2007 Balkan MO Shortlist

– Algebra

A1 Find the minimum and maximum value of the function

$$f(x,y) = ax^2 + cy^2$$

Under the condition $ax^2 - bxy + cy^2 = d$, where a, b, c, d are positive real numbers such that $b^2 - 4ac < 0$

A2 Find all values of $a \in \mathbb{R}$ for which the polynomial

$$f(x) = x^4 - 2x^3 + \left(5 - 6a^2\right)x^2 + \left(2a^2 - 4\right)x + \left(a^2 - 2\right)^2$$

has exactly three real roots.

A3 For $n \in \mathbb{N}$, $n \geq 2$, $a_i, b_i \in \mathbb{R}$, $1 \leq i \leq n$, such that

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1, \sum_{i=1}^{n} a_i b_i = 0.$$

Prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \leq n.$$

Cezar Lupu & Tudorel Lupu

A4 Show that the sequence

$$a_n = \left\lfloor \left(\sqrt[3]{n-2} + \sqrt[3]{n+3}\right)^3 \right\rfloor$$

contains infinitely many terms of the form $a_n^{a_n}$

A5 find all the function $f, g : R \to R$ such that
(1)for every $x, y \in R$ we have $f(xg(y+1)) + y = xf(y) + f(x + g(y))$
(2)$f(0) + g(0) = 0$

A6 Find all real functions f defined on $\mathbb{R}$, such that

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y,$$

for all real numbers x, y.

A7 Find all positive integers n such that there exist a permutation σ on the set $\{1, 2, 3, \ldots, n\}$ for which

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\ldots + \sqrt{\sigma(n-1) + \sqrt{\sigma(n)}}}}}$$

is a rational number.

A8 Let $c > 2$ and $a_0, a_1, \ldots$ be a sequence of real numbers such that

$$a_n = a_{n-1}^2 - a_{n-1} < \frac{1}{\sqrt{cn}}$$

for any $n \in \mathbb{N}$. Prove, $a_1 = 0$

– Combinatorics

C1 For a given positive integer $n>2$, let C_1, C_2, C_3 be the boundaries of three convex $n-$ gons in the plane , such that
$C_1 \cap C_2, C_2 \cap C_3, C_1 \cap C_3$ are finite. Find the maximum number of points of the sets $C_1 \cap C_2 \cap C_3$.

C2 Let $\mathcal{F}$ be the set of all the functions $f : \mathcal{P}(S) \longrightarrow \mathbb{R}$ such that for all $X, Y \subseteq S$, we have $f(X \cap Y) = \min(f(X), f(Y))$, where S is a finite set (and $\mathcal{P}(S)$ is the set of its subsets). Find

$$\max_{f \in \mathcal{F}} |\mathrm{Im}(f)|.$$

C3 Three travel companies provide transportation between n cities, such that each connection between a pair of cities is covered by one company only. Prove that, for $n \geq 11$, there must exist a round-trip through some four cities, using the services of a same company, while for $n < 11$ this is not anymore necessarily true.

Dan Schwarz

– Geometry

G1 Let ω be a circle with center O and let A be a point outside ω. The tangents from A touch ω at points B, and C. Let D be the point at which the line AO intersects the circle such that O is between A and D. Denote by X the orthogonal projection of B onto CD, by Y the midpoint of the segment BX and by Z the second point of intersection of the line DY with ω. Prove that ZA and ZC are perpendicular to each other.

G2 Let $ABCD$ a convex quadrilateral with $AB = BC = CD$, with AC not equal to BD and E be the intersection point of it's diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120$.

G3 Let $A_1A_2A_3A_4A_5$ be a convex pentagon, such that

$$[A_1A_2A_3] = [A_2A_3A_4] = [A_3A_4A_5] = [A_4A_5A_1] = [A_5A_1A_2].$$

Prove that there exists a point M in the plane of the pentagon such that

$$[A_1MA_2] = [A_2MA_3] = [A_3MA_4] = [A_4MA_5] = [A_5MA_1].$$

Here $[XYZ]$ stands for the area of the triangle ΔXYZ.

G4 Points M, N and P on the sides BC, CA and AB of $\triangle\ ABC$ are such that $\triangle\ MNP$ is acute. Denote by h and H the lengths of the shortest altitude of $\triangle\ ABC$ and the longest altitude of $\triangle\ MNP$. Prove that $h \leq 2H$.

– Number Theory

N1 Solve the given system in prime numbers

$$x^2 + yu = (x + u)^v$$

$$x^2 + yz = u^4$$

N2 Prove that there are no distinct positive integers x and y such that

$x^{2007} + y! = y^{2007} + x!$

N3 i thought that this problem was in mathlinks but when i searched i didn't find it.so here it is: Find all positive integers m for which for all $\alpha, \beta \in \mathbb{Z} - \{0\}$

$$\frac{2^m \alpha^m - (\alpha + \beta)^m - (\alpha - \beta)^m}{3\alpha^2 + \beta^2} \in \mathbb{Z}$$

N4 Find all infinite arithmetic progressions formed with positive integers such that there exists a number $N \in \mathbb{N}$, such that for any prime p, $p > N$,
the p-th term of the progression is also prime.

N5 Let $p \geq 5$ be a prime and let

$$(p-1)^p + 1 = \prod_{i=1}^{n} q_i^{\beta_i}$$

where q_i are primes. Prove,

$$\sum_{i=1}^{n} q_i \beta_i > p^2$$