ИМО2022

Задача 1. Банката во Осло кова два типови од монети: алуминиумски (обележуваме A) и бронзени (обележуваме B). Маријана има n алуминиумски монети и n бронзени монети, подредени во ред во некој произволен почетен редослед. *Синџир* е произволна подниза од последователни монети од ист тип. За даден позитивен цел број $k \leqslant 2n$, Маријана ја повторува следната постапка: таа го наоѓа најдолгиот синџир кој ја содржи k-тата монета гледано од лево, и ги поместува сите парички од овој синџир на левата страна од редот. На пример за $n = 4$ и $k = 4$, постапката почнувајќи од подредувањето $AABBBABA$ ќе биде

$$AAB\underline{B}BABA \to BBB\underline{A}AABA \to AAA\underline{B}BBBA \to BBB\underline{B}AAAA \to BBB\underline{B}AAAA \to \cdots.$$

Најди ги сите парови (n, k) каде $1 \leqslant k \leqslant 2n$ такви што за секое почетно подредување, во некој момент во постапката, првите n монети од лево ќе бидат сите од ист тип.

Задача 2. Нека $\mathbb{R}^+$ го означува множеството на позитивни реални броеви. Најди ги сите функции $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ такви што за секој $x \in \mathbb{R}^+$, постои точно еден $y \in \mathbb{R}^+$ за кој важи

$$xf(y) + yf(x) \leqslant 2.$$

Задача 3. Нека k е позитивен цел број и S е конечно множество од непарни прости броеви. Докажи дека постои најмногу еден начин (до ротација и осна симетрија) да се постават елементите од S околу кружница така што производот од било кои два соседи на кружницата е од облик $x^2 + x + k$ за некој позитивен цел број x.

Задача 4. Нека $ABCDE$ е конвексен петаголник таков што $BC = DE$. Претпоставуваме дека постои точка T во внатрешноста на $ABCDE$ за која $TB = TD$, $TC = TE$ и $\angle ABT = \angle TEA$. Нека правата AB ги сече правите CD и CT во точки P и Q, соодветно. Претпоставуваме дека точките P, B, A, Q се поставени на правата во овој редослед. Нека правата AE ги сече правите CD и DT во точки R и S, соодветно. Претпоставуваме дека точките R, E, A, S се поставени на правата во овој редослед. Докажи дека точките P, S, Q, R лежат на кружница.

Задача 5. Најди ги сите тројки (a, b, p) од позитивни цели броеви, каде p е прост број и важи

$$a^p = b! + p.$$

Задача 6. Нека n е позитивен цел број. *Нордиски квадрат* е $n \times n$ табла на која се напишани сите цели броеви од 1 до n^2 така што на секое поле е напишан точно еден број. Две полиња се соседни ако имаат заедничка страна. Секое поле кое е соседно само со полиња на кои се напишани поголеми броеви го нарекуваме *котлина*. *Нагорница* е низа од едно или повеќе полиња таква што:

(i) првото поле во низата е котлина,

(ii) секое следно поле во низата е соседно со претходното поле, и

(iii) броевите кои се напишани во полињата од низата се во растечки редослед.

Најди го, како функција од n, најмалиот можен вкупен број на нагорници во Нордиски квадрат.

Problem 1. The Bank of Oslo issues two types of coin: aluminium (denoted A) and bronze (denoted B). Marianne has n aluminium coins and n bronze coins, arranged in a row in some arbitrary initial order. A *chain* is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leqslant 2n$, Marianne repeatedly performs the following operation: she identifies the longest chain containing the k^{th} coin from the left, and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be

$$AAB\underline{B}BABA \to BBB\underline{A}AABA \to AAA\underline{B}BBBA \to BBB\underline{B}AAAA \to BBB\underline{B}AAAA \to \cdots .$$

Find all pairs (n, k) with $1 \leqslant k \leqslant 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Problem 2. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leqslant 2.$$

Problem 3. Let k be a positive integer and let S be a finite set of odd prime numbers. Prove that there is at most one way (up to rotation and reflection) to place the elements of S around a circle such that the product of any two neighbours is of the form $x^2 + x + k$ for some positive integer x.

Problem 4. Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line in that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Problem 5. Find all triples (a, b, p) of positive integers with p prime and

$$a^p = b! + p.$$

Problem 6. Let n be a positive integer. A *Nordic square* is an $n \times n$ board containing all the integers from 1 to n^2 so that each cell contains exactly one number. Two different cells are considered adjacent if they share a common side. Every cell that is adjacent only to cells containing larger numbers is called a *valley*. An *uphill path* is a sequence of one or more cells such that:

(i) the first cell in the sequence is a valley,

(ii) each subsequent cell in the sequence is adjacent to the previous cell, and

(iii) the numbers written in the cells in the sequence are in increasing order.

Find, as a function of n, the smallest possible total number of uphill paths in a Nordic square.

1 The Bank of Oslo issues coins made out of two types of metal: aluminium (denoted A) and copper (denoted C). Morgane has n aluminium coins, and n copper coins, and arranges her $2n$ coins in a row in some arbitrary initial order. Given a fixed positive integer $k \leqslant 2n$, she repeatedly performs the following operation: identify the largest subsequence containing the k-th coin from the left which consists of consecutive coins made of the same metal, and move all coins in that subsequence to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the configuration $AACCCACA$ would be

$$AACCCACA \to CCCAAACA \to AAACCCCA \to CCCCAAAA \to \cdots .$$

Find all pairs (n, k) with $1 \leqslant k \leqslant 2n$ such that for every initial configuration, at some point of the process there will be at most one aluminium coin adjacent to a copper coin.

(*France*)

Answer: All pairs (n, k) such that $n \leqslant k \leqslant \frac{3n+1}{2}$.

Solution. Define a *block* to be a maximal subsequence of consecutive coins made out of the same metal, and let M^b denote a block of b coins of metal M. The property that there is at most one aluminium coin adjacent to a copper coin is clearly equivalent to the configuration having two blocks, one consisting of all A-s and one consisting of all C-s.

First, notice that if $k < n$, the sequence $A^{n-1}C^{n-1}AC$ remains fixed under the operation, and will therefore always have 4 blocks. Next, if $k > \frac{3n+1}{2}$, let $a = k - n - 1, b = 2n - k + 1$. Then $k > 2a + b$, $k > 2b + a$, so the configuration $A^aC^bA^bC^a$ will always have four blocks:

$$A^aC^bA^bC^a \to C^aA^aC^bA^b \to A^bC^aA^aC^b \to C^bA^bC^aA^a \to A^aC^bA^bC^a \to \dots$$

Therefore a pair (n, k) can have the desired property only if $n \leqslant k \leqslant \frac{3n+1}{2}$. We claim that all such pairs in fact do have the desired property. Clearly, the number of blocks in a configuration cannot increase, so whenever the operation is applied, it either decreases or remains constant. We show that unless there are only two blocks, after a finite amount of steps the number of blocks will decrease.

Consider an arbitrary configuration with $c \geqslant 3$ blocks. We note that as $k \geqslant n$, the leftmost block cannot be moved, because in this case all n coins of one type are in the leftmost block, meaning there are only two blocks. If a block which is not the leftmost or rightmost block is moved, its neighbor blocks will be merged, causing the number of blocks to decrease.

Hence the only case in which the number of blocks does not decrease in the next step is if the rightmost block is moved. If c is odd, the leftmost and the rightmost blocks are made of the same metal, so this would merge two blocks. Hence $c \geqslant 4$ must be even. Suppose there is a configuration of c blocks with the i-th block having size a_i so that the operation always moves the rightmost block:

$$A^{a_1} \dots A^{a_{c-1}}C^{a_c} \to C^{a_c}A^{a_1} \dots A^{a_{c-1}} \to A^{a_{c-1}}C^{a_c}A^{a_1} \dots C^{a_{c-2}} \to \dots$$

Because the rightmost block is always moved, $k \geqslant 2n + 1 - a_i$ for all i. Because $\sum a_i = 2n$, summing this over all i we get $ck \geqslant 2cn + c - \sum a_i = 2cn + c - 2n$, so $k \geqslant 2n + 1 - \frac{2n}{c} \geqslant \frac{3n}{2} + 1$. But this contradicts $k \leqslant \frac{3n+1}{2}$. Hence at some point the operation will not move the rightmost block, meaning that the number of blocks will decrease, as desired.

2 Let $\mathbb{R}_{>0}$ be the set of positive real numbers. Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that, for every $x \in \mathbb{R}_{>0}$, there exists a unique $y \in \mathbb{R}_{>0}$ satisfying

$$xf(y) + yf(x) \leqslant 2.$$

(Netherlands)

Answer: The function $f(x) = 1/x$ is the only solution.

Solution 1. First we prove that the function $f(x) = 1/x$ satisfies the condition of the problem statement. The AM-GM inequality gives

$$\frac{x}{y} + \frac{y}{x} \geqslant 2$$

for every $x, y > 0$, with equality if and only if $x = y$. This means that, for every $x > 0$, there exists a unique $y > 0$ such that

$$\frac{x}{y} + \frac{y}{x} \leqslant 2,$$

namely $y = x$.

Let now $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a function that satisfies the condition of the problem statement. We say that a pair of positive real numbers (x, y) is *good* if $xf(y) + yf(x) \leqslant 2$. Observe that if (x, y) is good, then so is (y, x).

Lemma 1.0. If (x, y) is good, then $x = y$.

Proof. Assume that there exist positive real numbers $x \neq y$ such that (x, y) is good. The uniqueness assumption says that y is the unique positive real number such that (x, y) is good. In particular, (x, x) is not a good pair. This means that

$$xf(x) + xf(x) > 2$$

and thus $xf(x) > 1$. Similarly, (y, x) is a good pair , so (y, y) is not a good pair, which implies $yf(y) > 1$. We apply the AM-GM inequality to obtain

$$xf(y) + yf(x) \geqslant 2\sqrt{xf(y) \cdot yf(x)} = 2\sqrt{xf(x) \cdot yf(y)} > 2.$$

This is a contradiction, since (x, y) is a good pair.

By assumption, for any $x > 0$, there always exists a good pair containing x, however Lemma 1 implies that the only good pair that can contain x is (x, x), so

$$xf(x) \leqslant 1 \quad \iff \quad f(x) \leqslant \frac{1}{x},$$

for every $x > 0$.

In particular, with $x = 1/f(t)$ for $t > 0$, we obtain

$$\frac{1}{f(t)} \cdot f\left(\frac{1}{f(t)}\right) \leqslant 1.$$

Hence

$$t \cdot f\left(\frac{1}{f(t)}\right) \leqslant tf(t) \leqslant 1.$$

We claim that $(t, 1/f(t))$ is a good pair for every $t > 0$. Indeed,

$$t \cdot f\left(\frac{1}{f(t)}\right) + \frac{1}{f(t)}f(t) = t \cdot f\left(\frac{1}{f(t)}\right) + 1 \leqslant 2.$$

Lemma 1 implies that $t = 1/f(t) \iff f(t) = 1/t$ for every $t > 0$.

Solution 1.1. We give an alternative way to prove that $f(x) = 1/x$ assuming $f(x) \leqslant 1/x$ for every $x > 0$.

Indeed, if $f(x) < 1/x$ then for every $a > 0$ with $f(x) < 1/a < 1/x$ (and there are at least two of them), we have

$$af(x) + xf(a) < 1 + \frac{x}{a} < 2.$$

Hence (x, a) is a good pair for every such a, a contradiction. We conclude that $f(x) = 1/x$.

Solution 1.2. We can also conclude from Lemma 1 and $f(x) \leqslant 1/x$ as follows.

Lemma 2. The function f is decreasing.

Proof. Let $y > x > 0$. Lemma 1 says that (x, y) is not a good pair, but (y, y) is. Hence

$$xf(y) + yf(x) > 2 \geqslant 2yf(y) > yf(y) + xf(y),$$

where we used $y > x$ (and $f(y) > 0$) in the last inequality. This implies that $f(x) > f(y)$, showing that f is decreasing.

We now prove that $f(x) = 1/x$ for all x. Fix a value of x and note that for $y > x$ we must have $xf(x) + yf(x) > xf(y) + yf(x) > 2$ (using that f is decreasing for the first step), hence $f(x) > \frac{2}{x+y}$. The last inequality is true for every $y > x > 0$. If we fix x and look for the supremum of the expression $\frac{2}{x+y}$ over all $y > x$, we get

$$f(x) \geqslant \frac{2}{x + x} = \frac{1}{x}.$$

Since we already know that $f(x) \leqslant 1/x$, we conclude that $f(x) = 1/x$.

Solution 2.0. As in the first solution, we note that $f(x) = 1/x$ is a solution, and we set out to prove that it is the only one. We write $g(x)$ for the unique positive real number such that $(x, g(x))$ is a good pair. In this solution, we prove Lemma 2 without assuming Lemma 1.

Lemma 2. The function f is decreasing.

Proof. Consider $x < y$. It holds that $yf(g(y)) + g(y)f(y) \leqslant 2$. Moreover, because y is the only positive real number such that $(g(y), y)$ is a good pair and $x \neq y$, we have $xf(g(y))+g(y)f(x) > 2$. Combining these two inequalities yields

$$xf(g(y)) + g(y)f(x) > 2 \geqslant yf(g(y)) + g(y)f(y),$$

or $f(g(y))(x - y) > g(y)(f(y) - f(x))$. Because $g(y)$ and $f(g(y))$ are both positive while $x - y$ is negative, it follows that $f(y) < f(x)$, showing that f is decreasing.

We now prove Lemma 1 using Lemma 2. Suppose that $x \neq y$ but $xf(y) + yf(x) \leqslant 2$. As in the first solution, we get $xf(x) + xf(x) > 2$ and $yf(y) + yf(y) > 2$, which implies $xf(x) + yf(y) > 2$. Now

$$xf(x) + yf(y) > 2 \geqslant xf(y) + yf(x)$$

implies $(x - y)(f(x) - f(y)) > 0$, which contradicts the fact that f is decreasing. So $y = x$ is the unique y such that (x, y) is a good pair, and in particular we have $f(x) \leqslant 1/x$.

We can now conclude the proof as in any of the Solutions 1.x.

Solution 3.0. As in the other solutions we verify that the function $f(x) = 1/x$ is a solution. We first want to prove the following lemma:

Lemma 3. For all $x \in \mathbb{R}_{>0}$ we actually have $xf(g(x)) + g(x)f(x) = 2$ (that is: the inequality is actually an equality).

Proof. We proceed by contradiction: Assume there exists some number $x > 0$ such that for $y = g(x)$ we have $xf(y) + yf(x) < 2$. Then for any $0 < \epsilon < \frac{2 - xf(y) - yf(x)}{2f(x)}$ we have, by uniqueness of y, that $xf(y+\epsilon) + (y+\epsilon)f(x) > 2$. Therefore

$$\begin{aligned} f(y+\epsilon) &> \frac{2-(y+\epsilon)f(x)}{x} = \frac{2 - yf(x) - \epsilon f(x)}{x} \\ &> \frac{2 - yf(x) - \frac{2-xf(y)-yf(x)}{2}}{x} \\ &= \frac{2 - xf(y) - yf(x)}{2x} + f(y) > f(y). \end{aligned} \tag{1}$$

Furthermore, for every such ϵ we have $g(y+\epsilon)f(y+\epsilon) + (y+\epsilon)f(g(y+\epsilon)) \leqslant 2$ and $g(y+\epsilon)f(y) + yf(g(y+\epsilon)) > 2$ (since $y \neq y + \epsilon = g(g(y+\epsilon))$). This gives us the two inequalities

$$f(g(y+\epsilon)) \leqslant \frac{2 - g(y+\epsilon)f(y+\epsilon)}{y+\epsilon} \qquad \text{and} \qquad f(g(y+\epsilon)) > \frac{2 - g(y+\epsilon)f(y)}{y}.$$

Combining these two inequalities and rearranging the terms leads to the inequality

$$2\epsilon < g(y+\epsilon)[(y+\epsilon)f(y) - yf(y+\epsilon)].$$

Moreover combining with the inequality (1) we obtain

$$2\epsilon < g(y+\epsilon)\left[(y+\epsilon)f(y) - y\left(\frac{2-xf(y)-yf(x)}{2x} + f(y)\right)\right] = g(y+\epsilon)\left[\epsilon f(y) - y\frac{2-xf(y)-yf(x)}{2x}\right].$$

We now reach the desired contradiction, since for ϵ sufficiently small we have that the left hand side is positive while the right hand side is negative.

With this lemma it then follows that for all $x, y \in \mathbb{R}_{>0}$ we have

$$xf(y) + yf(x) \geqslant 2,$$

since for $y = g(x)$ we have equality and by uniqueness for $y \neq g(x)$ the inequality is strict.

In particular for every $x \in \mathbb{R}_{>0}$ and for $y = x$ we have $2xf(x) \geqslant 2$, or equivalently $f(x) \geqslant 1/x$ for all $x \in \mathbb{R}_{>0}$. With this inequality we obtain for all $x \in \mathbb{R}_{>0}$

$$2 \geqslant xf(g(x)) + g(x)f(x) \geqslant \frac{x}{g(x)} + \frac{g(x)}{x} \geqslant 2,$$

where the first inequality comes from the problem statement. Consequently each of these inequalities must actually be an equality, and in particular we obtain $f(x) = 1/x$ for all $x \in \mathbb{R}_{>0}$.

Solution 4. Again, let us prove that $f(x) = 1/x$ is the only solution. Let again $g(x)$ be the unique positive real number such that $(x, g(x))$ is a good pair.

Lemma 4. The function f is strictly convex.

Proof. Consider the function $q_s(x) = f(x) + sx$ for some real number s. If f is not strictly convex, then there exist $u < v$ and $t \in (0,1)$ such that

$$f(tu + (1-t)v) \geqslant tf(u) + (1-t)f(v).$$

Hence

$$\begin{aligned} q_s(tu + (1-t)v) &\geqslant tf(u) + (1-t)f(v) + s(tu + (1-t)v) \\ &= tq_s(u) + (1-t)q_s(v). \end{aligned}$$

Let $w = tu+(1-t)v$ and consider the case $s = f(g(w))/g(w)$. For that particular choice of s, the function $q_s(x)$ has a unique minimum at $x = w$. However, since $q_s(w) \geqslant tq_s(u)+(1-t)q_s(v)$, it must hold $q_s(u) \leqslant q_s(w)$ or $q_s(v) \leqslant q_s(w)$, a contradiction.

Lemma 5. The function f is continuous.

Proof. Since f is strictly convex and defined on an open interval, it is also continuous.

As in Solution 1, we can prove that $f(x) \leqslant 1/x$. If $f(x) < 1/x$, then we consider the function $h(y) = xf(y) + yf(x)$ which is continuous. Since $h(x) < 2$, there exist at least two distinct $z \neq x$ such that $h(z) < 2$ giving that (x, z) is good pair for both values of z, a contradiction. We conclude that $f(x) = 1/x$ as desired.

Comment. Lemma 5 implies Lemma 3, using an argument similar as in the end of Solution 4.

3 Let k be a positive integer and let S be a finite set of odd prime numbers. Prove that there is at most one way (modulo rotation and reflection) to place the elements of S around a circle such that the product of any two neighbors is of the form $x^2 + x + k$ for some positive integer x.

(U.S.A.)

Solution. Let us allow the value $x = 0$ as well; we prove the same statement under this more general constraint. Obviously that implies the statement with the original conditions.

Call a pair $\{p, q\}$ of primes with $p \neq q$ *special* if $pq = x^2 + x + k$ for some nonnegative integer x. The following claim is the key mechanism of the problem:

Claim.

(a) For every prime r, there are at most two primes less than r forming a special pair with r.

(b) If such p and q exist, then $\{p, q\}$ is itself special.

We present two proofs of the claim.

Proof 1. We are interested in integers $1 \leqslant x < r$ satsfying

$$x^2 + x + k \equiv 0 \pmod{r}. \tag{1}$$

Since there are at most two residues modulo r that can satisfy that quadratic congruence, there are at most two possible values of x. That proves (a).

Now suppose there are primes p, q with $p < q < r$ and nonnegative integers x, y such that

$$x^2 + x + k = pr$$
$$y^2 + y + k = qr.$$

From $p < q < r$ we can see that $0 \leqslant x < y \leqslant r - 1$. The numbers x, y are the two solutions of (1); by Vieta's formulas, we should have $x + y \equiv -1 \pmod{r}$, so $x + y = r - 1$.

Letting $K = 4k - 1$, $X = 2x + 1$, and $Y = 2y + 1$, we obtain

$$4pr = X^2 + K,$$
$$4qr = Y^2 + K$$

with $X + Y = 2r$. Multiplying the two above equations,

$$\begin{aligned} 16pqr^2 &= (X^2 + K)(Y^2 + K) \\ &= (XY - K)^2 + K(X + Y)^2 \\ &= (XY - K)^2 + 4Kr^2, \\ 4pq &= \left(\frac{XY - K}{2r}\right)^2 + K. \end{aligned}$$

In particular, the number $Z = \frac{XY-K}{2r}$ should be an integer, and so $4pq = Z^2 + K$. By parity, Z is odd, and thus

$$pq = z^2 + z + k \quad \text{where } z = \frac{Z - 1}{2},$$

so $\{p, q\}$ is special. $\square$

Proof 2. As before, we suppose that

$$x^2 + x + k = pr$$
$$y^2 + y + k = qr.$$

Subtracting, we have

$$(x + y + 1)(x - y) = r(p - q).$$

As before, we have $x + y = r - 1$, so $x - y = p - q$, and

$$x = \tfrac{1}{2}(r + p - q - 1)$$
$$y = \tfrac{1}{2}(r + q - p - 1).$$

Then,

$$\begin{aligned} k = pr - x^2 - x &= \tfrac{1}{4}(4pr - (r + p - q - 1)^2 - 2(r + p - q - 1)) \\ &= \tfrac{1}{4}(4pr - (r + p - q)^2 + 1) \\ &= \tfrac{1}{4}(2pq + 2pr + 2qr - p^2 - q^2 - r^2 + 1), \end{aligned}$$

which is symmetric in p, q, r, so

$$pq = z^2 + z + k \quad \text{where } z = \tfrac{1}{2}(p + q - r - 1),$$

and $\{p, q\}$ is special. $\square$

Now we settle the problem by induction on $|S|$, with $|S| \leqslant 3$ clear.

Suppose we have proven it for $|S| = n$ and consider $|S| = n + 1$. Let r be the largest prime in S; the claim tells us that in any valid cycle of primes:

- the neighbors of r are uniquely determined, and
- removing r from the cycle results in a smaller valid cycle.

It follows that there is at most one valid cycle, completing the inductive step.

Comment. The statement is not as inapplicable as it might seem. For example, for $k = 41$, the following 385 primes form a valid cycle of primes:

53, 4357, 104173, 65921, 36383, 99527, 193789, 2089123, 1010357, 2465263, 319169, 15559, 3449, 2647, 1951, 152297, 542189, 119773, 91151, 66431, 222137, 1336799, 469069, 45613, 1047941, 656291, 355867, 146669, 874879, 2213327, 305119, 3336209, 1623467, 520963, 794201, 1124833, 28697, 15683, 42557, 6571, 39607, 1238833, 835421, 2653681, 5494387, 9357539, 511223, 1515317, 8868173, 114079681, 59334071, 22324807, 3051889, 5120939, 7722467, 266239, 693809, 3931783, 1322317, 100469, 13913, 74419, 23977, 1361, 62983, 935021, 512657, 1394849, 216259, 45827, 31393, 100787, 1193989, 600979, 209543, 357661, 545141, 19681, 10691, 28867, 165089, 2118023, 6271891, 12626693, 21182429, 1100467, 413089, 772867, 1244423, 1827757, 55889, 1558873, 5110711, 1024427, 601759, 290869, 91757, 951109, 452033, 136471, 190031, 4423, 9239, 15809, 24133, 115811, 275911, 34211, 877, 6653, 88001, 46261, 317741, 121523, 232439, 379009, 17827, 2699, 15937, 497729, 335539, 205223, 106781, 1394413, 4140947, 8346383, 43984757, 14010721, 21133961, 729451, 4997297, 1908223, 278051, 529747, 40213, 768107, 456821, 1325351, 225961, 1501921, 562763, 75527, 5519, 9337, 14153, 499, 1399, 2753, 14401, 94583, 245107, 35171, 397093, 195907, 2505623, 34680911, 18542791, 7415917, 144797293, 455529251, 86675291, 252704911, 43385123, 109207907, 204884269, 330414209, 14926789, 1300289, 486769, 2723989, 907757, 1458871, 65063, 4561, 124427, 81343, 252887, 2980139, 1496779, 3779057, 519193, 47381, 135283, 268267, 446333, 669481, 22541, 54167, 99439, 158357, 6823, 32497, 1390709, 998029, 670343, 5180017, 13936673, 2123491, 4391941, 407651, 209953, 77249, 867653, 427117, 141079, 9539, 227, 1439, 18679, 9749, 25453, 3697, 42139, 122327, 712303, 244261, 20873, 52051, 589997, 4310569, 1711069, 291563, 3731527, 11045429, 129098443, 64620427, 162661963, 22233269, 37295047, 1936969, 5033449, 725537, 1353973, 6964457, 2176871, 97231, 7001, 11351, 55673, 16747, 169003, 1218571, 479957, 2779783, 949609, 4975787, 1577959, 2365007, 3310753, 79349, 23189, 107209, 688907, 252583, 30677, 523, 941, 25981, 205103, 85087, 1011233, 509659, 178259, 950479, 6262847, 2333693, 305497, 3199319, 9148267, 1527563, 466801, 17033, 9967, 323003, 4724099, 14278309, 2576557, 1075021, 6462593, 2266021, 63922471, 209814503, 42117791, 131659867, 270892249, 24845153, 12104557, 3896003, 219491, 135913, 406397, 72269, 191689, 2197697, 1091273, 2727311, 368227, 1911661, 601883, 892657, 28559, 4783, 60497, 31259, 80909, 457697, 153733, 11587, 1481, 26161, 15193, 7187, 2143, 21517, 10079, 207643, 1604381, 657661, 126227, 372313, 2176331, 748337, 64969, 844867, 2507291, 29317943, 14677801, 36952793, 69332267, 111816223, 5052241, 8479717, 441263, 3020431, 1152751, 13179611, 38280013, 6536771, 16319657, 91442699, 30501409, 49082027, 72061511, 2199433, 167597, 317963, 23869, 2927, 3833, 17327, 110879, 285517, 40543, 4861, 21683, 50527, 565319, 277829, 687917, 3846023, 25542677, 174261149, 66370753, 9565711, 1280791, 91393, 6011, 7283, 31859, 8677, 10193, 43987, 11831, 13591, 127843, 358229, 58067, 15473, 65839, 17477, 74099, 19603, 82847, 21851, 61.

4 Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle TBA = \angle AET$. Let lines CD and CT intersect line AB at points P and Q, respectively, and let lines CD and DT intersect line AE at points R and S, respectively. Assume that points P, B, A, Q and R, E, A, S respectively, are collinear and occur on their lines in this order. Prove that the points P, S, Q, R are concyclic.

(Slovakia)

Solution 1. By the conditions we have $BC = DE$, $CT = ET$ and $TB = TD$, so the triangles TBC and TDE are congruent, in particular $\angle BTC = \angle DTE$.

In triangles TBQ and TES we have $\angle TBQ = \angle SET$ and $\angle QTB = 180^\circ - \angle BTC = 180^\circ - \angle DTE = \angle ETS$, so these triangles are similar to each other. It follows that $\angle TSE = \angle BQT$ and

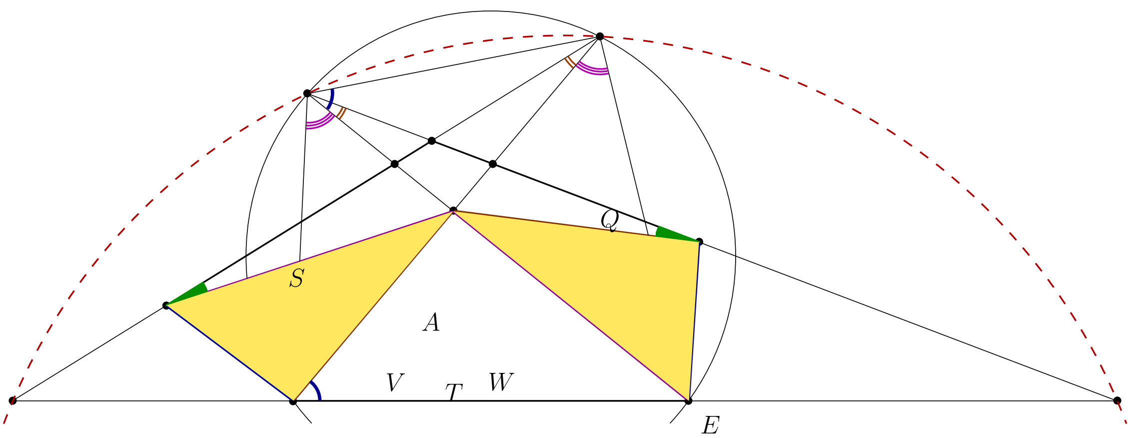
$$\frac{TD}{TQ} = \frac{TB}{TQ} = \frac{TE}{TS} = \frac{TC}{TS}.$$

By rearranging this relation we get $TD \cdot TS = TC \cdot TQ$, so C, D, Q and S are concyclic. (Alternatively, we can get $\angle CQD = \angle CSD$ from the similar triangles TCS and TDQ.) Hence, $\angle DCQ = \angle DSQ$.

Finally, from the angles of triangle CQP we get

$$\angle RPQ = \angle RCQ - \angle PQC = \angle DSQ - \angle DSR = \angle RSQ,$$

which proves that P, Q, R and S are concyclic.



Solution 2. As in the previous solution, we note that triangles TBC and TDE are congruent. Denote the intersection point of DT and BA by V, and the intersection point of CT and EA by W. From triangles BCQ and DES we then have

$$\begin{aligned}\angle VSW = \angle DSE &= 180^\circ - \angle SED - \angle EDS = 180^\circ - \angle AET - \angle TED - \angle EDT \\ &= 180^\circ - \angle TBA - \angle TCB - \angle CBT = 180^\circ - \angle QCB - \angle CBQ = \angle BQC = \angle VQW,\end{aligned}$$

meaning that $VSQW$ is cyclic, and in particular $\angle WVQ = \angle WSQ$. Since

$$\angle VTB = 180^\circ - \angle BTC - \angle CTD = 180^\circ - \angle CTD - \angle DTE = \angle ETW,$$

and $\angle TBV = \angle WET$ by assumption, we have that the triangles VTB and WTE are similar, hence

$$\frac{VT}{WT} = \frac{BT}{ET} = \frac{DT}{CT}.$$

Thus $CD \parallel VW$, and angle chasing yields

$$\angle RPQ = \angle WVQ = \angle WSQ = \angle RSQ,$$

concluding the proof.

5 Find all triples of positive integers (a, b, p) with p prime and

$$a^p = b! + p.$$

(Belgium)

Answer: $(2,2,2)$ and $(3,4,3)$.

Solution 1. Clearly, $a > 1$. We consider three cases.

Case 1: We have $a < p$. Then we either have $a \leqslant b$ which implies $a \mid a^p - b! = p$ leading to a contradiction, or $a > b$ which is also impossible since in this case we have $b! \leqslant a! < a^p - p$, where the last inequality is true for any $p > a > 1$.

Case 2: We have $a > p$. In this case $b! = a^p - p > p^p - p \geqslant p!$ so $b > p$ which means that $a^p = b! + p$ is divisible by p. Hence, a is divisible by p and $b! = a^p - p$ is not divisible by p^2. This means that $b < 2p$. If $a < p^2$ then $a/p < p$ divides both a^p and $b!$ and hence it also divides $p = a^p - b!$ which is impossible. On the other hand, the case $a \geqslant p^2$ is also impossible since then $a^p \geqslant (p^2)^p > (2p-1)! + p \geqslant b! + p$.

Comment. The inequality $p^{2p} > (2p-1)! + p$ can be shown e.g. by using

$$(2p-1)! = [1 \cdot (2p-1)] \cdot [2 \cdot (2p-2)] \cdot \cdots \cdot [(p-1)(p+1)] \cdot p < \left(\left(\frac{2p}{2} \right)^2 \right)^{p-1} \cdot p = p^{2p-1},$$

where the inequality comes from applying AM-GM to each of the terms in square brackets.

Case 3: We have $a = p$. In this case $b! = p^p - p$. One can check that the values $p = 2, 3$ lead to the claimed solutions and $p = 5$ does not lead to a solution. So we now assume that $p \geqslant 7$. We have $b! = p^p - p > p!$ and so $b \geqslant p + 1$ which implies that

$$v_2\big((p+1)!\big) \leqslant v_2(b!) = v_2(p^{p-1} - 1) \overset{LTE}{=} 2v_2(p-1) + v_2(p+1) - 1 = v_2\left(\frac{p-1}{2} \cdot (p-1) \cdot (p+1) \right),$$

where in the middle we used lifting-the-exponent lemma. On the RHS we have three factors of $(p+1)!$. But, due to $p + 1 \geqslant 8$, there are at least 4 even numbers among $1, 2, \ldots, p+1$, so this case is not possible.

Solution 2. The cases $a \neq p$ are covered as in solution 1, as are $p = 2, 3$. For $p \geqslant 5$ we have $b! = p(p^{p-1} - 1)$. By Zsigmondy's Theorem there exists some prime q that divides $p^{p-1} - 1$ but does not divide $p^k - 1$ for $k < p - 1$. It follows that $\operatorname{ord}_q(p) = p - 1$, and hence $q \equiv 1 \mod (p-1)$. Note that $p \neq q$. But then we must have $q \geqslant 2p - 1$, giving

$$b! \geqslant (2p-1)! = [1 \cdot (2p-1)] \cdot [2 \cdot (2p-2)] \cdot \cdots \cdot [(p-1) \cdot (p+1)] \cdot p > (2p-1)^{p-1} p > p^p > p^p - p,$$

a contradiction.

Solution 3. The cases $a \neq p$ are covered as in solution 1, as are $p = 2, 3$. Also $b > p$, as $p^p > p! + p$ for $p > 2$. The cases $p = 5, 7, 11$ are also checked manually, so assume $p \geqslant 13$.
Let $q | p + 1$ be an odd prime. By LTE

$$v_q(p^p - p) = v_q\left((p^2)^{\frac{p-1}{2}} - 1 \right) = v_q(p^2 - 1) + v_q\left(\frac{p-1}{2} \right) = v_q(p+1).$$

But $b \geqslant p + 1$, so then $v_q(b!) > v_q(p+1)$, since $q < p + 1$, a contradiction. This means that $p + 1$ has no odd prime divisor, i.e. $p + 1 = 2^k$ for some k.

Now let $q|p-1$ be an odd prime. By LTE

$$v_q(p^p - p) = 2v_q(p-1).$$

Let $d = v_q(p-1)$. Then $p \geqslant 1 + q^d$, so

$$v_q(b!) \geqslant v_q(p!) \geqslant v_q(q^d!) > q^{d-1} \geqslant 2d$$

provided $d \geqslant 2$ and $q > 3$, or $d \geqslant 3$.
If $q = 3$, $d = 2$ and $p \geqslant 13$ then $v_q(b!) \geqslant v_q(p!) \geqslant v_q(13!) = 5 > 2d$. Either way, $d \leqslant 1$.
If $p > 2q+1$ (so $p > 3q$, as $q|p-1$) then

$$v_q(b!) \geqslant v_q((3q)!) = 3,$$

so we must have $q \geqslant \frac{p}{2}$, in other words, $p-1 = 2q$. This implies that $p = 2^k - 1$ and $q = 2^{k-1} - 1$ are both prime, but it is not possible to have two consecutive Mersenne primes.

Solution 4. Let $a = p$, $b > p$ and $p \geqslant 5$ (the remaining cases are dealt with as in solution 3). Modulo $(p+1)^2$ it holds that

$$p^p - p = (p+1-1)^p - p \equiv \binom{p}{1}(p+1)(-1)^{p-1} + (-1)^p - p = p(p+1) - 1 - p = p^2 - 1 \not\equiv 0 \mod \left((p+1)^2\right).$$

Since $p \geqslant 5$, the numbers 2 and $\frac{p+1}{2}$ are distinct and less than or equal to p. Therefore, $p+1|p!$, and so $(p+1)^2|(p+1)!$.
But $b \geqslant p+1$, so $b! \equiv 0 \not\equiv p^p - p \mod (p+1)^2$, a contradiction.

66

Alice fills the fields of an $n \times n$ board with numbers from 1 to n^2, each number being used exactly once. She then counts the total number of good paths on the board. *A good path* is a sequence of fields of arbitrary length (including 1) such that:

(i) The first field in the sequence is one that is only adjacent to fields with larger numbers,

(ii) Each subsequent field in the sequence is adjacent to the previous field,

(iii) The numbers written on the fields in the sequence are in increasing order.

Two fields are considered adjacent if they share a common side. Find the smallest possible number of good paths Alice can obtain, as a function of n.

(Serbia)

Answer: $2n^2 - 2n + 1$.

Solution.

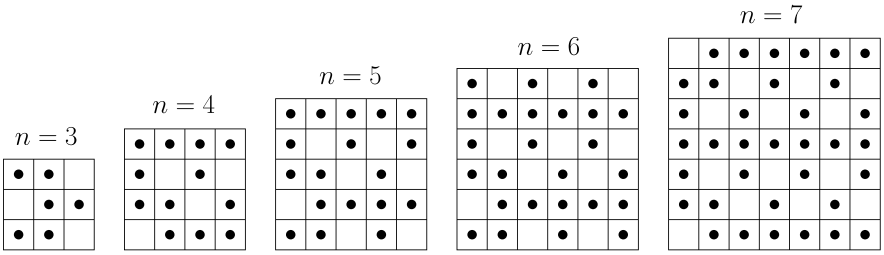
We will call any field that is only adjacent to fields with larger numbers a well. Other fields will be called non-wells. Let us make a second $n \times n$ board B where in each field we will write the number of good sequences which end on the corresponding field in the original board A. We will thus look for the minimal possible value of the sum of all entries in B.

We note that any well has just one good path ending in it, consisting of just the well, and that any other field has the number of good paths ending in it equal to the sum of this quantity for all the adjacent fields with smaller values, since a good path can only come into the field from a field of lower value. Therefore, if we fill in the fields in B in increasing order with respect to their values in A, it follows that each field not adjacent to any already filled field will receive a 1, while each field adjacent to already filled fields will receive the sum of the numbers already written on these adjacent fields.

We note that there is at least one well in A, that corresponding with the field with the entry 1 in A. Hence, the sum of values of fields in B corresponding to wells in A is at least 1. We will now try to minimize the sum of the non-well entries, i.e. of the entries in B corresponding to the non-wells in A. We note that we can ascribe to each pair of adjacent fields the value of the lower assigned number and that the sum of non-well entries will then equal to the sum of the ascribed numbers. Since the lower number is still at least 1, the sum of non-well entries will at least equal the number of pairs of adjacent fields, which is $2n(n-1)$. Hence, the total minimum sum of entries in B is at least $2n(n-1)+1 = 2n^2-2n+1$. The necessary conditions for the minimum to be achieved is for there to be only one well and for no two entries in B larger than 1 to be adjacent to each other.

We will now prove that the lower limit of $2n^2-2n+1$ entries can be achieved. This amounts to finding a way of marking a certain set of squares, those that have a value of 1 in B, such that no two unmarked squares are adjacent and that the marked squares form a connected tree with respect to adjacency.

For $n = 1$ and $n = 2$ the markings are respectively the lone field and the L-trimino. Now, for $n > 2$, let $s = 2$ for $n \equiv 0, 2 \mod 3$ and $s = 1$ for $n \equiv 1 \mod 3$. We will take indices k and l to be arbitrary non-negative integers. For $n \geqslant 3$ we will construct a path of marked squares in the first two columns consisting of all squares of the form $(1, i)$ where i is not of the form $6k + s$ and $(2, j)$ where j is of the form $6k + s - 1$, $6k + s$ or $6 + s + 1$. Obviously, this path is connected. Now, let us consider the fields $(2, 6k + s)$ and $(1, 6k + s + 3)$. For each considered field (i, j) we will mark all squares of the form (l, j) for $l > i$ and $(i + 2k, j \pm 1)$. One can easily see that no set of marked fields will produce a cycle, that the only fields of the unmarked form $(1, 6k + s)$, $(2 + 2l + 1, 6k + s \pm 1)$ and $(2 + 2l, 6k + s + 3 \pm 1)$ and that no two are adjacent, since the consecutive considered fields are in columns of opposite parity. Examples of markings are given for $n = 3, 4, 5, 6, 7$, and the corresponding constructions for A and B are given for $n = 5$.



$n = 5$:

A :

12	13	14	15	16
11	24	17	25	18
10	9	22	8	23
21	3	4	5	6
1	2	19	7	20

B :

1	1	1	1	1
1	4	1	4	1
1	1	4	1	3
3	1	1	1	1
1	1	3	1	2

Common remarks.

- The construction can be achieved in different ways. For example, it can also be done recursively; we can complete any construction for n to a construction for $n + 1$.
- It is a natural idea to change the direction of the path: that way it can start anywhere, but only can end in a well, which exactly means that we cannot extend the path. This is just a reformulation of the problem, but can give some intuitions.