Balkan MO Shortlist 2013

– Algebra

A1 Positive real numbers a, b, c satisfy $ab+bc+ca=3$. Prove the inequality

$$\frac{1}{4+(a+b)^2}+\frac{1}{4+(b+c)^2}+\frac{1}{4+(c+a)^2}\le\frac{3}{8}$$

A2 Let a, b, c and d are positive real numbers so that $abcd=\frac{1}{4}$. Prove that holds

$$\left(16ac+\frac{a}{c^2b}+\frac{16c}{a^2d}+\frac{4}{ac}\right)\left(bd+\frac{b}{256d^2c}+\frac{d}{b^2a}+\frac{1}{64bd}\right)\ge\frac{81}{4}$$

When does the equality hold?

A3 Prove that the polynomial $P(x)=(x^2-8x+25)(x^2-16x+100)...(x^2-8nx+25n^2)-1, n\in N^*$, cannot be written as the product of two polynomials with integer coefficients of degree greater or equal to 1.

A4 Find all positive integers n such that there exist non-constant polynomials with integer coefficients $f_1(x), ..., f_n(x)$ (not necessarily distinct) and $g(x)$ such that

$$1+\prod_{k=1}^{n}\left(f_k^2(x)-1\right)=(x^2+2013)^2g^2(x)$$

A5 Determine all positive integers$n$ such that $f_n(x,y,z)=x^{2n}+y^{2n}+z^{2n}-xy-yz-zx$ divides $g_n(x,y,z)=(x-y)^{5n}+(y-z)^{5n}+(z-x)^{5n}$, as polynomials in x, y, z with integer coefficients.

A6 Let S be the set of positive real numbers. Find all functions $f\colon S^3\to S$ such that, for all positive real numbers x, y, z and k, the following three conditions are satisfied:

(a) $xf(x,y,z)=zf(z,y,x)$,

(b) $f(x,ky,k^2z)=kf(x,y,z)$,

(c) $f(1,k,k+1)=k+1$.

(*United Kingdom*)

A7 Suppose that k is a positive integer. A bijective map $f: Z\to Z$ is said to be k-*jumpy* if $|f(z)-z|\le k$ for all integers z.
Is it that case that for every k, each k-jumpy map is a composition of 1-jumpy maps?
It is well known that this is the case when the support of the map is finite.

– Combinatorics

C1 In a mathematical competition, some competitors are friends; friendship is mutual, that is, when A is a friend of B, then B is also a friend of A.
We say that $n \geq 3$ different competitors $A_1, A_2, \ldots, A_n$ form a *weakly-friendly cycle* if A_i is not a friend of A_{i+1} for $1 \leq i \leq n$ (where $A_{n+1} = A_1$), and there are no other pairs of non-friends among the components of the cycle.

The following property is satisfied:

"for every competitor C and every weakly-friendly cycle $\mathcal{S}$ of competitors not including C, the set of competitors D in $\mathcal{S}$ which are not friends of C has at most one element"

Prove that all competitors of this mathematical competition can be arranged into three rooms, such that every two competitors in the same room are friends.

(*Serbia*)

C2 Some squares of an $n \times n$ chessboard have been marked ($n \in N^*$). Prove that if the number of marked squares is at least $n\left(\sqrt{n} + \frac{1}{2}\right)$, then there exists a rectangle whose vertices are centers of marked squares.

C3 The square $ABCD$ is divided into n^2 equal small (elementary) squares by parallel lines to its sides, (see the figure for the case $n = 4$). A spider starts from point$A$ and moving only to the right and up tries to arrive at point C. Every movement of the spider consists of: k steps to the right and m steps up or m steps to the right and k steps up (which can be performed in any way). The spider first makes l movements and in then, moves to the right or up without any restriction. If $n = m \cdot l$, find all possible ways the spider can approach the point C, where n, m, k, l are positive integers with $k < m$.

C4 A closed, non-self-intersecting broken line L is drawn over a $(2n+1) \times (2n+1)$ chessboard in such a way that the set of L's vertices coincides with the set of the vertices of the boards squares and every edge in L is a side of some board square. All board squares lying in the interior of L are coloured in red. Prove that the number of neighbouring pairs of red squares in every row of the board is even.

C5 The cells of an $n \times n$ chessboard are coloured in several colours so that no 2×2 square contains four cells of the same colour. A *proper path* of length m is a sequence $a_1, a_2, ..., a_m$ of distinct cells in which the cells a_i and a_{i+1} have a common side and are coloured in different colours for all $1 \leq i < m$. Show that there exists a proper path of length n.

–	Geometry
G1	In a triangle ABC, the excircle ω_a opposite A touches AB at P and AC at Q, while the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN and let L be the projection of C onto PQ. Show that the quadrilateral $MKLP$ is cyclic. (*Bulgaria*)
G2	Let $ABCD$ be a quadrilateral, let O be the intersection point of diagonals AC and BD, and let P be the intersection point of sides AB and CD. Consider the parallelograms $AODE$ and $BOCF$. Prove that E, F and P are collinear.
G3	Two circles Γ_1 and Γ_2 intersect at points M, N. A line ℓ is tangent to Γ_1, Γ_2 at A and B, respectively. The lines passing through A and B and perpendicular to ℓ intersects MN at C and D respectively. Prove that $ABCD$ is a parallelogram.
G4	Let $c(O, R)$ be a circle, AB a diameter and C an arbitrary point on the circle different than A and B such that $\angle AOC > 90^o$. On the radius OC we consider point K and the circle $c_1(K, KC)$. The extension of the segment KB meets the circle (c) at point E. From E we consider the tangents ES and ET to the circle (c_1). Prove that the lines BE, ST and AC are concurrent.
G5	Let ABC be an acute triangle with $AB < AC < BC$ inscribed in a circle (c) and let E be an arbitrary point on its altitude CD. The circle (c_1) with diameter EC, intersects the circle (c) at point K (different than C), the line AC at point L and the line BC at point M. Finally the line KE intersects AB at point N. Prove that the quadrilateral $DLMN$ is cyclic.

–	Number Theory
N1	Let p be a prime number. Determine all triples (a, b, c) of positive integers such that $a+b+c < 2p\sqrt{p}$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{p}$
N2	Determine all positive integers x, y and z such that $x^5 + 4^y = 2013^z$. (*Serbia*)
N3	Determine all quadruplets (x, y, z, t) of positive integers, such that $12^x + 13^y - 14^z = 2013^t$.
N4	Let p be a prime number greater than 3. Prove that the sum $1^{p+2} + 2^{p+2} + ... + (p-1)^{p+2}$ is divisible by p^2.
N5	Prove that there do not exist distinct prime numbers p and q and a positive integer n satisfying the equation $p^{q-1} - q^{p-1} = 4n^2$
N6	Prove that there do not exist distinct prime numbers p and q and a positive integer n satisfying the equation $p^{q-1} - q^{p-1} = 4n^3$
N7	Two distinct positive integers are called *close* if their greatest common divisor equals their difference. Show that for any n, there exists a set S of n elements such that any two elements of S are close.
N8	Suppose that a and b are integers. Prove that there are integers c and d such that $a+b+c+d=0$ and $ac+bd=0$, if and only if $a-b$ divides $2ab$.
N9	Let $n \geq 2$ be a given integer. Determine all sequences $x_1, ..., x_n$ of positive rational numbers such that $x_1^{x_2} = x_2^{x_3} = ... = x_{n-1}^{x_n} = x_n^{x_1}$