JBMO Shortlist 2013

– Algebra

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1 $\boxed{A1}$ Find all ordered triplets of (x, y, z) real numbers that satisfy the following system of equation $x^3 = \frac{z}{y} - \frac{2y}{z}$ $y^3 = \frac{x}{z} - \frac{2z}{x}$ $z^3 = \frac{y}{x} - \frac{2x}{y}$

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2 $\boxed{\text{A2}}$ Find the maximum value of $|\sqrt{x^2 + 4x + 8} - \sqrt{x^2 + 8x + 17}|$ where x is a real number.

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3 Show that

$$\left(a + 2b + \frac{2}{a+1}\right)\left(b + 2a + \frac{2}{b+1}\right) \geq 16$$

for all positive real numbers a and b such that $ab \geq 1$.

– Combinatorics

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1 Find the maximum number of different integers that can be selected from the set $\{1, 2, ..., 2013\}$ so that no two exist that their difference equals to 17.

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2 In a billiard with shape of a rectangle $ABCD$ with $AB = 2013$ and $AD = 1000$, a ball is launched along the line of the bisector of $\angle BAD$. Supposing that the ball is reflected on the sides with the same angle at the impact point as the angle shot , examine if it shall ever reach at vertex B.

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3 Let n be a positive integer. Two players, Alice and Bob, are playing the following game:
- Alice chooses n real numbers; not necessarily distinct.
- Alice writes all pairwise sums on a sheet of paper and gives it to Bob. (There are $\frac{n(n-1)}{2}$ such sums; not necessarily distinct.)
- Bob wins if he finds correctly the initial n numbers chosen by Alice with only one guess.
Can Bob be sure to win for the following cases?

a. $n = 5$
b. $n = 6$
c. $n = 8$

Justify your answer(s).

[For example, when $n = 4$, Alice may choose the numbers 1, 5, 7, 9, which have the same pairwise sums as the numbers 2, 4, 6, 10, and hence Bob cannot be sure to win.]

–	Geometry
1	Let AB be a diameter of a circle ω and center O , OC a radius of ω perpendicular to AB,M be a point of the segment (OC) . Let N be the second intersection point of line AM with ω and P the intersection point of the tangents of ω at points N and B. Prove that points M, O, P, N are cocyclic. (Albania)
2	Circles ω_1 , ω_2 are externally tangent at point M and tangent internally with circle ω_3 at points K and L respectively. Let A and Bbe the points that their common tangent at point M of circles ω_1 and ω_2 intersect with circle ω_3. Prove that if $\angle KAB = \angle LAB$ then the segment AB is diameter of circle ω_3. Theoklitos Paragyiou (Cyprus)
3	Let ABC be an acute-angled triangle with $AB < AC$ and let O be the centre of its circumcircle ω. Let D be a point on the line segment BC such that $\angle BAD = \angle CAO$. Let E be the second point of intersection of ω and the line AD. If M, N and P are the midpoints of the line segments BE, OD and AC, respectively, show that the points M, N and P are collinear.
4	Let I be the incenter and AB the shortest side of the triangle ABC. The circle centered at I passing through C intersects the ray AB in P and the ray BA in Q. Let D be the point of tangency of the A-excircle of the triangle ABC with the side BC. Let E be the reflection of C with respect to the point D. Prove that $PE \perp CQ$.
5	A circle passing through the midpoint M of the side BC and the vertex A of the triangle ABC intersects the segments AB and AC for the second time in the points P and Q, respectively. Prove that if $\angle BAC = 60°$, then $AP + AQ + PQ < AB + AC + \frac{1}{2}BC$.
6	Let P and Q be the midpoints of the sides BC and CD, respectively in a rectangle $ABCD$. Let K and M be the intersections of the line PD with the lines QB and QA, respectively, and let N be the intersection of the lines PA and QB. Let X, Y and Z be the midpoints of the segments AN, KN and AM, respectively. Let ℓ_1 be the line passing through X and perpendicular to MK, ℓ_2 be the line passing through Y and perpendicular to AM and ℓ_3 the line passing through Z and perpendicular to KN. Prove that the lines ℓ_1, ℓ_2 and ℓ_3 are concurrent.

– Number Theory

1 $\boxed{N1}$ find all positive integers n for which $1^3 + 2^3 + \cdots + 16^3 + 17^n$ is a perfect square.

2 Solve in integers $20^x + 13^y = 2013^z$.

3 Find all ordered pairs (a, b) of positive integers for which the numbers $\dfrac{a^3b-1}{a+1}$ and $\dfrac{b^3a+1}{b-1}$ are both positive integers.

4 A rectangle in xy Cartesian System is called latticed if all it's vertices have integer coordinates.
a) Find a latticed rectangle of area 2013, whose sides are not parallel to the axes.
b) Show that if a latticed rectangle has area 2011, then their sides are parallel to the axes.

5 Solve in positive integers: $\frac{1}{x^2} + \frac{y}{xz} + \frac{1}{z^2} = \frac{1}{2013}$.

6 Solve in integers the system of equations:

$$x^2 - y^2 = z$$

$$3xy + (x-y)z = z^2$$