XV APMO: Solutions and Marking Schemes

1. Let a, b, c, d, e, f be real numbers such that the polynomial

$$p(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \ldots, 8$. Determine all possible values of f.

Solution.

From

$$x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = (x - x_1)(x - x_2)\ldots(x - x_8)$$

we have

$$\sum_{i=1}^{8} x_i = 4 \qquad \text{and} \qquad \sum x_i x_j = 7,$$

where the second sum is over all pairs (i, j) of integers where $1 \le i < j \le 8$. Since this sum can also be written

$$\frac{1}{2}\left[\left(\sum_{i=1}^{8} x_i\right)^2 - \sum_{i=1}^{8} x_i^2\right],$$

we get

$$14 = \left(\sum_{i=1}^{8} x_i\right)^2 - \sum_{i=1}^{8} x_i^2 = 16 - \sum_{i=1}^{8} x_i^2,$$

so

$$\sum_{i=1}^{8} x_i^2 = 2 \qquad \text{while} \qquad \sum_{i=1}^{8} x_i = 4. \qquad \textbf{[3 marks]} \tag{1}$$

Now

$$\sum_{i=1}^{8}(2x_i - 1)^2 = 4\sum_{i=1}^{8} x_i^2 - 4\sum_{i=1}^{8} x_i + 8 = 4(2) - 4(4) + 8 = 0,$$

which forces $x_i = 1/2$ for all i. **[3 marks]** Therefore

$$f = \prod_{i=1}^{8} x_i = \left(\frac{1}{2}\right)^8 = \frac{1}{256}. \qquad \textbf{[1 mark]}$$

Alternate solution: After obtaining (1) **[3 marks]**, use Cauchy's inequality to get

$$16 = (x_1 \cdot 1 + x_2 \cdot 1 + \cdots + x_8 \cdot 1)^2 \le (x_1^2 + x_2^2 + \cdots + x_8^2)(1^2 + 1^2 + \cdots + 1^2) = 8 \cdot 2 = 16\ ;$$

or the power mean inequality to get

$$\frac{1}{2} = \frac{1}{8}\sum_{i=1}^{8} x_i \le \left(\frac{1}{8}\sum_{i=1}^{8} x_i^2\right)^{1/2} = \frac{1}{2}. \qquad \textbf{[2 marks]}$$

Either way, equality must hold, which can only happen if all the terms x_i are equal, that is, if $x_i = 1/2$ for all i. **[1 mark]** Thus $f = 1/256$ as above. **[1 mark]**

2. Suppose $ABCD$ is a square piece of cardboard with side length a. On a plane are two parallel lines ℓ_1 and ℓ_2, which are also a units apart. The square $ABCD$ is placed on the plane so that sides AB and AD intersect ℓ_1 at E and F respectively. Also, sides CB and CD intersect ℓ_2 at G and H respectively. Let the perimeters of $\triangle AEF$ and $\triangle CGH$ be m_1 and m_2 respectively. Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

Solution 1.

Let EH intersect FG at O. The distance from G to line FD and line EF are both a. So FG bisects $\angle EFD$. Similarly, EH bisects $\angle BEF$. So O is an excentre of $\triangle AEF$. Similarly, O is an excentre of $\triangle CGH$. **[2 marks]** Construct these excircles with centre O. Let M, N, P, Q be on sides AB, BC, CD, DA respectively, where these excircles touch the square. Then $OM \perp AB$, $ON \perp BC$, $OP \perp CD$, and $OQ \perp DA$. Since $AB \parallel CD$ and $AD \parallel BC$, M, O, P are collinear and N, O, Q are collinear. Now $MP = NQ = a$. **[2 marks]** Using the fact that the two tangents from a point to a circle have the same length, we get $EF = EM + FQ$ and $GH = GN + HP$. **[1 mark]** Then

$$m_1 = AE + AF + EF = AE + AF + (EM + FQ) = AM + AQ = OQ + OM$$

and

$$m_2 = CG + CH + GH = CG + CH + (GN + HP) = CN + CP = OP + ON. \qquad \textbf{[1 mark]}$$

Therefore

$$m_1 + m_2 = (OQ + OM) + (OP + ON) = MP + NQ = 2a. \qquad \textbf{[1 mark]}$$

Solution 2.

Extend AB to I and DC to J so that $AE = BI = CJ$. Let ℓ_2 intersect IJ at M, and let K lie on IJ so that $GK \perp IJ$. Then, since $AE = GK$, $\triangle AEF$ and $\triangle KGM$ are congruent. **[1 mark]** Thus, since $GK = CJ$ and $GC = KJ$,

$$m_1 + m_2 = \text{perimeter}(KGM) + \text{perimeter}(CGH) = \text{perimeter}(HMJ). \qquad \textbf{[2 marks]}$$

Let L lie on CD so that $EL \perp CD$. Then a circle with centre E and radius a will touch DC at L, IJ at I, and the interior of HM at some point N, so

$$\text{perimeter}(HMJ) = JH + (HN + NM) + JM = (JH + HL) + (MI + JM) = JL + IJ = a + a = 2a.$$

[4 marks] Thus $m_1 + m_2 = 2a$.

Solution 3.

Without loss of generality, assume the square has side $a = 1$. Let θ be the acute angle between ℓ_1 (or ℓ_2) and the sides AB and CD of the square. Then, letting $EF = x$ and $GH = y$, we have

$$EA = x\cos\theta, \quad AF = x\sin\theta, \qquad CH = y\cos\theta, \quad CG = y\sin\theta.$$

Thus

$$m_1 + m_2 = (x + y)(\sin\theta + \cos\theta + 1). \qquad \textbf{[2 marks]} \qquad (1)$$

Draw lines parallel to ℓ_1, ℓ_2 through A and C respectively. The distance between these lines is $\sin\theta + \cos\theta$ **[1 mark]**, as can be seen by drawing a mutual perpendicular to these lines through B, say. Also, the altitudes from A to EF and from C to GH have lengths $x\sin\theta\cos\theta$ and $y\sin\theta\cos\theta$ respectively **[1 mark]**. Therefore the distance between ℓ_1 and ℓ_2 must be

$$(\sin\theta + \cos\theta) - x\sin\theta\cos\theta - y\sin\theta\cos\theta.$$

But we are given that this distance is $a = 1$, so

$$(x + y)\sin\theta\cos\theta + 1 = \sin\theta + \cos\theta,$$

or

$$x + y = \frac{\sin\theta + \cos\theta - 1}{\sin\theta\cos\theta}. \qquad \textbf{[1 mark]}$$

Therefore, by (1),

$$\begin{aligned} m_1 + m_2 &= \frac{(\sin\theta + \cos\theta - 1)(\sin\theta + \cos\theta + 1)}{\sin\theta\cos\theta} \\ &= \frac{(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta) - 1}{\sin\theta\cos\theta} \\ &= \frac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta} = 2. \qquad \textbf{[2 marks]} \end{aligned}$$

3. Let $k \geq 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \geq 3k/4$. Let n be a composite integer. Prove:

(a) if $n = 2p_k$, then n does not divide $(n-k)!$;

(b) if $n > 2p_k$, then n divides $(n-k)!$.

Solution.

(a) Note that $n - k = 2p_k - k < 2p_k - p_k = p_k$, so $p_k \not| (n-k)!$, so $2p_k \not| (n-k)!$. **[1 mark]**

(b) Note that $n > 2p_k \geq 3k/2$ implies $k < 2n/3$, so $n - k > n/3$. So if we can find integers $a, b \geq 3$ such that $n = ab$ and $a \neq b$, then both a and b will appear separately in the product $(n-k)! = 1 \times 2 \times \cdots \times (n-k)$, which means $n|(n-k)!$. Observe that $k \geq 14$ implies $p_k \geq 13$, so that $n > 2p_k \geq 26$.

If $n = 2^\alpha$ for some integer $\alpha \geq 5$, then take $a = 2^2$, $b = 2^{\alpha-2}$. **[1 mark]** Otherwise, since $n \geq 26 > 16$, we can take a to be an odd prime factor of n and $b = n/a$ **[1 mark]**, unless $b < 3$ or $b = a$.

Case (i): $b < 3$. Since n is composite, this means $b = 2$, so that $2a = n > 2p_k$. As a is a prime number and p_k is the largest prime number which is strictly less than k, it follows that $a \geq k$. From $n - k = 2a - k \geq 2a - a = a > 2$ we see that $n = 2a$ divides into $(n-k)!$. **[2 marks]**

Case (ii): $b = a$. Then $n = a^2$ and $a > 6$ since $n \geq 26$. Thus $n - k > n/3 = a^2/3 > 2a$, so that both a and $2a$ appear among $\{1, 2, \ldots, n-k\}$. Hence $n = a^2$ divides into $(n-k)!$. **[2 marks]**

4. Let a, b, c be the sides of a triangle, with $a + b + c = 1$, and let $n \geq 2$ be an integer. Show that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2} .$$

Solution.

Without loss of generality, assume $a \leq b \leq c$. As $a + b > c$, we have

$$\frac{\sqrt[n]{2}}{2} = \frac{\sqrt[n]{2}}{2}(a+b+c) > \frac{\sqrt[n]{2}}{2}(c+c) = \sqrt[n]{2c^n} \geq \sqrt[n]{b^n + c^n}. \qquad \textbf{[2 marks]} \qquad (1)$$

As $a \leq c$ and $n \geq 2$, we have

$$\begin{aligned}(c^n + a^n) - \left(c + \frac{a}{2}\right)^n &= a^n - \sum_{k=1}^{n} \binom{n}{k} c^{n-k} \left(\frac{a}{2}\right)^k \\ &\leq \left[1 - \sum_{k=1}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^k\right] a^n \qquad (\text{since } c^{n-k} \geq a^{n-k}) \\ &= \left[\left(1 - \frac{n}{2}\right) - \sum_{k=2}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^k\right] a^n < 0.\end{aligned}$$

Thus

$$\sqrt[n]{c^n + a^n} < c + \frac{a}{2} . \qquad \textbf{[3 marks]} \qquad (2)$$

Likewise

$$\sqrt[n]{b^n + a^n} < b + \frac{a}{2} . \qquad \textbf{[1 mark]} \qquad (3)$$

Adding (1), (2) and (3), we get

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < \frac{\sqrt[n]{2}}{2} + c + \frac{a}{2} + b + \frac{a}{2} = 1 + \frac{\sqrt[n]{2}}{2} . \qquad \textbf{[1 mark]}$$

5. Given two positive integers m and n, find the smallest positive integer k such that among any k people, either there are $2m$ of them who form m pairs of mutually acquainted people or there are $2n$ of them forming n pairs of mutually unacquainted people.

Solution.

Let the smallest positive integer k satisfying the condition of the problem be denoted $r(m,n)$. We shall show that

$$r(m,n) = 2(m+n) - \min\{m,n\} - 1.$$

Observe that, by symmetry, $r(m,n) = r(n,m)$. Therefore it suffices to consider the case where $m \geq n$, and to prove that

$$r(m,n) = 2m + n - 1. \qquad \textbf{[1 mark]} \tag{1}$$

First we prove that

$$r(m,n) \geq 2m + n - 1$$

by an example. Call a group of k people, every two of whom are mutually acquainted, a *k-clique.* Consider a set of $2m + n - 2$ people consisting of a $(2m-1)$-clique together with an additional $n-1$ people none of whom know anyone else. (Call such people *isolated.*) Then there are not $2m$ people forming m mutually acquainted pairs, and there also are not $2n$ people forming n mutually unacquainted pairs. Thus $r(m,n) \geq (2m-1) + (n-1) + 1 = 2m + n - 1$ by the definition of $r(m,n)$. **[1 mark]**

To establish (1), we need to prove that $r(m,n) \leq 2m + n - 1$. To do this, we now show that

$$r(m,n) \leq r(m-1, n-1) + 3 \qquad \text{for all } m \geq n \geq 2. \tag{2}$$

Let G be a group of $t = r(m-1, n-1) + 3$ people. Notice that

$$t \geq 2(m-1) + (n-1) - 1 + 3 = 2m + n - 1 \geq 2m \geq 2n.$$

If G is a t-clique, then G contains $2m$ people forming m mutually acquainted pairs, and if G has only isolated people, then G contains $2n$ people forming n mutually unacquainted pairs. Otherwise, there are three people in G, say a, b and c, such that a, b are acquainted but a, c are not. Now consider the group A obtained by removing a, b and c from G. A has $t - 3 = r(m-1, n-1)$ people, so by the definition of $r(m-1, n-1)$, A either contains $2(m-1)$ people forming $m-1$ mutually acquainted pairs, or else contains $2(n-1)$ people forming $n-1$ mutually unacquainted pairs. In the former case, we add the acquainted pair a, b to A to form m mutually acquainted pairs in G. In the latter case, we add the unacquainted pair a, c to A to form n mutually unacquainted pairs in G. This proves (2). **[3 marks]**

Trivially, $r(s,1) = 2s$ for all s **[1 mark]**, so $r(m,n) \leq 2m + n - 1$ holds whenever $n = 1$. Proceeding by induction on n, by (2) we obtain

$$r(m,n) \leq r(m-1, n-1) + 3 \leq 2(m-1) + (n-1) - 1 + 3 = 2m + n - 1,$$

which completes the proof. **[1 mark]**

Note. Give an additional 1 mark to any student who gets at most 5 marks by the above marking scheme, but in addition gives a valid argument that $r(2,2) = 5$.