove that

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca)$$

for all real numbers $a, b, c > 0$.

Notes:
Unless specific, $[ABC]$ means the area of ABC.

Problem 1.
Determine all finite nonempty sets S of positive integers satisfying

$$\frac{i+j}{(i,j)} \text{ is an element of } S \text{ for all } i, j \text{ in } S,$$

Where (i, j) is the greatest common divisor of i and j.

Solution. Let $k \in S$. Then $\frac{k+k}{(k,k)} = 2 \in S$.

Let M be the largest odd element of S. Then, $\frac{M+2}{(M,2)} = M+2 \in S$ which is a contradiction.

Therefore, all elements of S must be even.

Let $m = 2n$ be the smallest element of S greater than 2. Then $\frac{m+2}{2} = n+1 \in S$. However, $n > 1$ (or $m = 2$) must be true. By minimality of m, $2n = 2$ and $n = 1$ which is a contradiction.

Hence, $S = \{2\}$.

Problem 2.
Let O be the circumcentre and H the orthocentre of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

Solution. Let G be the centroid. Note that OH is an Euler line, so G lies on OH.

Without loss of generality, we may assume A lies on a side of OH as well as B and C lie on the opposite side.

Let M be the midpoint of BC. Then the length of the perpendicular line from M to OH is the mean of the lengths of the perpendicular lines from B and C. Hence,

$$[BOH]+[COH] = OH \cdot GM \cdot \sin\measuredangle MGO = \frac{OH \cdot AG}{2} \cdot \sin\measuredangle MGO = [AOH].$$

Problem 3.
Let a set S of 2004 points in the plane be given, no three of which are collinear. Let $\Im$ denote the set of all lines (extended indefinitely in both directions) determined by pairs of points from the set. Show that it is possible to colour the points of S with at most two colours, such that for any points p, q of S, the number of lines in $\Im$ which separate p from q is odd if and only if p and q have the same colour.

Note: A line l separates two points p and q if p and q lie on opposite sides of l with neither point on l.

Solution. Let d_{XY} be the number of lines separating the points X and Y. Take a point X and colour it blue. For every other point Y, colour it blue iff d_{XY} is odd. For any three points A, B and C, $d_{AB} + d_{BC} + d_{AC}$ is odd. If Y and Z are the same colour, then d_{XY} and d_{XZ} have the same parity. Therefore, d_{YZ} is odd, which is true. Similarly, if Y and Z are coloured with opposite colours, then d_{YZ} is even, which is true.

Consider the lines which pass through an interior point of one or more of AB, BC and AC. Lines cannot pass through all three lines. If two lines are passed through, then the parity of $d_{AB} + d_{BC} + d_{AC}$ does not change. Consider the lines which pass through point A and the interior of BC, and the cases for point B and C are similar. Let $n_1, n_2, ..., n_7$ be the number of points, excluding point A, B and C, in the various regions as shown. The number of lines passing through point A and BC is $n_1+n_2+n_3$. Therefore,

$$\begin{aligned} d_{AB}+d_{BC}+d_{AC} &= (n_1+n_2+n_3)+(n_1+n_4+n_5)+(n_1+n_6+n_7) \\ &\equiv n_1+n_2+n_3+n_4+n_5+n_6+n_7 = 2004-3 \equiv 1 \pmod 2). \end{aligned}$$

Hence, it is possible to colour the points of S with at most two colours, such that for any points p, q of S, the number of lines in $\Im$ which separate p from q is odd if and only if p and q have the same colour.

Problem 4.

For a real number x, let $\lfloor x \rfloor$ stand for the largest integer that is less than or equal to x. Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integers n.

Solution. For $n=1,2,3,4,5$, $\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor = 0$. Consider $n \geq 6$.

If n and $n+1$ are composite, $n+1 \mid (n-1)!$ or $n \mid (n-1)!$. As n and $n+1$ are coprime, $n(n+1) \mid (n-1)!$. Note that only one of n or $n+1$ is even. For $m \geq 6$, $(m-2)!$ is divisible by more powers of 2 than m. Therefore $\frac{(n-1)!}{n(n+1)}$ is even.

Now, we consider two cases: $n+1=p$ and $n=p$ where p is a prime number.

If $n+1=p$, then $p-1$ is composite, so $p-1 \mid (p-2)!$. Let $k = \frac{(p-2)!}{p-1}$.

By Wilson's theorem, $(p-2)! \equiv 1 \pmod p$. Hence, $k(p-1) \equiv 1 \pmod p$ and $k \equiv -1 \pmod p$ .Therefore, $\frac{k+1}{p} \in \mathbb{Z}$. As k is even, $k+1$ is odd and hence $\frac{k+1}{p}$ is odd. Then, we obtain that $\left\lfloor \frac{k}{p} \right\rfloor = \frac{k+1}{p-1}$.

Therefore, $\left\lfloor \frac{k}{p} \right\rfloor = \left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$ is even.

If $n=p$, then $k = \frac{(p-1)!}{p+1}$ is an even integer. Then $k+1$ is an odd integer. By Wilson's theorem, $k(p+1) \equiv -1 \pmod p$. Then $\frac{k+1}{p} \in \mathbb{Z}$ is an integer and it is an odd integer. Hence, $\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor = \left\lfloor \frac{k}{p} \right\rfloor = \frac{k+1}{p-1}$ is even.

Problem 5.

Prove that

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca),$$

for all real numbers $a, b, c > 0$.

Solution. Note that

$$(a^2+2)(b^2+2)(c^2+2) = a^2b^2c^2 + 2(a^2b^2+b^2c^2+c^2a^2) + 4(a^2+b^2+c^2) + 8$$

By AM-GM inequality,

$$3(a^2+b^2+c^2) \geq 3(ab+bc+ca)$$

$$(2a^2b^2+2)+(2b^2c^2+2)+(2c^2a^2+2) \geq 4(ab+bc+ca)$$

Then, we want to show

$$a^2b^2c^2 + a^2 + b^2 + c^2 + 2 \geq 2(ab + bc + ca) .$$

By AM-GM inequality,

$$(x + y)(y + z)(z + x) \geq 8xyz .$$

Let $2x = -u + v + w, 2y = u - v + w, 2z = u + v - w$. If $-u + v + w, u - v + w, u + v - w$ are all non-negative, then,

$$uvw \geq (-u + v + w)(u - v + w)(u + v - w)$$

If u, v, w are positive, then at most one of $-u + v + w, u - v + w, u + v - w$ is negative. Trivally,

$$uvw \geq (-u + v + w)(u - v + w)(u + v - w)$$

and it holds for all positive u, v, w. Then, we obtain that

$$u^3 + v^3 + w^3 + 3uvw \geq uv(u + v) + vw(v + w) + uw(u + w)$$

By AM-GM inequality,

$$u^3 + v^3 + w^3 + 3uvw \geq 2(uv)^{\frac{3}{2}} + 2(vw)^{\frac{3}{2}} + 2(uw)^{\frac{3}{2}}$$

Then, let $u = a^{\frac{2}{3}}, v = b^{\frac{2}{3}}, w = c^{\frac{2}{3}}$, we obtain that

$$a^2b^2c^2 + a^2 + b^2 + c^2 + 2 \geq a^2 + b^2 + c^2 + 3(abc)^{\frac{2}{3}} \geq 2(ab + bc + ca) .$$