SRB

36th BALKAN MATHEMATICAL OLYMPIAD

Chisinau, Republic of Moldova

April 30 – May 5, 2019

SHORTLIST
OF SELECTED PROBLEMS

SRB

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until BMO 2020.

Contributing Countries

The Organizing Committee and the Problem Selection Committee of BMO 2019 thank the following 6 countries for contributing 20 problem proposals:

Albania

Bularia

Cyprus

Greece

Romania

United Kingdom

Problem Selection Committee

Chairman: ***Baltag Valeriu***

Members: ***Costaş Ana***

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Izbaş Vladimir

Rusu Galina

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ALGEBRA

ACB **A1.** Let a_0 be an arbitrary positive integer. Let $\{a_n\}$ be an infinite sequence of positive integers such that for every positive integer n the term a_n is the smallest positive integer such that $a_0 + a_1 + ... + a_n$ is divisible by n. Prove that there is a positive integer N such that $a_{n+1} = a_n$ for all $n \geq N$.

A1b.[1] Let a_0 be an arbitrary positive integer. Consider the infinite sequence $(a_n)_{n\geq 1}$, defined inductively as follows: given $a_0, a_1, ..., a_{n-1}$ define the term a_n as the smallest positive integer such that $a_0 + a_1 + ... + a_n$ is divisible by n. Prove that there exists a positive integer M such that $a_{n+1} = a_n$ for all $n \geq M$.

UNK **A2.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(xy) = yf(x) + x + f(f(y) - f(x))$$

for all $x, y \in \mathbb{R}$.

ROU **A3.** Let a, b, c be real numbers such that $0 \leq a \leq b \leq c$. Prove that if

$$a + b + c = ab + bc + ca > 0,$$

then $\sqrt{bc}(a + 1) \geq 2$. When does the equality hold?

HEL **A4.** Let a_{ij}, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, be positive real numbers. Prove that

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} \frac{1}{a_{ij}} \right)^{-1} \leq \left(\sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right)^{-1} \right)^{-1}.$$

When does the equality hold?

ROU **A5.** Let a, b, c be positive real numbers, such that $(ab)^2 + (bc)^2 + (ca)^2 = 3$. Prove that

$$(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq 1.$$

[1]Proposed by PSC.

GEOMETRY

ROU **G1.** Let $ABCD$ be a square of center O and let M be the symmetric of the point B with respect to the point A. Let E be the intersection of CM and BD, and let S be the intersection of MO and AE. Show that SO is the angle bisector of $\angle ESB$.

ROU **G2.** Let be a triangle $\triangle ABC$ with $m(\angle ABC) = 75°$ and $m(\angle ACB) = 45°$. The angle bisector of $\angle CAB$ intersects CB at the point D. We consider the point $E \in (AB)$, such that $DE = DC$. Let P be the intersection of the lines AD and CE. Prove that P is the midpoint of the segment AD.

UNK **G3.** Let ABC be a scalene and acute triangle, with circumcentre O. Let ω be the circle with centre A, tangent to BC at D. Suppose there are two points F and G on ω such that $FG \perp AO$, $\angle BFD = \angle DGC$ and the couples of points (B, F) and (C, G) are in different halfplanes with respect to the line AD. Show that the tangents to ω at F and G meet on the circumcircle of ABC.

UNK **G4.** Given an acute triangle ABC, let M be the midpoint of BC and H the orthocentre. Let Γ be the circle with diameter HM, and let X, Y be distinct points on Γ such that AX, AY are tangent to Γ. Prove that $BXYC$ is cyclic.

BGR **G5.** Let ABC ($BC > AC$) be an acute triangle with circumcircle k centered at O. The tangent to k at C intersects the line AB at the point D. The circumcircles of triangles BCD, OCD and AOB intersect the ray CA (beyond A) at the points Q, P and K, respectively, such that $P \in (AK)$ and $K \in (PQ)$. The line PD intersects the circumcircle of triangle BKQ at the point T, so that P and T are in different halfplanes with respect to BQ. Prove that $TB = TQ$.

HEL **G6.** Let ABC be an acute triangle, and AX, AY two isogonal lines. Also, suppose that K, S are the feet of perpendiculars from B to AX, AY, and T, L are the feet of perpendiculars from C to AX, AY respectively. Prove that KL and ST intersect on BC.

ALB **G7.** Let AD, BE, and CF denote the altitudes of triangle $\triangle ABC$. Points E' and F' are the reflections of E and F over AD, respectively. The lines BF' and CE' intersect at X, while the lines BE' and CF' intersect at the point Y. Prove that if H is the orthocenter of $\triangle ABC$, then the lines AX, YH, and BC are concurrent.

CYP **G8.** Given an acute triangle ABC, (c) is circumcircle with center O and H the orthocenter of the triangle ABC. The line AO intersects (c) at the point D. Let D_1, D_2 and H_2, H_3 be the symmetrical points of the points D and H with respect to the lines AB, AC respectively. Let (c_1) be the circumcircle of the triangle AD_1D_2. Suppose that the line AH intersects again (c_1) at the point U, the line H_2H_3 intersects the segment D_1D_2 at the point K_1 and the line DH_3 intersects the segment UD_2 at the point L_1. Prove that one of the intersection points of the circumcircles of the triangles $D_1K_1H_2$ and UDL_1 lies on the line K_1L_1.

CYP **G9.** Given semicircle (c) with diameter AB and center O. On the (c) we take point C such that the tangent at the C intersects the line AB at the point E. The perpendicular line from C to AB intersects the diameter AB at the point D. On the (c) we get the points H, Z such that $CD = CH = CZ$. The line HZ intersects the lines CO, CD, AB at the points S, I, K respectively and the parallel line from I to the line AB intersects the lines CO, CK at the points L, M respectively. We consider the circumcircle(k) of the triangle LMD, which intersects again the lines AB, CK at the points P, U respectively. Let (e_1), (e_2), (e_3) be the tangents of the (k) at the points L, M, P respectively and $R = (e_1) \cap (e_2)$, $X = (e_2) \cap (e_3)$, $T = (e_1) \cap (e_3)$. Prove that if Q is the center of (k), then the lines RD, TU, XS pass through the same point, which lies in the line IQ.

NUMBER THEORY

ALB **TN1.** Let $\mathbb{P}$ be the set of all prime numbers. Find all functions $f : \mathbb{P} \to \mathbb{P}$ such that

$$f(p)^{f(q)} + q^p = f(q)^{f(p)} + p^q$$

holds for all $p, q \in \mathbb{P}$.

ROU **TN2.** Let $S \subset \{1, \dots, n\}$ be a nonempty set, where n is a positive integer. We denote by s the greatest common divisor of the elements of the set S. We assume that $s \neq 1$ and let d be its smallest divisor greater than 1. Let $T \subset \{1, \dots, n\}$ be a set such that $S \subset T$ and $|T| \geq 1 + \left[\frac{n}{d}\right]$. Prove that the greatest common divisor of the elements in T is 1.

TN2b.[2] Let $n \, (n \geq 1)$ be a positive integer and $U = \{1, \dots, n\}$. Let S be a nonempty subset of U and let d $(d \neq 1)$ be the smallest common divisor of all elements of the set S. Find the smallest positive integer k such that for any subset T of U, consisting of k elements, with $S \subset T$, the greatest common divisor of all elements of T is equal to 1.

[2]Proposed by PSC.

COMBINATORICS

UNK **C1.** 100 couples are invited to a traditional Moldovan dance. The 200 people stand in a line, and then in a *step*, two of them (not necessarily adjacent) may swap positions. Find the least C such that whatever the initial order, they can arrive at an ordering where everyone is dancing next to their partner in at most C steps.

HEL **C2.** Suppose that the numbers $\{1, 2, \ldots, 25\}$ are written in some order in an 5×5 array. Find the maximal positive integer k, such that the following holds. There is always an 2×2 subarray whose numbers have a sum not less than k.

C2b.[3] An 5×5 array must be completed with all numbers $\{1, 2, \ldots, 25\}$, one number in each cell. Find the maximal positive integer k, such that for any completion of the array there is a 2×2 square (subarray), whose numbers have a sum not less than k.

CYP **C3.** Anna and Bob play a game on the set of all points of the form (m, n) where m, n are integers with $|m|, |n| \leqslant 2019$. Let us call the lines $x = \pm 2019$ and $y = \pm 2019$ the *boundary lines* of the game. The points of these lines are called the *boundary points*. The *neighbours* of point (m, n) are the points $(m+1, n), (m-1, n), (m, n+1), (m, n-1)$.

Anna starts with a token at the origin $(0, 0)$. With Bob playing first, they alternately perform the following steps: At his turn, Bob *deletes* two points on each boundary line. On her turn Anna makes a sequences of three moves of the token, where a *move* of the token consists of picking up the token from its current position and placing it in one of its neighbours.

To win the game Anna must place her token on a boundary point before it is deleted by Bob. Does Anna have a winning strategy?

[**Note:** At every turn except perhaps her last, Anna **must** make **exactly** three moves.]

UNK **C4.** A town-planner has built an isolated city whose road network consists of $2N$ roundabouts, each connecting exactly three roads. A series of tunnels and bridges ensure that all roads in the town meet only at roundabouts. All roads are two-way, and each roundabout is oriented clockwise.

Vlad has recently passed his driving test, and is nervous about roundabouts. He starts driving from his house, and always takes the first exit at each roundabout he encounters. It turns out his journey includes every road in the town in both directions before he arrives back at the starting point in the starting direction. For what values of N is this possible?

[3] Proposed by PSC.

ALGEBRA

A1. Let a_0 be an arbitrary positive integer. Let $\{a_n\}$ be an infinite sequence of positive integers such that for every positive integer n the term a_n is the smallest positive integer such that $a_0 + a_1 + ... + a_n$ is divisible by n. Prove that there is a positive integer N such that $a_{n+1} = a_n$ for all $n \geq N$.

A1b.[4] Let a_0 be an arbitrary positive integer. Consider the infinite sequence $(a_n)_{n\geq 1}$, defined inductively as follows: given $a_0, a_1, ..., a_{n-1}$ define the term a_n as the smallest positive integer such that $a_0 + a_1 + ... + a_n$ is divisible by n. Prove that there exists a positive integer M such that $a_{n+1} = a_n$ for all $n \geq M$.

Solution. Define $b_n = \frac{a_0+a_1+...+a_n}{n}$ for every positive integer n. According to condition, b_n is a positive integer for every positive integer n.

Since a_{n+1} is the smallest positive integer such that $\frac{a_0+a_1+...+a_n+a_{n+1}}{n+1}$ is a positive integer and

$$\frac{a_0 + a_1 + ... + a_n + b_n}{n+1} = \frac{a_0 + a_1 + ... + a_n + \frac{a_0+a_1+...+a_n}{n}}{n+1} = \frac{a_0 + a_1 + ... + a_n}{n} = b_n,$$

which is a positive integer, we get $a_{n+1} \leq b_n$ for every positive integer n.

Now from last result we have

$$b_{n+1} = \frac{a_0 + a_1 + ... + a_n + a_{n+1}}{n+1} \leq \frac{a_0 + a_1 + ... + a_n + b_n}{n+1} = b_n.$$

Hence the infinite sequence of positive integers $b_1, b_2, ...$ is non-increasing. So there exists a positive integer T such that for all $n \geq T$ we have

$$b_{n+1} = b_n \Rightarrow \frac{a_0 + a_1 + ... + a_n + a_{n+1}}{n+1} = \frac{a_0 + a_1 + ... + a_n}{n} \Rightarrow$$

$$n(a_0 + a_1 + ... + a_n + a_{n+1}) = (n+1)(a_0 + a_1 + ... + a_n) \Rightarrow$$

$$na_{n+1} = a_0 + a_1 + ... + a_n \Rightarrow a_{n+1} = \frac{a_0 + a_1 + ... + a_n}{n} = b_n.$$

Similarly we get $a_{n+2} = b_{n+1}$, which follows that $a_{n+2} = b_{n+1} = b_n = a_{n+1}$. Hence, taking $M = T + 1$, we can state that $a_{n+1} = a_n$ for every $n \geq M$. □

[4] Proposed by PSC.

A2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(xy) = yf(x) + x + f(f(y) - f(x))$$

for all $x, y \in \mathbb{R}$.

Solution. Firstly, considering $(x, y) = (1, 1)$ we get $f(0) = -1$.

Then, setting $y = 1$, we see that $-x = f(f(1) - f(x))$, so f must be surjective.

Now let $(x, y) = (a, 0)$ and $(0, a)$ to get

$$-1 = a + f(-1 - f(a)) \quad \text{and} \quad -1 = -a + f(f(a) + 1).$$

Since f is surjective, for any real z we may write $z = f(a) + 1$ and then adding these two results gives $f(z) + f(-z) = -2$.

Letting $(x, y) = (a, 1)$ and $(1, a)$ we get

$$-a = f(f(1) - f(a)) \quad \text{and} \quad f(a) = af(1) + 1 + f(f(a) - f(1)).$$

Adding these, and using the previous result with $z = f(a) - f(1)$ gives

$$f(a) = af(1) + a - 1.$$

So $f(x) = kx - 1$ for all x, for some fixed k. Substituting back into the original equation we see that 1 and -1 are the only possibilities for k and that both of these values do give a function that works. □

Alternative solution. We prove that $f(x) = x - 1$ and $f(x) = -x - 1$ are the only solutions. Let $x = y = 1$; this gives $f(1) = f(1) + 1 + f(0)$, so $f(0) = -1$. Then let $(x, y) = (0, a + 1)$, $(-a - 1, 0)$, and $(-a, 1)$ to give the three equalities

$$\begin{aligned} f(0) = (a+1)f(0) + f(f(a+1) - f(0)) \quad &\Rightarrow \quad a = f(f(a+1) + 1) \\ f(0) = -a - 1 + f(f(0) - f(-a-1)) \quad &\Rightarrow \quad a = f(-f(-a-1) - 1) \\ f(-a) = f(-a) - a + f(f(1) - f(a)) \quad &\Rightarrow \quad a = f(f(1) - f(-a)). \end{aligned}$$

The last of these three implies f is bijective, hence we have

$$f(a+1) + 1 = -f(-a-1) - 1 = f(1) - f(-a)$$

From the second of these equalities we can deduce the recurrence relation $f(x) = f(x-1) + f(1) + 1$, so if $c = f(1) + 1$, we have $f(x) = cx - 1$ for all $x \in \mathbb{Z}$. Substituting into the original equation we see that $c^2 = 1$, so $f(x) = x - 1$ or $f(x) = -x - 1$ for $x \in \mathbb{Z}$.

In the first case, let $x = 1$. Then $f(y) = 1 + f(f(y))$, which implies $f(x) = x - 1$ for all x as f is surjective. In the second case, set $x = -1$, so $f(-y) = -1 + f(f(y))$. However from above we have $f(a+1) + f(-a-1) = 2$, so $f(f(y)) - 1 = f(-y) = -f(y) - 2$, and we have $f(x) = -x - 1$ by surjectivity. □

A3. Let a, b, c be real numbers such that $0 \le a \le b \le c$. Prove that if

$$a+b+c=ab+bc+ca>0,$$

then $\sqrt{bc}(a+1) \ge 2$. When does the equality hold?

Solution. Let $a+b+c=ab+bc+ca=k$. Since $(a+b+c)^2 \ge 3(ab+bc+ca)$, we get that $k^2 \ge 3k$. Since $k>0$, we obtain that $k \ge 3$.

We have $bc \ge ca \ge ab$, so from the above relation we deduce that $bc \ge 1$.

By AM-GM, $b+c \ge 2\sqrt{bc}$ and consequently $b+c \ge 2$. The equality holds iff $b=c$.

The constraint gives us

$$a=\frac{b+c-bc}{b+c-1}=1-\frac{bc-1}{b+c-1} \ge 1-\frac{bc-1}{2\sqrt{bc}-1}=\frac{\sqrt{bc}(2-\sqrt{bc})}{2\sqrt{bc}-1}.$$

For $\sqrt{bc}=2$ condition $a \ge 0$ gives $\sqrt{bc}(a+1) \ge 2$ with equality iff $a=0$ and $b=c=2$.

For $\sqrt{bc}<2$, taking into account the estimation for a, we get

$$a\sqrt{bc} \ge \frac{bc(2-\sqrt{bc})}{2\sqrt{bc}-1}=\frac{bc}{2\sqrt{bc}-1}(2-\sqrt{bc}).$$

Since $\frac{bc}{2\sqrt{bc}-1} \ge 1$, with equality for $bc=1$, we get $\sqrt{bc}(a+1) \ge 2$ with equality iff $a=b=c=1$.

For $\sqrt{bc}>2$ we have $\sqrt{bc}(a+1)>2(a+1) \ge 2$.

The proof is complete.

The equality holds iff $a=b=c=1$ or $a=0$ and $b=c=2$. $\square$

A4. Let a_{ij}, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, be positive real numbers. Prove that

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\frac{1}{a_{ij}}\right)^{-1} \leq \left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}\right)^{-1}\right)^{-1}.$$

When does the equality hold?

Solution. We will use the following

Lemma. *If $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ are positive real numbers then*

$$\frac{1}{\sum\limits_{j=1}^{n}\frac{1}{a_j}} + \frac{1}{\sum\limits_{j=1}^{n}\frac{1}{b_j}} \leq \frac{1}{\sum\limits_{j=1}^{n}\frac{1}{a_j+b_j}}.$$

The equality holds when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}$.

Proof. Set $x_j = \frac{1}{a_j}$ and $y_j = \frac{1}{b_j}$ for each $j = 1, 2, \ldots, n$. Then we have to prove that

$$\frac{1}{\sum\limits_{j=1}^{n}x_j} + \frac{1}{\sum\limits_{j=1}^{n}y_j} \leq \frac{1}{\sum\limits_{j=1}^{n}\frac{x_jy_j}{x_j+y_j}} \quad \text{or} \quad \sum_{j=1}^{n}\frac{x_jy_j}{x_j+y_j} \leq \frac{\left(\sum\limits_{j=1}^{n}x_j\right)\left(\sum\limits_{j=1}^{n}y_j\right)}{\sum\limits_{j=1}^{n}x_j + \sum\limits_{j=1}^{n}y_j}.$$

Subtract $\sum\limits_{j=1}^{n}x_j$, and we have to prove that

$$\sum_{j=1}^{n}\left(x_j - \frac{x_jy_j}{x_j+y_j}\right) \geq \sum_{j=1}^{n}x_j - \frac{\left(\sum\limits_{j=1}^{n}x_j\right)\left(\sum\limits_{j=1}^{n}y_j\right)}{\sum\limits_{j=1}^{n}x_j + \sum\limits_{j=1}^{n}y_j}$$

or

$$\sum_{j=1}^{n}\left(\frac{x_j^2}{x_j+y_j}\right) \geq \frac{\left(\sum\limits_{j=1}^{n}x_j\right)^2}{\sum\limits_{j=1}^{n}x_j + \sum\limits_{j=1}^{n}y_j}.$$

The last one is a consequence of Cauchy-Schwarz inequality and thus the lemma is proved.

We will now prove that repeating the lemma we will get the desired inequality. For example, if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n$ are positive reals then by repeating lemma two times we get

$$\frac{1}{\sum\limits_{j=1}^{n}\frac{1}{a_j}} + \frac{1}{\sum\limits_{j=1}^{n}\frac{1}{b_j}} + \frac{1}{\sum\limits_{j=1}^{n}\frac{1}{c_j}} \leq \frac{1}{\sum\limits_{j=1}^{n}\frac{1}{a_j+b_j}} + \frac{1}{\sum\limits_{j=1}^{n}\frac{1}{c_j}} \leq \frac{1}{\sum\limits_{j=1}^{n}\frac{1}{(a_j+b_j)+c_j}} = \frac{1}{\sum\limits_{j=1}^{n}\frac{1}{a_j+b_j+c_j}}.$$

Using similar reasoning we can prove by induction that

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\frac{1}{a_{ij}}\right)^{-1}=\sum_{i=1}^{m}\frac{1}{\sum\limits_{j=1}^{n}\frac{1}{a_{ij}}}\leq\frac{1}{\sum\limits_{j=1}^{n}\frac{1}{\sum\limits_{i=1}^{m}a_{ij}}}=\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m}a_{ij}\right)^{-1}\right)^{-1},$$

which is the desired result.

The equality holds iff

$$\frac{a_{i1}}{a_{11}}=\frac{a_{i2}}{a_{12}}=\ldots=\frac{a_{in}}{a_{1n}}$$

for all $i=1,2,\ldots,m$. $\square$

A5. Let a, b, c be positive real numbers, such that $(ab)^2 + (bc)^2 + (ca)^2 = 3$. Prove that

$$(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq 1.$$

Solution. The inequality is equivalent with

$$(a^2-a+1)(b^2-b+1)(c^2-c+1) \geq 1 \Leftrightarrow (a^3+1)(b^3+1)(c^3+1) \geq (a+1)(b+1)(c+1).$$

Thus:

$$\prod_{cyc}(a^3+1) - \prod_{cyc}(a+1) = \sum_{cyc} a^3 + \sum_{cyc}(ab)^3 + (abc)^3 - \sum_{cyc} a - \sum_{cyc} ab - abc =$$

$$\sum_{cyc}(a^3+a) + \sum_{cyc}(a^3b^3+ab) + [(abc)^3+1+1] - 2\sum_{cyc} a - 2\sum_{cyc} ab - abc - 2 \overset{AM \geq GM}{\geq}$$

$$2\sum_{cyc} a^2 + 2\sum_{cyc} a^2b^2 + 2abc - 2\sum_{cyc} a - 2\sum_{cyc} ab - 2 \overset{\sum_{cyc} a^2b^2=3}{=}$$

$$\sum_{cyc}(a^2-2a+1) + \left(\sum_{cyc} a^2 + 2abc + 1 - 2\sum_{cyc} ab\right) =$$

$$\sum_{cyc}(a-1)^2 + \left(\sum_{cyc} a^2 + 2abc + 1 - 2\sum_{cyc} ab\right) \geq \left(\sum_{cyc} a^2 + 2abc + 1 - 2\sum_{cyc} ab\right).$$

We will show that $\sum_{cyc} a^2 + 2abc + 1 - 2\sum_{cyc} ab \geq 0$ **(1)** for every $a, b, c \geq 0$.

Firstly, let us observe that

$$(1+2abc)(a+b+c) = (1+abc+abc)(a+b+c) \geq 9\sqrt[3]{a^2b^2c^2abc} = 9abc,$$

implying

$$1 + 2abc \geq \frac{9abc}{a+b+c}.$$

Then, using Schur's Inequality, (i.e. $\sum_{cyc} a(a-b)(a-c) \geq 0$, for any $a, b, c \geq 0$) we obtain that

$$\sum_{cyc} a^2 \geq 2\sum_{cyc} ab - \frac{9abc}{a+b+c}.$$

Returning to (1), we get:

$$\sum_{cyc} a^2 + 2abc + 1 - 2\sum_{cyc} ab \geq (2\sum_{cyc} ab - \frac{9abc}{a+b+c}) + 2abc + 1 - 2\sum_{cyc} ab =$$

$$(1+2abc) - \frac{9abc}{a+b+c} \geq 0,$$

which gives us $\prod_{cyc}(a^3+1) - \prod_{cyc}(a+1) \geq 0$ and, respectively, $\prod_{cyc}(a^2-a+1) \geq 1$. $\square$

GEOMETRY

G1. Let $ABCD$ be a square of center O and let M be the symmetric of the point B with respect to the point A. Let E be the intersection of CM and BD, and let S be the intersection of MO and AE. Show that SO is the angle bisector of $\angle ESB$.

Solution. We have

$$\begin{cases} DC \equiv DA \\ \angle EDC \equiv \angle EDA \\ DE \equiv DE \end{cases} \Rightarrow \triangle DEC \equiv \triangle DEA \Rightarrow \angle DAE \equiv \angle DCE\ (*).$$

Let $CM \cap AD = \{P\}$, then follows $\triangle CDP \equiv \triangle BAP$ and $\angle PCD \equiv \angle PBA\ (**)$.

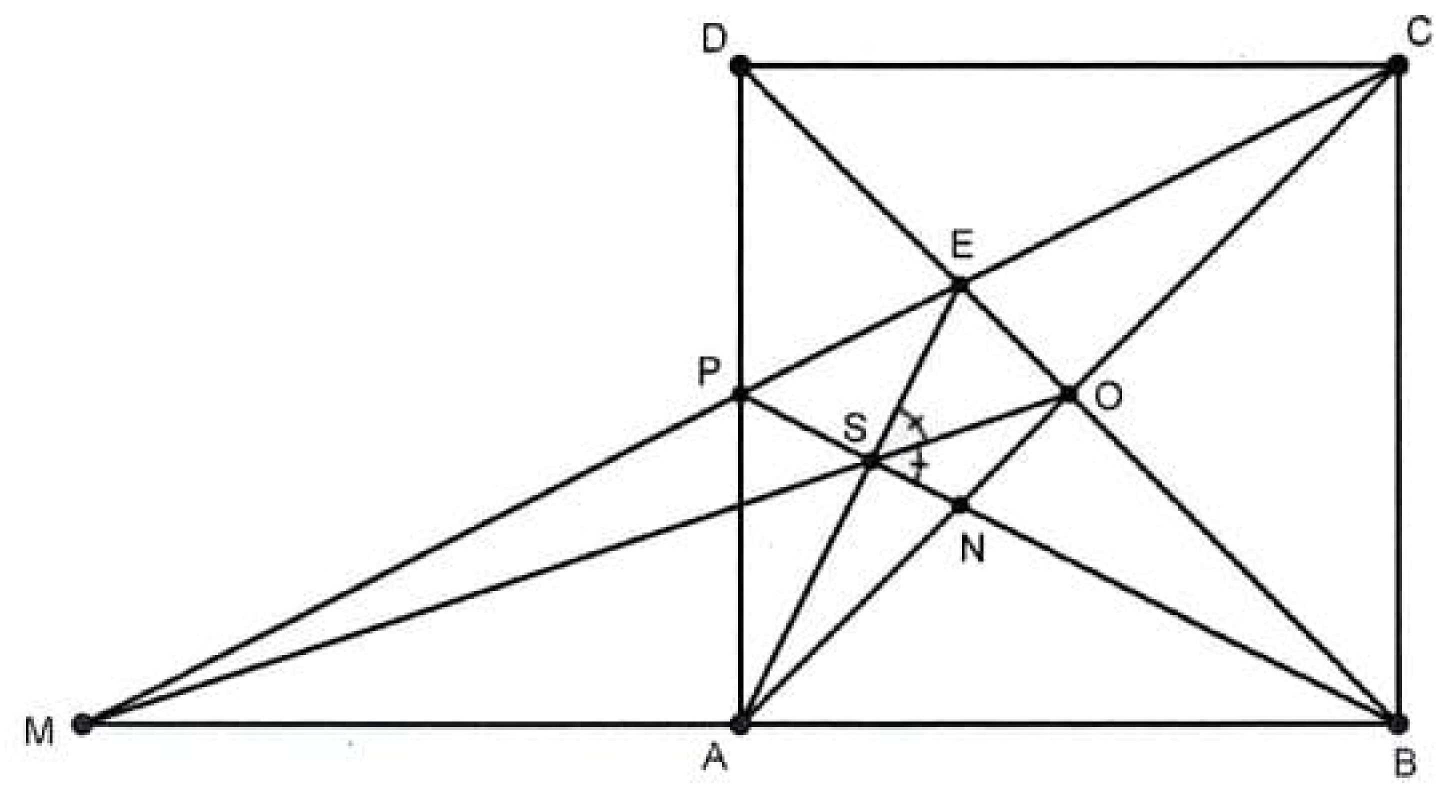

Figure 1: G1

From $(*)$ and $(**)$ follows $\angle DCP \equiv \angle DAE \equiv \angle PBA$.
Now, let $S' = AE \cap PB$.
In the triangle $S'AB$ we have

$$m(\angle S'AB) + m(\angle S'BA) = m(\angle S'AB) + m(\angle PAS') = m(\angle PAB) = 90°,$$

so $m(\angle BS'A) = 90°$.

We show that AE, BP and MO are concurrent.

In the triangle $\triangle EMB$ we apply the Ceva theorem, so

$$\frac{EP}{PM} \cdot \frac{MA}{AB} \cdot \frac{BO}{OE} = 1 \Leftrightarrow \frac{EP}{PM} = \frac{OE}{BO}$$

is true because PO is a midsegment in the triangle DAB ($PO||AB$).

According to the Thales theorem in the triangle EMB, $\frac{EP}{PM} = \frac{EO}{OB}$ and AE, BP, MO are concurrent in S', which is in fact S.

Let $PB \cap CA = \{N\}$. Because $ESNO$ has $m(\angle EON) + m(\angle ESN) = 180°$, it follows $ESNO$ cyclic and $m(\angle ESO) = m(\angle ENO) = m(\angle DAO) = 45°$. $\square$

G2. Let be a triangle $\triangle ABC$ with $m(\angle ABC) = 75°$ and $m(\angle ACB) = 45°$. The angle bisector of $\angle CAB$ intersects CB at the point D. We consider the point $E \in (AB)$, such that $DE = DC$. Let P be the intersection of the lines AD and CE. Prove that P is the midpoint of the segment AD.

Solution. Let P' be the midpoint of the segment AD. We will prove that $P' = P$. Let $F \in AC$ such that $DF \perp AC$. The triangle CDF is isosceles with $FD = FC$ and the triangle $DP'F$ is equilateral as $m(\angle ADF) = 60°$. Thus, the triangle FCP' is isosceles ($FP' = FC$) and $m(\angle FCP') = m(\angle FP'C) = 15°$.

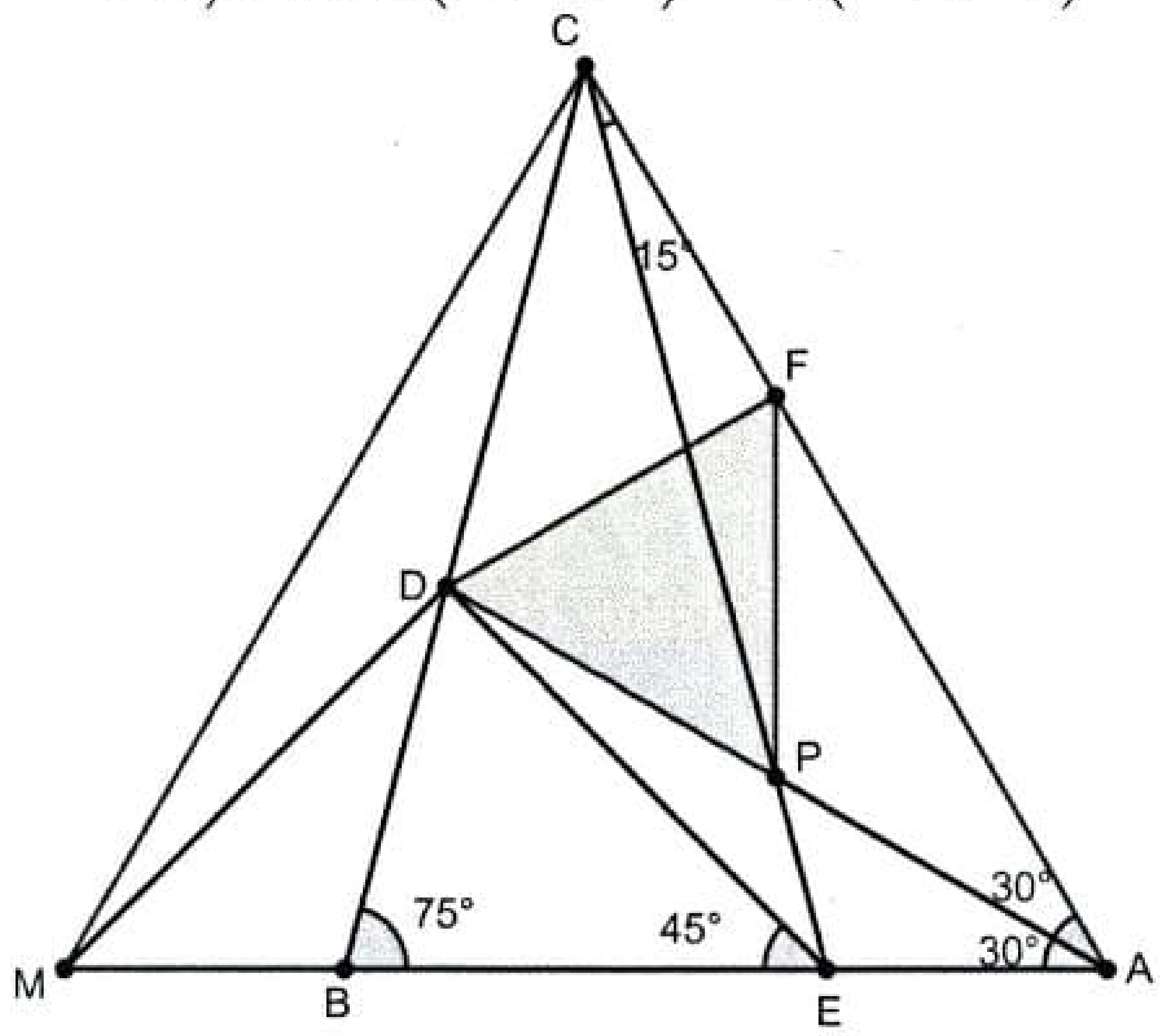

Figure 2: G2

We prove now that $m(\angle FCE) = 15°$.

Let M be the point on $[AB$ such that the triangle ACM is equilateral. As $\triangle ADC \equiv \triangle ADM(SAS) \Rightarrow DC = DM(= DE)$ and $m(\angle AMD) = m(\angle ACD) = 45°$. It follows that the triangle $\triangle DME$ is isosceles with $m(\angle DME) = m(\angle DEM) = 45°$. In the triangle $\triangle BDE$ we have $m(\angle BDE) = 60°$and thus $m(\angle CDE) = 120°$.As the triangle DCE is isoscel with $m(\angle DCE) = m(\angle DEC) = 30°$. Finaly $m(\angle ACE) = m(\angle ACB) - m(\angle BCE) = 45° - 30° = 15°$.

Thus $m(\angle FCP') = 15° = m(\angle FCE)$, and therefore $P' \in CE$ and $P' = P$, which means that P is the midpoint of the segment AD.

Alternative solution: In the way as above we prove that $m(\angle BCE) = 15°$.

So the quadrilateral $ACDE$ is inscribed in a circle. Now, applying the sine rules to $\triangle DPE$ and $\triangle APE$ we get

$$\frac{DP}{\sin 30°} = \frac{PE}{\sin 15°}, \quad \frac{AP}{\sin 105°} = \frac{PE}{\sin 30°} \Rightarrow \frac{DP}{\sin 30°} \cdot \frac{\sin 105°}{AP} = \frac{PE}{\sin 15°} \cdot \frac{\sin 30°}{PE},$$

$$\frac{DP}{AP} = \frac{\sin 30°}{\sin 105° \cdot \sin 15°} = \frac{1}{4 \cdot \sin 105° \cdot \sin 15°} = \frac{1}{2 \cdot (\cos 90° - \cos 120°)} = \frac{1}{2 \cdot \frac{1}{2}} = 1.$$

Thus, $QP = AP$. $\square$

G3. Let ABC be a scalene and acute triangle, with circumcentre O. Let ω be the circle with centre A, tangent to BC at D. Suppose there are two points F and G on ω such that $FG \perp AO$, $\angle BFD = \angle DGC$ and the couples of points (B,F) and (C,G) are in different halfplanes with respect to the line AD. Show that the tangents to ω at F and G meet on the circumcircle of ABC.

Solution. Consider any two points F, G on ω such that $\angle BFD = \angle DGC$. Exploiting the isosceles triangles $\triangle AFG$, $\triangle AFD$, and $\triangle ADG$, we deduce (using directed angles throughout):

$$\angle DBF - \angle GCD = 180^\circ - \angle BFD - \angle BDF - (180^\circ - \angle DGC - \angle CDG) \overset{(*)}{=}$$

$$\angle CDG - \angle FDB = \frac{1}{2}\cdot(\angle DAG - \angle DAF) = \frac{1}{2}\cdot[(180^\circ - 2\cdot\angle ADG) - (180^\circ - 2\cdot\angle ADF)] =$$

$$\angle ADF - \angle GDA = \angle DFA - \angle AGD = \angle DFG - \angle FGD \overset{(*)}{=} \angle BFG - \angle FGC,$$

where we use $\angle BFD = \angle DGC$ at (*). Thus $BFGC$ is cyclic.

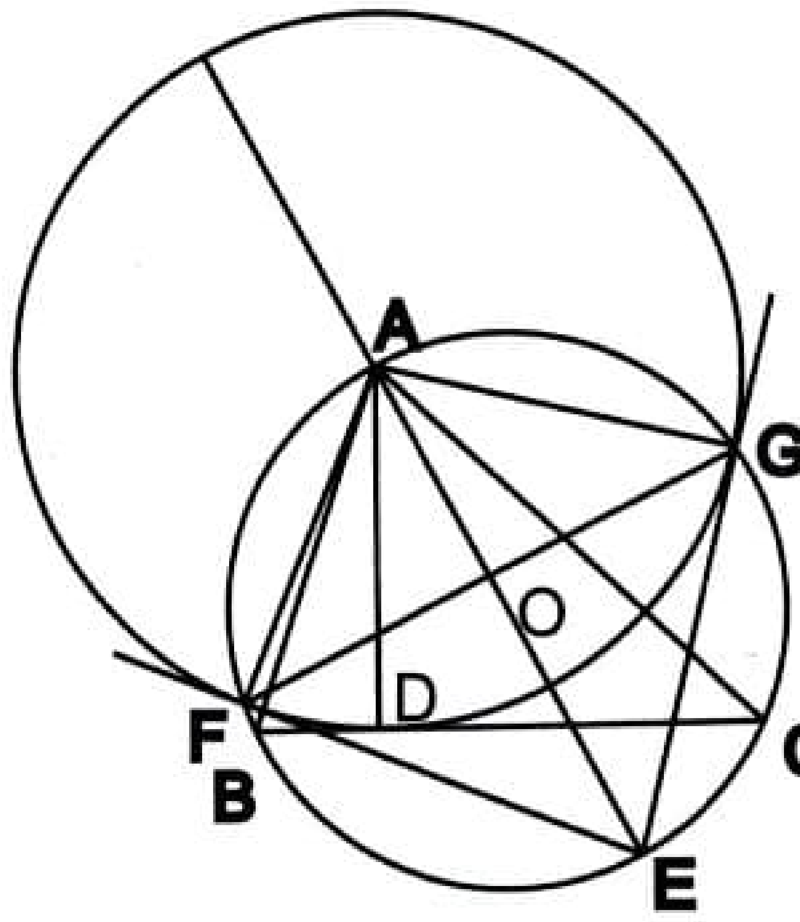

Figure 3: G3

Now, if in addition $FG \perp AO$, then since A is the centre of ω, in fact AO is the perpendicular bisector of FG. But by definition, since ABC is scalene, AO meets the perpendicular bisector of BC at O. Hence O is the centre of $BFGC$, and thus in fact $BFAGC$ is cyclic. But then the lines perpendicular to AF at F, and AG at G (the tangents to ω) must intersect at E, the point antipodal to A on $\odot BFAGC$. □

Alternative solution: Let the circumcircle of ABC be Γ. From the conditions, G is the reflection of F in the line AO. Let B', D' be the reflections of B, D across this same line AO. Clearly D' also lies on ω and B' lies on Γ.

Then, using directed angles, $\angle CGD = \angle DFB = \angle B'GD'$ so

$$\angle B'GC = \angle B'GD' - \angle CGD' = \angle CGD - \angle CGD' = \angle D'GD = \frac{1}{2}\angle D'AD = \angle OAD.$$

Then, exploiting the isogonality property that $\angle DAB = \angle CAO$, we have

$$\angle OAD = \angle CAB - 2\angle DAB = \angle ABC - \angle BCA = \angle ABC - \angle B'BA = \angle B'BC.$$

So G lies on Γ, and by the reflection property so does F.

But then, as in the previous solution, the tangents at F and G to ω must intersect at E, the point antipodal to A on Γ. □

G4. Given an acute triangle ABC, let M be the midpoint of BC and H the orthocentre. Let Γ be the circle with diameter HM, and let X, Y be distinct points on Γ such that AX, AY are tangent to Γ. Prove that $BXYC$ is cyclic.

Solution. Let D be the foot of the altitude from A to BC, which also lies on Γ. Let O be the circumcentre of $\triangle ABC$. Since $\angle HDM = 90°$, note that rays HD and HM meet the circumcircle at points which are reflections in OM. Then, since $\angle BAD = \angle OAC$, we recover the well-known fact that ray HM meets the circumcircle at A', the point antipodal to A. Therefore, the ray MH meets the circumcircle at a point T such that $\angle MTA = 90°$. Note that T, D lie on the circle with diameter AM.

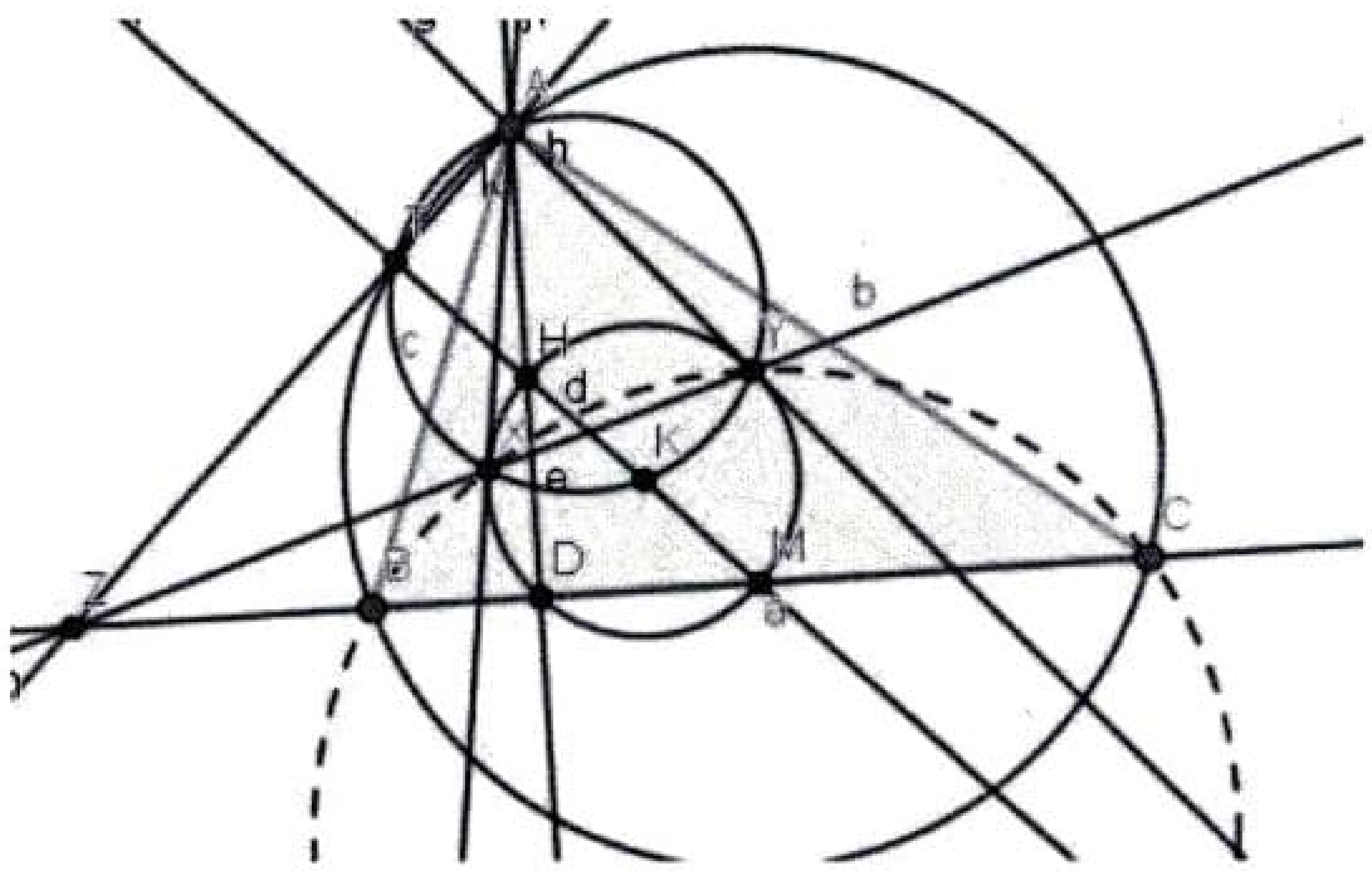

Figure 4: G4

Now, study K, the centre of Γ. Clearly $AXKY$ is cyclic, with diameter AK, so T also lies on this circle. We can now apply the radical axis theorem to the three circles $\odot ATXKY, \odot ATDM, \odot HXDMY$ to deduce that AT, XY, DM concur at a point, Z.

Then, by power of a point in $\odot ATXY$, we have $ZX \cdot ZY = ZT \cdot ZA$; but also by power of a point in the circumcircle, we have $ZA \cdot ZT = ZB \cdot ZC$. Therefore

$$ZX \cdot ZY = ZB \cdot ZC,$$

and the result follows. □

G5. Let ABC ($BC > AC$) be an acute triangle with circumcircle k centered at O. The tangent to k at C intersects the line AB at the point D. The circumcircles of triangles BCD, OCD and AOB intersect the ray CA (beyond A) at the points Q, P and K, respectively, such that $P \in (AK)$ and $K \in (PQ)$. The line PD intersects the circumcircle of triangle BKQ at the point T, so that P and T are in different halfplanes with respect to BQ. Prove that $TB = TQ$.

Solution. As DC is tangent to k at C then $\angle OCD = 90°$. Denote by X the midpoint of AB. Then $\angle OXA = 90°$ because of OX is the perpendicular bisector of the side AB. The pentagon $PXOCD$ is inscribed in the circle with diameter OD, hence $\angle PXA = \angle PXD = \angle PCD = \angle QCD = \angle QBA$ (the latter is due to $QBCD$ being cyclic). We deduce that $PX \parallel QB$ and that P is the midpoint of AQ, so $AP = PQ$.

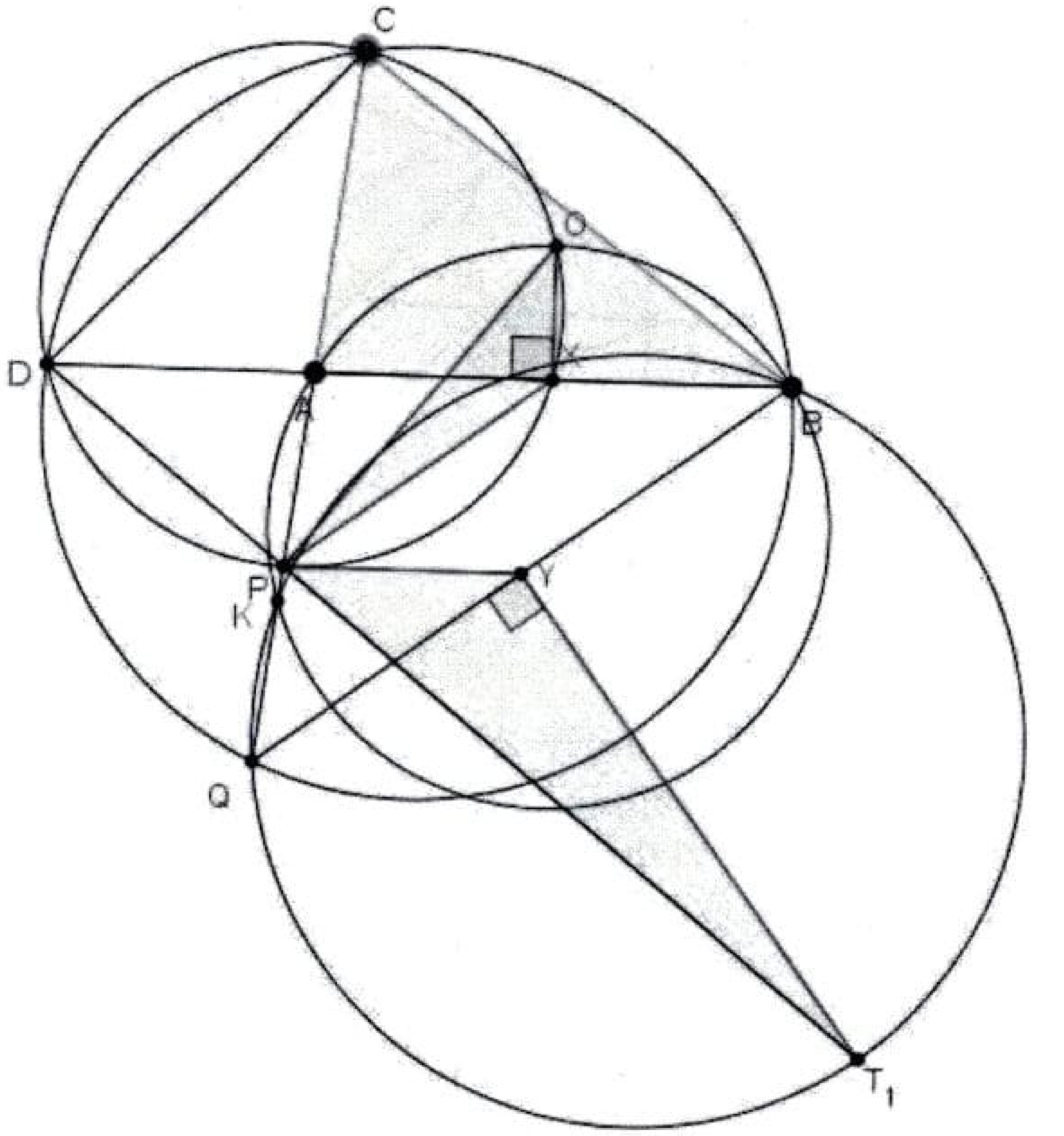

Figure 5: G5

Now let T_1 be the midpoint of the arc BQ, not containing K, from the circumcircle of $\triangle BKQ$, then $T_1B = T_1Q$. Due to $\angle DPO = 90°$, it suffices to show that $\angle OPT_1 = 90°$ – indeed, $T \equiv T_1$ and $TB = TQ$ would follow.

Denote by Y the midpoint of BQ. Then $\angle OXB = \angle T_1YB = 90°$. The quadrilateral $QKBT_1$ is inscribed in a circle, hence $\angle BT_1Q = 180 - \angle BKQ = \angle AKB$. Then $\angle XBO = \frac{1}{2}\angle AKB = \frac{1}{2}\angle BT_1Q = \angle BT_1Y$ and thus $\triangle OXB \sim \triangle BYT_1$. The quadrilaterals $PXBY$

and $AXYP$ are paralellograms, since XY and PY are middle lines of the triangle AQB. Consequently,

$$\frac{OX}{XP} = \frac{OX}{BY} = \frac{XB}{T_1Y} = \frac{PY}{T_1Y},$$

which along with $\angle PXB = \angle PYB$ and $\angle OXB = \angle T_1YB$ gives $\angle OXP = \angle PYT_1$ and $\triangle OXP \sim \triangle PYT_1$. Thus $\angle XPO = \angle YT_1P$ and $\angle POX = \angle T_1PY$.

In conclusion,

$$\angle OPT_1 = \angle XPY + \angle XPO + \angle YPT_1 = \angle PXA + \angle XPO + \angle XOP = 90^\circ.$$

□

G6. Let ABC be an acute triangle, and AX, AY two isogonal lines. Also, suppose that K, S are the feet of perpendiculars from B to AX, AY, and T, L are the feet of perpendiculars from C to AX, AY respectively. Prove that KL and ST intersect on BC.

Solution. Denote $\phi = \widehat{XAB} = \widehat{YAC}, \alpha = \widehat{CAX} = \widehat{BAY}$. Then, because the quadrilaterals ABSK and ACTL are cyclic, we have

$$\widehat{BSK} + \widehat{BAK} = 180^\circ = \widehat{BSK} + \phi = \widehat{LAC} + \widehat{LTC} = \widehat{LTC} + \phi,$$

so, due to the 90-degree angles formed, we have $\widehat{KSL} = \widehat{KTL}$. Thus, KLST is cyclic.

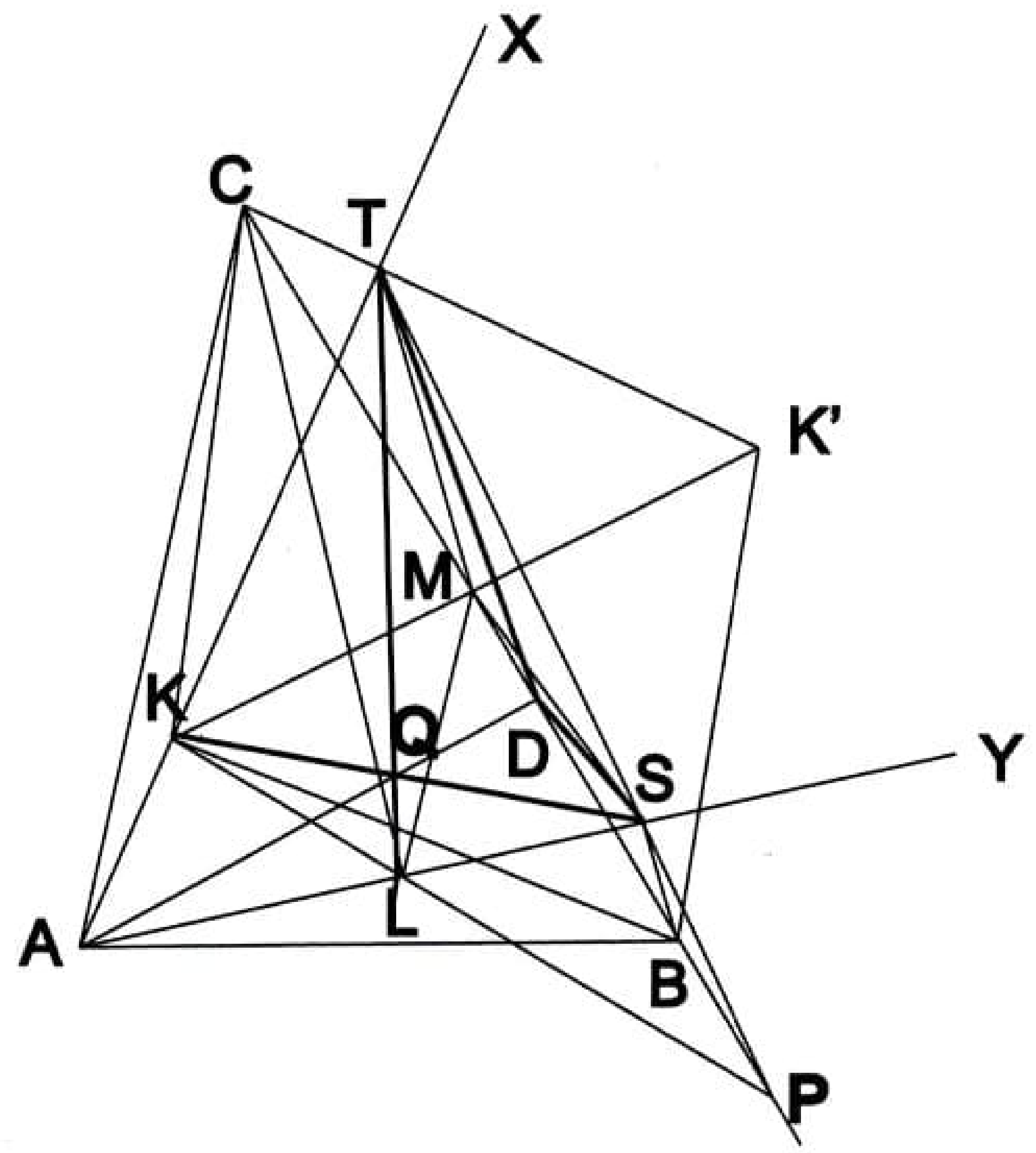

Figure 6: G6

Consider M to be the midpoint of BC and K' to be the symmetric point of K with respect to M. Then, $BKCK'$ is a parallelogram, and so $BK \| CK'$. But $BK \| CT$, because they are both perpendicular to AX. So, K' lies on CT and, as $\widehat{KTK'} = 90$ and M is the midpoint of KK', $MK = MT$. In a similar way, we have that $MS = ML$. Thus, the center of $(KLST)$ is M.

Consider D to be the foot of altitude from A to BC. Then, D belongs in both $(ABKS)$ and $(ACLT)$. So,

$$\widehat{ADT} + \widehat{ACT} = 180^\circ = \widehat{ABS} + \widehat{ADS} = \widehat{ADT} + 90^\circ - \alpha = \widehat{ADS} + 90^\circ - \alpha,$$

and AD is the bisector of $\widehat{SDT}$.

Because DM is perpendicular to AD, DM is the external bisector of this angle, and, as $MS = MT$, it follows that $DMST$ is cyclic. In a similar way, we have that $DMLK$ is also cyclic.

So, we have that ST, KL and DM are the radical axes of these three circles, $(KLST)$, $(DMST)$, $(DMKL)$. These lines are, therefore, concurrent, and we have proved the desired result. □

Alternative solution. We continue after proving that M is the center of $(KLST)$. If D is the foot of perpendicular from A to BC, then $ASDKB$ is cyclic, as well as $ATDLC$. The radical axes of those two circles and $(KLST)$ are concurrent, thus KS and LT intersect on point $Q \in AD$. So, if P is the intersection point of KL and TS, due to Brokard's theorem, AQ is perpendicular to MP. This is, of course, equivalent to proving that P belongs on BC. □

G7. Let AD, BE, and CF denote the altitudes of triangle $\triangle ABC$. Points E' and F' are the reflections of E and F over AD, respectively. The lines BF' and CE' intersect at X, while the lines BE' and CF' intersect at the point Y. Prove that if H is the orthocenter of $\triangle ABC$, then the lines AX, YH, and BC are concurrent.

Solution. We will prove that the desired point of concurrency is the midpoint of BC. Assume that $\triangle ABC$ is acute. Let (ABC)[5] intersect (AEF) at the point Y'; we will prove that $Y = Y'$.

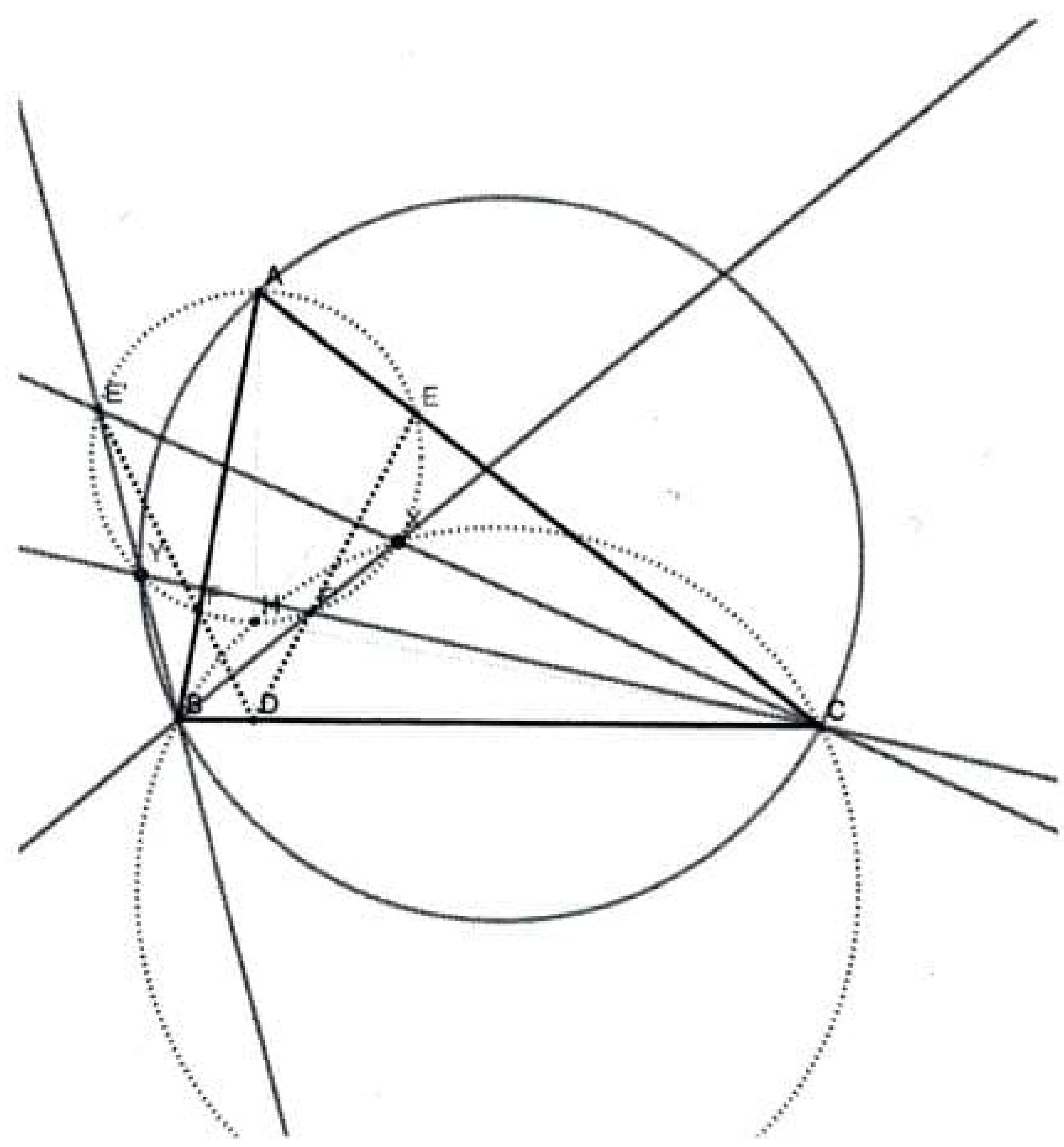

Figure 7: G7

Using the fact that H is the incenter of $\triangle DEF$ we get that D, E', F and D, F', E are triples of collinear points. Furthermore,

$$90^\circ = \angle^6 AEH = \angle AF'H = \angle AE'H = \angle AFH \Rightarrow F', E', H \in (AEFY').$$

We will now prove that the points Y', B, D, F' are concyclic. Indeed,

$$\angle Y'BD = \angle Y'BC = \angle Y'AC = \angle Y'AE = \angle Y'F'E \ \Rightarrow (Y', B, D, F').$$

Now, as

$$\angle F'Y'B = \angle F'DC = \angle EDC = \angle CAB = \angle CY'B,$$

the points C, F', Y' are collinear. Similarly we get that B, E', Y' are collinear, which implies

$$Y' = Y = (ABC) \cap (AEF).$$

[5] (XYZ) denotes the circumcircle of $\triangle XYZ$
[6] $\angle$ denotes a directed angle modulo π

Since we proved this property using directed angles, we know that it is also true for obtuse triangles.

Notice that the points A, B, C, H form an orthocentric system; in other words H is the orthocenter of $\triangle ABC$ and A is the orthocenter $\triangle HBC$. Furthermore, notice that F' is to $\triangle ABC$ as E' is to $\triangle HBC$ and that E' is to $\triangle ABC$ as F' is to $\triangle HBC$. This means that X is to $\triangle HBC$ as Y is to $\triangle ABC$ and, as we know the proven property is also true for obtuse triangles, we get

$$X = (HBC) \cap (AEF).$$

By *Reflecting the Orthocenter Lemma* we know that in a triangle ABC, the reflection of its orthocenter over the midpoint of BC is the antipode of A w.r.t. (ABC). Applying this *Lemma* on the triangles ABC and HBC we get that YH and AX both go through the midpoint of BC, thus finishing the solution. □

Remark 1: The crucial part of this solution is defining the points X, Y as intersections of circles. This can also be achieved directly by using similar triangles or by using the Spiral Similarity Lemma on $\triangle HBC, \triangle HF'E'$ and $\triangle ABC, \triangle AE'F'$.

Remark 2: We can also invert around A with radius $\sqrt{AH \cdot AD}$ or around H with radius $\sqrt{HA \cdot HD}$ to prove that X or Y invert to the midpoint of BC by using the existence of the nine-point circle. □

G8. Given an acute triangle ABC, (c) is circumcircle with center O and H the orthocenter of the triangle ABC. The line AO intersects (c) at the point D. Let D_1, D_2 and H_2, H_3 be the symmetrical points of the points D and H with respect to the lines AB, AC respectively. Let (c_1) be the circumcircle of the triangle AD_1D_2. Suppose that the line AH intersects again (c_1) at the point U, the line H_2H_3 intersects the segment D_1D_2 at the point K_1 and the line DH_3 intersects the segment UD_2 at the point L_1. Prove that one of the intersection points of the circumcircles of the triangles $D_1K_1H_2$ and UDL_1 lies on the line K_1L_1.

Solution. It is well known that the symmetrical points H_1, H_2, H_3 of H with respect the sides BC, AB, AC of the triangle ABC respectively lie on the circle (c).

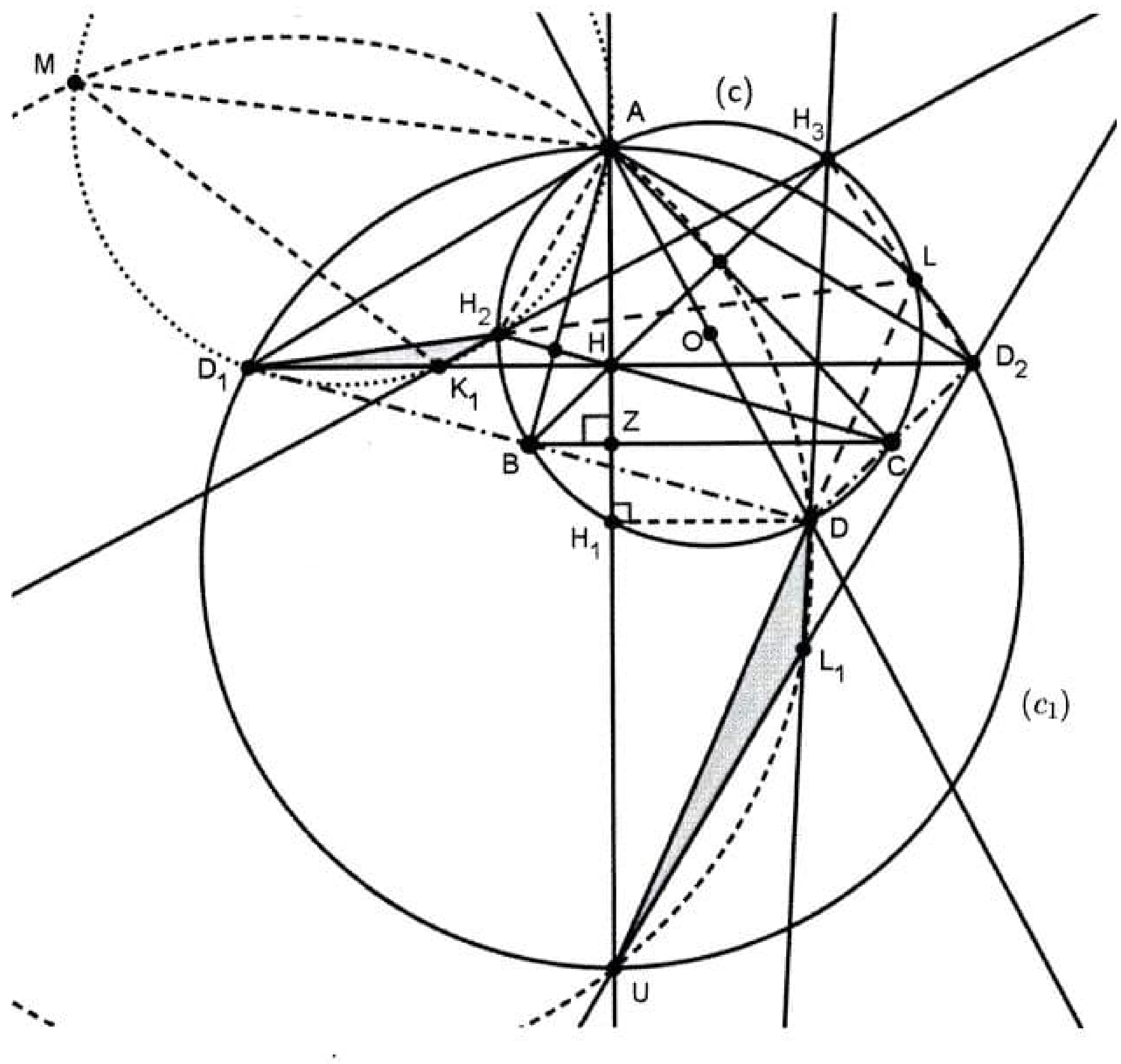

Figure 8: G8

Let L be the second point of intersection of (c) and (c_1). First we will prove that the lines D_1H_2, D_2H_3 and UD pass through the point L.

Suppose that the line AH intersects the side BC at the point Z. Since $H_1D \| BC \| D_1D_2$ and B, C are the midpoints of the segments D_1D, D_2D respectively, we get that Z is the midpoint of the segment HH_1, so the point H lies on D_1D_2. Therefore, $AH \perp D_1D_2$ and AU is a diameter of (c_1). Thus, $AL \perp UL$ and $AL \perp DL$. We have that the points U, D, L are collinear. (1)

Now, $\angle ALD_1 = \angle AD_2D_1$, $\angle ALH_2 = \angle ACH_2$. Since $AHCD_2$ is cyclic we get

$\angle ACH_2 = \angle AD_2D_1$. Therefore, $\angle ALH_2 = \angle ALD_1$. So the points D_1, H_2, L are collinear. (2)

Similarly,

$$\angle D_1LD_2 = \angle D_1AD_2 = 180^o - 2(\angle AD_1H).$$

Since AD_1BH is cyclic we have $\angle AD_1H = \angle ABH = \angle ABH_3$. Therefore, we get

$$\angle D_1LD_2 = 180^o - 2(\angle ABH_3) = 180^o - 2(\angle ADH_3) = 180^o - \angle H_2DH_3.$$

Thus,

$$\angle D_1LD_2 + \angle H_2DH_3 = 180^o \qquad \text{or } \angle D_1LD_2 + \angle H_3LH_2 = 180^o.$$

So the points H_3, L, D_2 are collinear. (3)

From (1), (2), (3) we have that the lines D_1H_2, D_2H_3 and UD are concurrent at the point L.

Also we have

$$\angle H_3DA = \angle D_2DA - \angle CDH_3 = \angle AD_2D - \angle CBH_3$$

and because BHD_2C is a parallelogram, we get $\angle CBH_3 = \angle HD_2C$. So

$$\angle H_3DA = \angle AD_2D - \angle HD_2C = \angle AD_2D_1 = \angle AD_1D_2 = \angle AUD_2.$$

Therefore, the circumcircle of the triangle UDL_1 passes through the point A. Also, $\angle AD_1K_1 = \angle D_2D_1A = \angle D_2UA$. But AUL_1D is cyclic and we have $\angle D_2UA = \angle H_3DA = \angle H_3BA = \angle H_3H_2A$. Therefore, $\angle AD_1K_1 = \angle H_3H_2A$. Thus, the circumcircle of the triangle $D_1K_1H_2$ passes through the point A.

Because the points H_3, L, D_2 are collinear by the Desargues theorem, the lines UD_1, L_1K_1, DH_2 are concurrent, let say in the point M.

From the similarity of the triangles UDL_1 and $D_1K_1H_2$ we conclude that M is the center of unique spiral similarity and because the circumcircles of the triangles $D_1K_1H_2$ and UDL_1 intersect at the point A, then the second point of intersection is M. Therefore, M lies on the line K_1L_1. □

Comment. We can prove the last part in a different way.

Let M be the point of intersection of the circumcircles of the triangles $D_1K_1H_2$ and UDL_1. Now, we have

$$\angle K_1MA = \angle H_3H_2A = \angle H_3BA = \angle ADH_3 = \angle L_1UA = \angle L_1MA.$$

Therefore, the points L_1, K_1, M are collinear. □

G9. Given semicircle (c) with diameter AB and center O. On the (c) we take point C such that the tangent at the C intersects the line AB at the point E. The perpendicular line from C to AB intersects the diameter AB at the point D. On the (c) we get the points H, Z such that $CD = CH = CZ$. The line HZ intersects the lines CO, CD, AB at the points S, I, K respectively and the parallel line from I to the line AB intersects the lines CO, CK at the points L, M respectively. We consider the circumcircle(k) of the triangle LMD, which intersects again the lines AB, CK at the points P, U respectively. Let (e_1), (e_2), (e_3) be the tangents of the (k) at the points L, M, P respectively and $R = (e_1) \cap (e_2)$, $X = (e_2) \cap (e_3)$, $T = (e_1) \cap (e_3)$. Prove that if Q is the center of (k), the lines RD, TU, XS pass through the same point, which lies in the line IQ.

Solution. Since $CH = CZ$ we have $OC \perp HZ$. So from the cyclic quadrilateral $SODI$ we get

$$CS \cdot CO = CI \cdot CD. \tag{1}$$

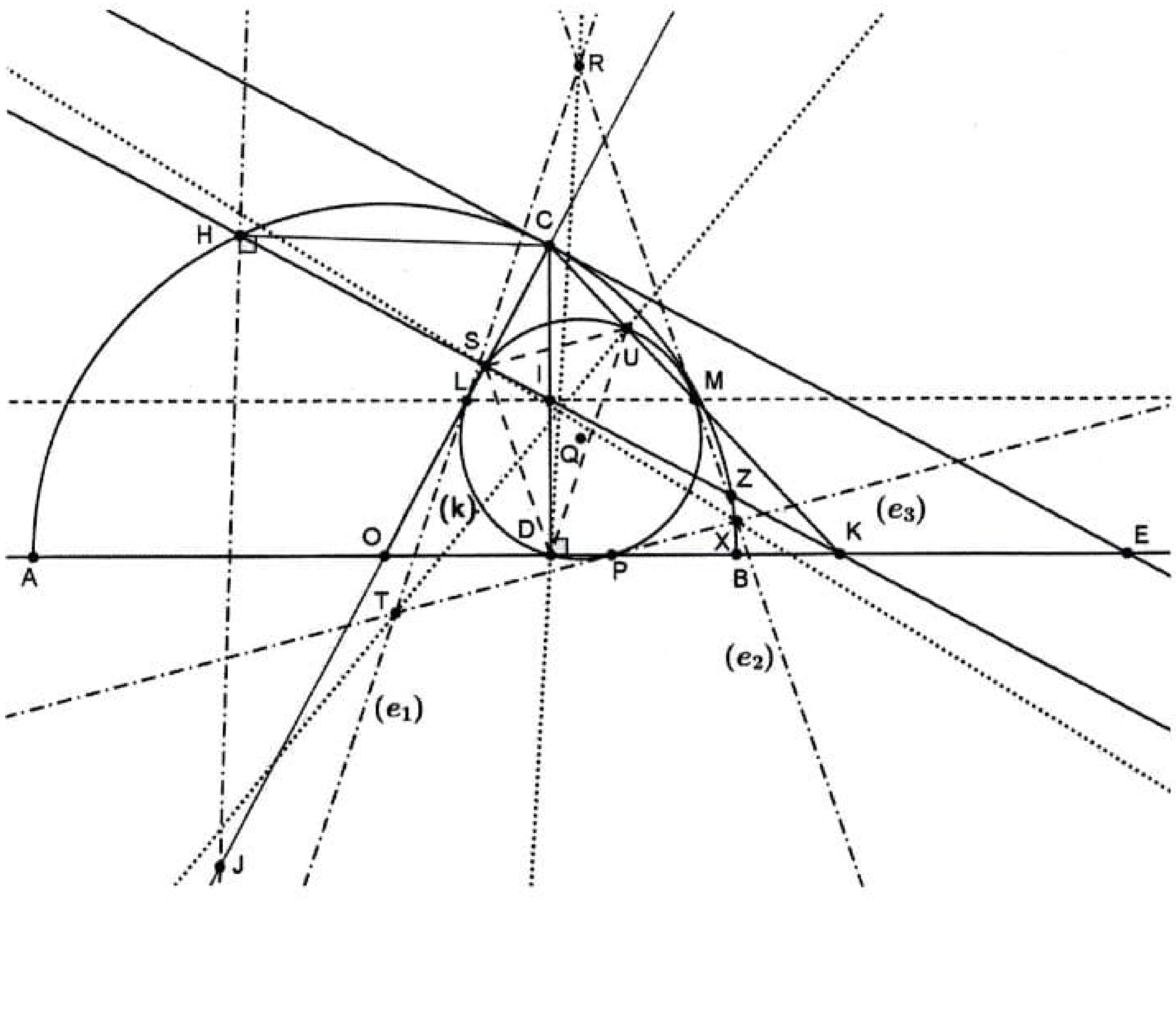

Figure 9: G9

We draw the perpendicular line (v) to HC at the point H. Let J be the intersection point of lines (v) and CO. Then CJ is diameter of the circle (O, OA) and

$$CJ = 2CO. \tag{2}$$

From the right triangle JHC we have

$$HC^2 = CS \cdot CJ. \tag{3}$$

Therefore, from (1), (2) and (3) we get

$$CS \cdot \frac{1}{2} CJ = CI \cdot CD \qquad \text{or} \qquad HC^2 = 2CI \cdot CD. \tag{3}$$

However $HC = CD$ and thus $CD = 2CI$. Thus, I is the midpoint of the segment CD. Nevertheless, $LM \| OK$, so the points L, M are the midpoints of the sides CO and CK respectively. Therefore, the circumcircle (k) of the triangle LMD is the *Euler circle* of the COK and thus it passes through the point S.

We have $QS = QU$ and from the right triangles OSK, OUK we get $PS = PU = \frac{OK}{2}$.

Therefore, the points P, Q are located on the perpendicular bisector of the segment SU. Now, we conclude that $SU \| TX$, because $QP \perp (e_3)$. Similarly, we prove that $DU \| RT$ and $SD \| RX$.

Since the triangles SUD and XTR are homothetic we get that the lines RD, TU, XS are concurrent at the center $\mathcal{M}$ of homothety.

The points I and Q atre the incenters of homothetic triangles SUD and XTR, respectively. Thus, the line IQ passes through the point $\mathcal{M}$. □

NUMBER THEORY

TN1. Let $\mathbb{P}$ be the set of all prime numbers. Find all functions $f : \mathbb{P} \to \mathbb{P}$ such that

$$f(p)^{f(q)} + q^p = f(q)^{f(p)} + p^q$$

holds for all $p, q \in \mathbb{P}$.

Solution. Obviously, the identical function $f(p) = p$ for all $p \in \mathbb{P}$ is a solution. We will show that this is the only one.

First we will show that $f(2) = 2$. Taking $q = 2$ and p any odd prime number, we have

$$f(p)^{f(2)} + 2^p = f(2)^{f(p)} + p^2.$$

Assume that $f(2) \neq 2$. It follows that $f(2)$ is odd and so $f(p) = 2$ for any odd prime number p.

Taking any two different odd prime numbers p, q we have

$$2^2 + q^p = 2^2 + p^q \;\Rightarrow\; p^q = q^p \;\Rightarrow p = q,$$

contradiction. Hence, $f(2) = 2$.

So for any odd prime number p we have

$$f(p)^2 + 2^p = 2^{f(p)} + p^2.$$

Copy this relation as

$$2^p - p^2 = 2^{f(p)} - f(p)^2. \tag{1}$$

Let T be the set of all positive integers greater than 2, i.e. $T = \{3, 4, 5, \ldots\}$. The function $g : T \to \mathbb{Z}$, $g(n) = 2^n - n^2$, is strictly increasing, i.e.

$$g(n+1) - g(n) = 2^n - 2n - 1 > 0 \tag{2}$$

for all $n \in T$. We show this by induction. Indeed, for $n = 3$ it is true, $2^3 - 2 \cdot 3 - 1 > 0$. Assume that $2^k - 2k - 1 > 0$. It follows that for $n = k + 1$ we have

$$2^{k+1} - 2(k+1) - 1 = (2^k - 2k - 1) + (2^k - 2) > 0$$

for any $k \geq 3$. Therefore, (2) is true for all $n \in T$.

As consequence, (1) holds if and only if $f(p) = p$ for all odd prime numbers p, as well as for $p = 2$.

Therefore, the only function that satisfies the given relation is $f(p) = p$, for all $p \in \mathbb{P}$.

□

TN2. Let $S \subset \{1, \ldots, n\}$ be a nonempty set, where n is a positive integer. We denote by s the greatest common divisor of the elements of the set S. We assume that $s \neq 1$ and let d be its smallest divisor greater than 1. Let $T \subset \{1, \ldots, n\}$ be a set such that $S \subset T$ and $|T| \geq 1 + \left[\frac{n}{d}\right]$. Prove that the greatest common divisor of the elements in T is 1.

Solution. Let t be the greatest common divisor of the elements in T. Due to the fact that $S \subset T$, we immediately get that t/s. Let us assume for the sake of contradiction that $t \neq 1$. From the previous observation we get that $t \geq d$.

By taking into account that $|T| \geq 1 + \left[\frac{n}{d}\right]$, we infer that we can find at least $1 + \left[\frac{n}{d}\right]$ elements in T. All of them will be divisible by t, and the largest of them, which we shall denote by M, will be at least $t \cdot \left(1 + \left[\frac{n}{d}\right]\right)$. On the other hand, $t \geq d$, hence

$$M \geq t \cdot \left(1 + \left[\frac{n}{d}\right]\right) \geq d \cdot \left(1 + \left[\frac{n}{d}\right]\right) > d \cdot \frac{n}{d} = n.$$

Therefore, $M > n$, which contradicts the fact that $M \in \{1, \ldots, n\}$.

In conclusion, $t = 1$, as desired. □

TN2b.[7] Let $n\,(n \geq 1)$ be a positive integer and $U = \{1, \ldots, n\}$. Let S be a nonempty subset of U and let d $(d \neq 1)$ be the smallest common divisor of all elements of the set S. Find the smallest positive integer k such that for any subset T of U, consisting of k elements, with $S \subset T$, the greatest common divisor of all elements of T is equal to 1.

Solution. We will show that $k_{\min} = 1 + \left[\frac{n}{d}\right]$ (here $[\cdot]$ denotes the integer part).

Obviously, the number of elements of S is not grater than $\left[\frac{n}{d}\right]$, i.e. $|S| \leq \left[\frac{n}{d}\right]$, and $S \neq U$.

If $S \subset T$ and the greatest common divisor of elements of T is equal to 1, then $|T| \geq |S| + 1$.

1) Assume that $|S| < \left[\frac{n}{d}\right]$. Let T be the subset of U, consisting of all multiples of d in U. Thus, $|T| = \left[\frac{n}{d}\right]$ and $S \subset T$. Therefore, the greatest common divisor of all elements of T is $d > 1$. Thus, $k \geq 1 + \left[\frac{n}{d}\right]$.

2) Assume $|S| = \left[\frac{n}{d}\right]$. Let T be any subset of U with $S \subset T, S \neq T$. Therefore, $|T| \geq 1 + \left[\frac{n}{d}\right]$. Let q be the greatest common divisor of all elements of T. Assume that $q > 1$. Therefore, q is a common divisor of all elements of S as well. Hence, $q \geq d$. It follows that $|T| \leq \left[\frac{n}{q}\right] \leq \left[\frac{n}{d}\right]$, contradiction. Hence, $q = 1$.

Therefore, the minimal possible value of k is $1 + \left[\frac{n}{d}\right]$. □

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[7] Proposed by PSC.

COMBINATORICS

C1. 100 couples are invited to a traditional Moldovan dance. The 200 people stand in a line, and then in a *step*, two of them (not necessarily adjacent) may swap positions. Find the least C such that whatever the initial order, they can arrive at an ordering where everyone is dancing next to their partner in at most C steps.

Solution. With 100 replaced by N, the answer is $C = C(N) = N - 1$. Throughout, we will say that the members of a couple have the same.

N=2: We use this as a base case for induction for both bounds. Up to labelling, there is one trivial initial order, and two non-trivial ones, namely

$$1, 1, 2, 2; \qquad 1, \overbracket{2, 2, 1}; \qquad 1, \overbracket{2, 1}, 2.$$

The brackets indicate how to arrive at a suitable final ordering with one step. Obviously one step is necessary in the second and third cases.

Upper bound: First we show $C(N) \leq N - 1$, by induction. The base case $N = 2$ has already been seen. Now suppose the claim is true for $N - 1$, and consider an initial arrangement of N couples. Suppose the types of the left-most couples in line are a and b. If $a \neq b$, then in the first step, swap the b in place two with the other person with type a. If $a = b$, skip this. In both cases, we now have $N - 1$ couples distributed among the final $2N - 2$ places, and we know that $N - 2$ steps suffices to order them appropriately, by induction. So $N - 1$ steps suffices for N couples.

Lower bound: We need to exhibit an example of an initial order for which $N - 1$ steps are necessary. Consider

$$\mathcal{A}_N := 1, 2, 2, 3, 3, \ldots, N-1, N-1, N, N, 1. \tag{1}$$

Proceed by induction, with the base case $N = 2$ trivial. Suppose there is a sequence of at most $N - 2$ steps which works. In any suitable final arrangement, a given type must be in positions (odd, even), whereas they start in positions (even, odd). So each type must be involved in at least one step. However, each step involves at most two types, so by the pigeonhole principle, at least four types are involved in at most one step. Pick one such type $a \neq 1$. The one step involving a must be one of

$$\ldots, \overbracket{?, a, a}, ?, \ldots \qquad \ldots, ?, \overbracket{a, a, ?}, \ldots.$$

Neither of these steps affects the relative order of the $2N - 2$ other people. So by ignoring this step involving the a, we have a sequence of at most $N - 3$ steps acting on the other $2N - 2$ people which appropriately sorts them. By induction, this is a contradiction. $\square$

Alternative lower bound I: Consider the graph with vertices given by pairs of positions $\{(1, 2), (3, 4), \ldots, (2N - 1, 2N)\}$. We add an edge between pairs of (different) vertices if we ever swap two people in places corresponding to those vertices. In particular, at the end, the two people with type k end up in places corresponding to a single vertex.

Suppose we start from the ordering (1) and have some number of steps leading to an ordering where everyone is next to their partner. Then, in the induced graph, there is a path between the vertices corresponding to the places $(2k-3, 2k-2)$ and $(2k-1, 2k)$ for each $2 \leq k \leq N$, and also between $(1, 2)$ and $(2N-1, 2N)$. In other words, the graph is connected, and so must have at least $N-1$ edges. □

Alternative lower bound II: Consider a bipartite multigraph with vertex classes $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$. Connect v_i to w_j if a person of type j is in positions $(2i-1, 2i)$ (if both positions are taken by the type j couple, then add two edges).

Each step in the dance consists of replacing edges $E = \{v_a \leftrightarrow w_c, v_b \leftrightarrow w_d)\}$ with $E' = \{v_a \leftrightarrow w_d, v_b \leftrightarrow w_c\}$. However, both before and after the step, the number of components in the graph which include $\{v_a, v_b, w_c, w_d\}$ is either one or two. The structure of other components which do not include these vertices is unaffected by the move.

Therefore, the number of connected components increases by at most 1 in each step.

Starting from configuration (1), the graph initially consists of a single (cyclic) component, so one requires at least $n-1$ steps to get to the final configuration for which there are n connected components. □

C2. Suppose that the numbers $\{1, 2, \ldots, 25\}$ are written in some order in an 5×5 array. Find the maximal positive integer k, such that the following holds. There is always an 2×2 subarray whose numbers have a sum not less than k.

C2b.[8] An 5×5 array must be completed with all numbers $\{1, 2, \ldots, 25\}$, one number in each cell. Find the maximal positive integer k, such that for any completion of the array there is a 2×2 square (subarray), whose numbers have a sum not less than k.

Solution. We will prove that $k_{\max} = 45$.

We number the columns and the rows and we select all possible $3^2 = 9$ choices of an odd column with an odd row.

Collecting all such pairs of an odd column with an odd row, we double count some squares. Indeed, we take some 3^2 squares 5 times, some 12 squares 3 times and there are some 4 squares (namely all the intersections of an even column with an even row) that we don't take in such pairs.

It follows that the maximal total sum over all 3^2 choices of an odd column with an odd row is

$$5 \times (17 + 18 + \cdots + 25) + 3 \times (5 + 6 + \cdots + 16) = 1323.$$

So, by an averaging argument, there exists a pair of an odd column with an odd row with sum at most $\dfrac{1323}{9} = 147$.

Then all the other squares of the array will have sum at least

$$(1 + 2 + \cdots + 25) - 147 = 178.$$

But for these squares there is a tiling with 2×2 arrays, which are 4 in total. So there is an 2×2 array, whose numbers have a sum at least $\dfrac{178}{4} > 44$. So, there is a 2×2 array whose numbers have a sum at least 45. This argument gives that

$$k_{\max} \geq 45. \qquad (1)$$

We are going now to give an example of an array, in which 45 is the best possible. We fill the rows of the array as follows:

25	5	24	6	23
11	4	12	3	13
22	7	21	8	20
14	2	15	1	16
19	9	18	10	17

We are going now to even rows:

In the above array, every 2×2 subarray has a sum, which is less or equal to 45. This gives that

$$k_{\max} \leq 45. \qquad (2)$$

A combination of (1) and (2) gives that $k_{\max} = 45$. □

[8]Proposed by PSC.

C3. Anna and Bob play a game on the set of all points of the form (m, n) where m, n are integers with $|m|, |n| \leqslant 2019$. Let us call the lines $x = \pm 2019$ and $y = \pm 2019$ the *boundary lines* of the game. The points of these lines are called the *boundary points*. The *neighbours* of point (m, n) are the points $(m+1, n), (m-1, n), (m, n+1), (m, n-1)$.

Anna starts with a token at the origin $(0, 0)$. With Bob playing first, they alternately perform the following steps: At his turn, Bob *deletes* two points on each boundary line. On her turn Anna makes a sequences of three moves of the token, where a *move* of the token consists of picking up the token from its current position and placing it in one of its neighbours.

To win the game Anna must place her token on a boundary point before it is deleted by Bob. Does Anna have a winning strategy?

[**Note:** At every turn except perhaps her last, Anna **must** make **exactly** three moves.]

Solution. Anna does not have a winning strategy. We will provide a winning strategy for Bob. It is enough to describe his strategy for the deletions on the line $y = 2019$.

Bob starts by deleting $(0, 2019)$ and $(-1, 2019)$. Once Anna completes her step, he deletes the next two available points on the left if Anna decreased her x-coordinate, the next two available points on the right if Anna increased her x-coordinate, and the next available point to the left and the next available point to the right if Anna did not change her x-coordinate. The only exception to the above rule is on the very first time Anna decreases x by exactly 1. In that step, Bob deletes the next available point to the left and the next available point to the right.

Bob's strategy guarantees the following: If Anna makes a sequence of steps reaching $(-x, y)$ with $x > 0$ and the exact opposite sequence of moves in the horizontal direction reaching (x, y) then Bob deletes at least as many points to the left of $(0, 2019)$ in the first sequence than points to the right of $(0, 2019)$ in the second sequence.

So we may assume for contradiction that Anna wins by placing her token at $(k, 2019)$ for some $k > 0$.

Define $\Delta = 3m - (2x + y)$ where m is the total number of points deleted by Bob to the right of $(0, 2019)$, and (x, y) is the position of Anna's token.

For each sequence of steps performed first by Anna and then by Bob, Δ does not decrease. This can be seen by looking at the following table exhibiting the changes in $3m$ and $2x + y$. We have excluded the cases where $2x + y < 0$.

Step	(0,3)	(1,2)	(-1,2)	(2,1)	(0,1)	(3,0)	(1,0)	(2,-1)	(1,-2)
m	1	2	0 (or 1)	2	1	2	2	2	2
$3m$	3	6	0 (or 3)	6	3	6	6	6	6
$2x+y$	3	4	0	5	1	6	2	3	0

The table also shows that if in this sequence of steps Anna changes y by $+1$ or -2 then Δ is increased by 1. Also, if Anna changes y by $+2$ or -1 then the first time this happens Δ is increased by 2. (This also holds if her move is $(0, -1)$ or $(-2, -1)$ which are not shown in the table.)

Since Anna wins by placing her token at $(k, 2019)$ we must have $m \leqslant k - 1$ and $k \leqslant 2018$. So at that exact moment we have:

$$\Delta = 3m - (2k + 2019) = k - 2022 \leqslant -4.$$

So in her last turn she must have decreased Δ by at least 4. So her last step must have been $(1,2)$ or $(2,1)$ which give a decrease of 4 and 5 respectively. (It could not be $(3,0)$ because then she must have already won. Also she could not have done just one or two moves in her last turn since this is not enough for the required decrease in Δ.)

If her last step was $(1,2)$ then just before doing it we had $y = 2017$ and $\Delta = 0$. This means that in one of her steps the total change in y was not 0 mod 3. However in that case we have seen that $\Delta > 0$, a contradiction.

If her last step was $(2,1)$ then just before doing it we had $y = 2018$ and $\Delta = 0$ or $\Delta = 1$. So she must have made at least two steps with the change of y being $+1$ or -2 or at least one step with the change of y being $+2$ or -1. In both cases, consulting the table, we get an increase of at least 2 in Δ, a contradiction.

Note 1: If Anna is allowed to make **at most** three moves at each step, then she actually has a winning strategy.

Note 2: If 2019 is replaced by $N > 1$ then Bob has a winning strategy if and only if $3 \mid N$. □

C4. A town-planner has built an isolated city whose road network consists of $2N$ roundabouts, each connecting exactly three roads. A series of tunnels and bridges ensure that all roads in the town meet only at roundabouts. All roads are two-way, and each roundabout is oriented clockwise.

Vlad has recently passed his driving test, and is nervous about roundabouts. He starts driving from his house, and always takes the first exit at each roundabout he encounters. It turns out his journey includes every road in the town in both directions before he arrives back at the starting point in the starting direction. For what values of N is this possible?

Solution. N odd. In fact, the number of trajectories has the same parity as N.

The setting is a (multi)graph where every vertex has degree three. Each vertex has an *orientation*, an ordering of its incident edges. We call Vlad's possible paths *trajectories*, and a *complete trajectory* if he traverses every edge in both directions. We may assume the multigraph is connected, as otherwise a complete trajectory is certainly not possible.

N odd (construction): There is an example when $N = 1$, as shown in Figure 10.

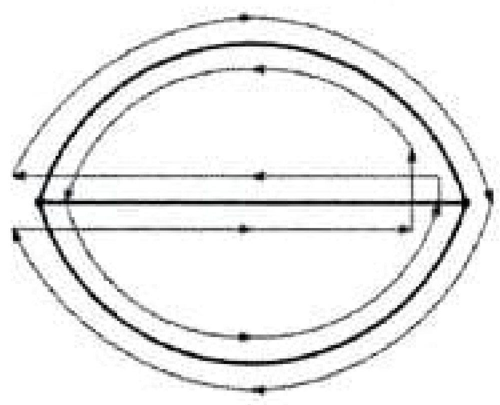

Figure 10: C4: $N = 1$

There are two 3-regular graphs on two vertices, the *handcuffs* and *theta*. The handcuffs fail since each self-loop has its own trajectory, but the theta does work for two of the four possible orientations.

We now construct examples for $N \geq 3$ odd by induction. Suppose we have a valid 3-regular graph on $2(N-2)$ vertices, such that Vlad's trajectory is complete. This has at least two (undirected) edges, so pick two of them, e and e'. (It *does not matter* if they share incident vertices.) Split both e and e' into three, by adding two new vertices to each, and connect as in Figure 11.

New vertices have degree three; other degrees are unchanged, so the graph is still 3-regular. For each edge e and e', pick a direction. (Both *up* in the figure.) These directed edges are part of the complete trajectory given by the induction hypothesis. Choose the orientations of the new vertices to preserve these two sections of the trajectory. The remaining two directed edges in the original graph will end up as partial trajectories in the new graph (see Figure 11).

However, because all the new partial trajectories start and finish at the same places and in the same directions in the original graph, and no other directed edges are changed, the trajectory remains complete. The result for N odd follows by induction.

N even: Split each edge e in the graph into two directed edges $\overleftarrow{e}$ and $\overrightarrow{e}$. Let D be the set of the $6N$ directed edges. Let α be the permutation of D which exchanges $\overleftarrow{e}$

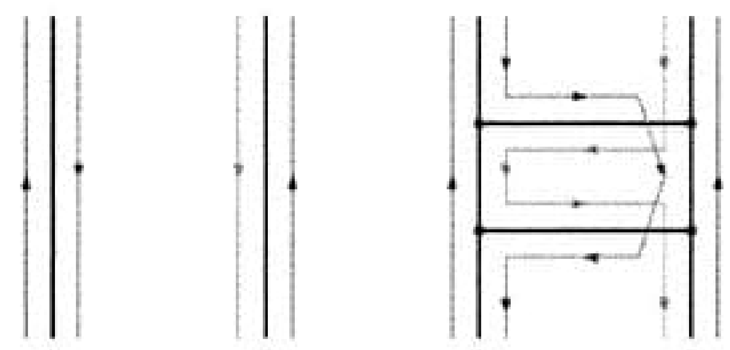

Figure 11: C4: Trajectories in the old and new graphs

and $\overrightarrow{e}$.

Now, for each roundabout v, let $\overleftarrow{e_1}, \overleftarrow{e_2}, \overleftarrow{e_3}$ be the three directed edges *into* v. The roundabout has a cyclic orientation, either $(\overleftarrow{e_1}, \overleftarrow{e_2}, \overleftarrow{e_3})$ or $(\overleftarrow{e_1}, \overleftarrow{e_3}, \overleftarrow{e_2})$. Let $\theta(\overleftarrow{e_1})$ describe the directed edge after $\overleftarrow{e_1}$ in this orientation. By considering all roundabouts, θ is also a permutation of D.

Note that $\theta(\overleftarrow{e_1})$ is directed *towards* v, so the directed edge after $\overleftarrow{e_1}$ in a trajectory is $\alpha(\theta(\overleftarrow{e_1}))$. So Vlad makes a complete trajectory precisely if $\alpha\theta$ is a cyclic permutation of D. Note that the cycle type of θ is $(3, 3, \dots, 3)$, and the cycle type of α is $(2, 2, \dots, 2)$. So θ is always an even permutation, while α is an even permutation precisely when N is even.

However, a cyclic permutation of D is always odd, since $|D| = 6N$ is even. So there is certainly no complete trajectory when N is even. □

Alternative I: We claim that in a graph with E edges, and V vertices, the number of trajectories, T, has the same parity as $V + E$. We allow degenerate cases of this statement, for example graphs that are disconnected, or trajectories that consist of only a single vertex, so that the graph that consists of V vertices and no edges has precisely V trajectories, and thus satisfies the given claim. This shows that N cannot be even.

We prove the claim by induction on E. Suppose we are given a graph with $E \geq 1$ edges and T trajectories. Then consider any edge e, and its two directions $\overrightarrow{e}, \overleftarrow{e}$. Let A be the sequence of directed edges starting from the one after $\overrightarrow{e}$ in its trajectory, ending at the edge before $\overleftarrow{e}$ or $\overrightarrow{e}$, whichever appears first. Similarly define B starting after $\overleftarrow{e}$. A and B are disjoint, and may be empty.

Figure 12: C4: (a) Initial trajectories. (b) After removing e

We consider removing e, but otherwise keep the orientations at its incident vertices the same. Then if $\overrightarrow{e}, \overleftarrow{e}$ are in different trajectories, these are the concatenations $(\overrightarrow{e}, A)$ and $(\overleftarrow{e}, B)$. After removing e, for each direction $\overleftarrow{e}, \overrightarrow{e}$, instead of proceeding onto this directed edge, the relevant trajectory moves to the *other* trajectory. In other words, the resulting trajectory is the concatenation (A, B). So T decreases by one.

Similarly, if both directions of e are part of the same trajectory, this is the concatenation $(\overrightarrow{e}, A, \overleftarrow{e}, B)$. Then when we remove e, this splits into the two trajectories (A) and (B), by an essentially identical argument. So T increases by one. Thus in both cases, removing one edge changes the parity of T, and so the claim follows by induction on E.

In the original setting we have $V = 2N$, $E = 3N$, so T must have the same parity as $5N$. Thus $T = 1$ is impossible when N is even. □

Alternative II: An alternative is to induct on N, using the following stronger claim.

Claim: You can't have exactly one trajectory for N even; nor exactly two trajectories for a connected graph with N odd.

Proof of claim: We have to check that the claim is true for $N = 1, 2$. Checking $N = 2$ requires a couple of case. Alternatively, one can argue that a single cyclic edge with no vertices (!) counts as the case $N = 0$. Now use strong induction by contradiction. If N is even, but has exactly one trajectory, then there are no self-loops, so pick any edge e, connecting vertices $v \neq w$. Remove e, then remove v, and connect v's other two incident edges (which are distinct from each other and e) to form a single edge. Do the same for w.

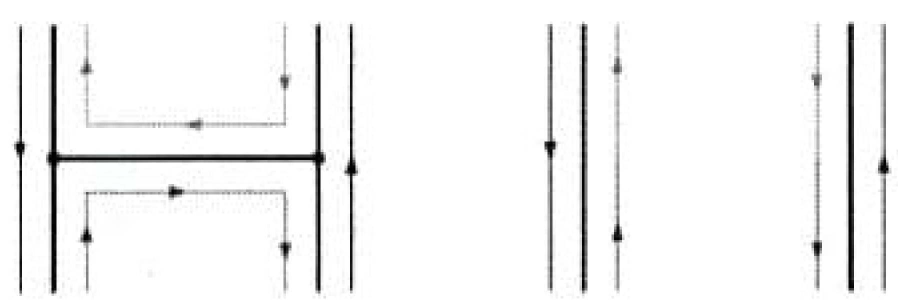

Figure 13: C4: Trajectories in the old and new graphs

The effect on the trajectories is shown in Figure 13. Note that the new graph is still 3-regular. We then argue as in Proof I that this operation splits the trajectory into two. So if the new graph is connected, this contradicts the hypothesis for $N - 1$. Alternately, the new graph might consist of two components. Since it is 3-regular, each component has an even number of vertices. The total number of vertices is $2(N - 1)$, which is 2 modulo 4, and so one of the components has a number of vertices which is a multiple of four, and a complete trajectory of this component, which also contradicts the induction hypothesis.

Now suppose N is odd, but the original oriented graph has exactly two trajectories. If there is a self-loop at some vertex v, then one of the trajectories involves only this self-loop. So remove this vertex, and consider the other vertex w connected to v. Remove w and join up its other two incident edges. The resulting graph corresponds to N even, and has a complete trajectory, which is a contradiction.

Otherwise, there are no self-loops, but the graph is connected hence there must be one edge e connecting vertices $v \neq w$ which has one trajectory in one direction, and the other trajectory in the other direction. Collapse this edge as in Figure 13, and again by the same argument as in Proof I, this merges the two trajectories, giving a complete trajectory for N even, and a contradiction. □