3RD EUROPEAN MATHEMATICAL CUP

6th December 2014–14th December 2014

Junior Category

MLADI NADARENI MATEMATIČARI **Marin Getaldić**

Problems and Solutions

Problem 1. Which of the following claims are true, and which of them are false? *If a fact is true you should prove it, if it isn't, find a counterexample.*

a) Let a, b, c be real numbers such that $a^{2013} + b^{2013} + c^{2013} = 0$. Then $a^{2014} + b^{2014} + c^{2014} = 0$.

b) Let a, b, c be real numbers such that $a^{2014} + b^{2014} + c^{2014} = 0$. Then $a^{2015} + b^{2015} + c^{2015} = 0$.

c) Let a, b, c be real numbers such that $a^{2013} + b^{2013} + c^{2013} = 0$ and $a^{2015} + b^{2015} + c^{2015} = 0$. Then $a^{2014} + b^{2014} + c^{2014} = 0$.

(Matko Ljulj)

Solution. Firstly, we know that for every real number x, $x^2 \geqslant 0$ holds.
The key idea in this problem is to realize that the expression $a^{2014} + b^{2014} + c^{2014}$ is a sum of squares (which are nonnegative numbers). Thus $a^{2014} + b^{2014} + c^{2014} = 0 \iff a = b = c = 0$.

a) NO: It is sufficient to find three real numbers whose sum equals 0, and then take their 2013^{th} roots. For example: $a = \sqrt[2013]{1}$, $b = \sqrt[2013]{2}$, $c = \sqrt[2013]{-3}$.

b) YES: From the key idea we conclude $a = b = c = 0$, and then we conclude $a^{2015} + b^{2015} + c^{2015} = 0 + 0 + 0 = 0$.

c) NO: Again we have to find a counterexample, for instance $a = 1, b = 0, c = -1$.

Problem 2. In each vertex of a regular n-gon $A_1A_2...A_n$ there is a unique pawn. In each step it is allowed:

1. to move all pawns one step in the clockwise direction or
2. to swap the pawns at vertices A_1 and A_2.

Prove that by a finite series of such steps it is possible to swap the pawns at vertices:

a) A_i and A_{i+1} for any $1 \leqslant i < n$ while leaving all other pawns in their initial place

b) A_i and A_j for any $1 \leqslant i < j \leqslant n$ leaving all other pawns in their initial place.

(Matija Bucić)

Solution. We denote a pawn that was initially at point A_i as i. We will prove part a) and then use it to show part b).

a) We apply first operation $i - 1$ times which will bring i and $i + 1$ to points A_1 and A_2 and move every other pawn $i - 1$ steps in clockwise direction.
We can now apply second operation to swap i and $i + 1$ as they are at points A_1 and A_2. This does not affect the position of any other pawn.
We now apply first operation $n - i + 1$ times returning pawn $k \neq i, i + 1$ to point A_k while moving pawn i to A_{i+1} and pawn $i + 1$ to A_i which is exactly what we wanted.

b) We present 2 possible solutions, one using induction and one not using induction.
Solution 1: By using the previous problem we can swap pawns $(i, i + 1)$ as they are at points (A_i, A_{i+1}) then $(i, i+2)$ as they are at points (A_{i+1}, A_{i+2}) and carry on until we swap (i, j) as they were at points (A_{j-1}, A_j). This brings us to the state where i is at A_j and each $i + 1 \leqslant k \leqslant j$ is at point A_{k-1}.
We can now apply part a to swap j with $j - 1$ and similarly carry on till we swap j with $i + 1$. This will place j at A_i and move each $i + 1 \leqslant k \leqslant j - 1$ to A_k.
This brings us to the state where we swapped pawns i and j leaving others where they were just as was desired. □

Solution 2**:** We use induction on n for the following claim:

We can swap any two pawns $1 \leqslant i < j \leqslant k$.

We note that the basis is exactly part a.

We assume we the claim holds for some k.

Hence we can swap any pawns $1 \leqslant i < j \leqslant k$ and only need to show that we can swap i and $k+1$ for any $1 \leqslant i \leqslant k$.

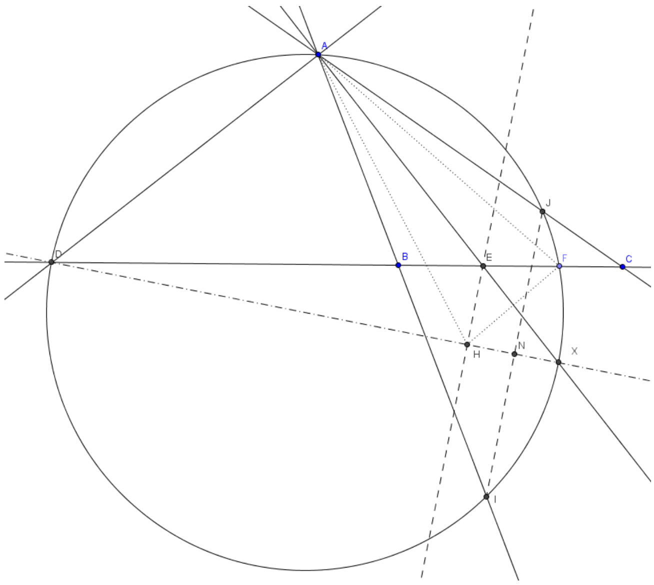
This follows as we can swap i and k then k and $k+1$ by part a). then again $k+1$ and i as they are now on points A_k and A_i.

□

Problem 3. Let ABC be a triangle. The external and internal angle bisectors of $\angle CAB$ intersect side BC at D and E, respectively. Let F be a point on the segment BC. The circumcircle of triangle ADF intersects AB and AC at I and J, respectively. Let N be the mid-point of IJ and H the foot of E on DN. Prove that E is the incenter of triangle AHF.

(Steve Dinh)

Solution. Denote by ω the circumcircle of $\triangle AHF$.



The key idea in the problem is to introduce a new point X which we define as the second intersection of DN and ω.

We now note that the $\angle JAD = \angle CAD = 90^\circ \pm \frac{\alpha}{2}$ and $\angle IAD = \angle BAD = 90^\circ \pm \frac{\alpha}{2}$ where $\alpha = \angle CAB$. As AD is an external bisector of $\angle CAB$.

The $\pm$ *signs depend on the picture and student shouldn't be deducted any points for not noticing this.*

Hence we have either $\angle JAD = \angle BAD$ or $\angle JAD + \angle IAD = 180^\circ$ so in both cases $DI = DJ$.

Now as N is midpoint of IJ this means that DN is bisector of IJ and hence pasess through the centre of the. This shows that DX is a diameter of ω and $EH||IJ$.

We also notice that $\angle EAD = 90^\circ$ as angle between bisectors and $\angle XAD = 90^\circ$ as DX is a diameter. Hence X, A, E are collinear.

Now this gives us $\angle DHE = \angle XHE = 90^\circ$ and $\angle XFE = \angle DFE = 90^\circ$ as DX is a diameter of ω and finally again $\angle EAD = 90^\circ$. All this gives us that quadrilaterals $XFEH$ and $ADEH$ are cyclic.

Final step is to use some angle chasing to get $\angle AHE = \angle ADH = \angle AXF = \angle EXF = \angle EHF$ where first, second and fourth equalities are due to cyclicity of $ADEH$, $ADXF$ and $XFEH$ respectively. Also $\angle DFH = \angle EFH = \angle EXH = \angle AFD = \angle AFE$ where the second and forth equalities are due to cyclicity of $XFEH$ and $ADXF$ respectively. This shows E is the incenter of $\triangle AFH$ as desired.

Problem 4. Find all infinite sequences $a_1, a_2, a_3, \ldots$ of positive integers such that

a) $a_{nm} = a_n a_m$, for all positive integers n, m, and

b) there are infinitely many positive integers n such that $\{1, 2, \ldots, n\} = \{a_1, a_2, \ldots, a_n\}$.

(Matko Ljulj)

Solution. Instead of sequence a_n, we'll use notation with the function $f(n)$ with same properties.

There exists only one such function: $f(n) = n$. We'll solve the problem with many separate facts.

Fact 1: $f(1) = 1$.

Proof: According to a) it holds $f(1) = f(1)f(1) = f(1)^2$. Since $f(1)$ is positive integer, it can't be $f(1) = 0$, so it must be $f(1) = 1$.

Fact 2: Function f is bijective.

Proof: Firstly, we'll show that f is injective. Let $a \neq b$ be two arbitrary positive integers and let's assume $f(a) = f(b)$. Since $\{1, 2, \dots, n\} = \{f(1), f(2), \dots, f(n)\}$ holds for infinitely many positive integers n, it holds for some integer greater than a and b. Then, since $f(a) = f(b)$, set $\{f(1), f(2), \dots, f(n)\}$ contains $n-1$ or less (different) elements, but according to b), it contains n elements.

Secondly, we'll show that f is surjective. Let c be arbitrary integer and let's assume that $f(n) \neq c$ for all positive integers n. Similarly as in first part of proof, let's take positive integer n such that $\{1, 2, \dots, n\} = \{f(1), f(2), \dots, f(n)\}$ holds. Since $c \in \{1, 2, \dots, n\}$, c is also element of the set $\{f(1), f(2), \dots, f(n)\}$, so there exists positive integer $m \leqslant n$ such that $f(m) = c$.

Fact 3: Positive integer n is prime if and only if $f(n)$ is prime.

Proof: Let's assume that n is prime, but $f(n)$ isn't. Then it must be $f(n) = a'b' = f(a)f(b) = f(ab)$, where a', b' are positive integers greater than 1, and a, b are unique positive integers such that $f(a) = a'$, $f(b) = b'$ (they exist since f is bijective). Since f is injective, $f(1) = 1$ and a', b' are not equal to 1, integers a, b are also not equal to 1. Since f is injective and $f(n) = f(ab)$, we have $n = ab$, so n is composite.

Let's assume that $f(n)$ is prime, but n isn't. Then there exist positive integers a, b greater than one such that $n = ab$. From there we have $f(n) = f(ab) = f(a)f(b)$. Again from injectivity of f and $f(1) = 1$, we see that $f(n)$ is product of two integers greater than 1.

Fact 4: If $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ is unique factorization of positive integer n, then

$$f(n) = f(p_1)^{a_1} f(p_2)^{a_2} \dots f(p_k)^{a_k}$$

is unique factorization of positive integer $f(n)$.

Proof: From multiple use of the condition a) we get identity $f(n) = f(p_1)^{a_1} f(p_2)^{a_2} \dots f(p_k)^{a_k}$. From Fact 3, numbers $f(p_i)$ are prime. Since f is injective, none of two numbers $f(p_i)$ and $f(p_j)$ are equal.

Fact 5: (Technical result) For all positive integers $y < x$ there exist positive integer n_0 such that for all positive integers $n \geqslant n_0$ holds inequality

$$y^{n+1} < x^n.$$

Proof: It is sufficient to prove the fact only for consecutive integers y and $y+1$ (because we'll have $y^{n+1} < (y+1)^n \leqslant x^n$). By binomial theorem we have

$$(y+1)^n \geqslant y^n + ny^{n-1} = y^{n-1}(y+n).$$

Thus if we define $n_0 = y^2 - y + 1$, then for all $n \geqslant n_0$ we have

$$(y+1)^n \geqslant y^{n-1}(y+n) \geqslant y^{n-1}(y+n_0) = y^{n-1}(y^2+1) > y^{n+1}.$$

Another proof: Inequality is equivalent to

$$\left(\frac{x}{y}\right)^n > y.$$

The fact follows from the fact that the expression on the left hand side is increasing and it is unbounded, while the right hand side is fixed.

Fact 6: For all prime numbers p we have $f(p) \leqslant p$.

Proof: Let $p_1, p_2, \dots, p_n, \dots$ be the increasing sequence $2, 3, 5, 7, \dots$ of all prime numbers. Let's take arbitrary prime number p_n. From the Fact 3 we have that $f(p_n)$ is also a prime. Let's take positive integer n_0 as the integer from the Fact 5, for positive integers $y = p_n < p_{n+1} = x$. Since b) holds for infinitely many positive integers, it holds for some positive integer N such that $\{1, 2, \dots, N\} = \{f(1), f(2), \dots, f(N)\}$, and such that $N \geqslant p_n^{n_0}$. Let α be the greatest positive integer such that $p_n^\alpha \leqslant N$. From definitions of N and α we have $\alpha \geqslant n_0$.

In set $\{1, 2, \dots, N\}$ we'll observe all positive integers which are α^{th} power of a prime number. Since $N \geqslant p_n^\alpha$, we have that p_n^α is in that set. It is easy to see that all numbers $p_1^\alpha, \dots, p_{n-1}^\alpha$ are also in that set. On the contrary, number p_{n+1}^α is not in that set, because from the definition of α and N respectively we have $N < p_n^{\alpha+1} \leqslant p_{n+1}^\alpha$ (remember Fact 5 and $\alpha \geqslant n_0$). Similarly, neither of the numbers p_m^α (for $m > n$) is not in the set $\{1, 2, \dots, N\}$.

Let us now observe all positive integers which are α^{th} power of a prime and they are in the set $\{f(1), f(2), \dots, f(N)\}$. According to Fact 4, we have that $f(n)$ is α^{th} power of a prime if and only if n is α^{th} power of a prime. From that and from previous paragraph we conclude that only such numbers are $f(p_1^\alpha), \dots, f(p_n^\alpha)$.

Now we have $\{p_1^\alpha, \dots, p_n^\alpha\} = \{f(p_1^\alpha), \dots, f(p_n^\alpha)\}$. Thus $f(p_n^\alpha) \in \{p_1^\alpha, \dots, p_n^\alpha\}$, so $f(p_n^\alpha) = p_i^\alpha$ for some $1 \leqslant i \leqslant n$, which implies $f(p_n)^\alpha = p_i^\alpha$, for some $1 \leqslant i \leqslant n \implies f(p_n) = p_i \leqslant p_n$, which completes the proof.

Fact 7: For every positive integer we have $f(n) = n$.

Proof: From Fact 3 we have that $f(p)$ if and only if p is prime. Let $p_1, p_2, \ldots, p_n, \ldots$ be the increasing sequence $2, 3, 5, 7, \ldots$ of all prime numbers. From Fact 6 we have $f(p_1) \leqslant p_1 \implies f(2) = 2$. For $n \geqslant 2$, inductively and from injectivity of f we have $f(p_n) > p_{n-1}$ and from Fact 6 we have $f(p_n) \leqslant p_n$, thus is must be $f(p_n) = p_n$, for all positive integers n.

Now for arbitrary positive integer n from Fact 4 we have

$$f(n) = f(p_1)^{a_1} f(p_2)^{a_2} \ldots f(p_k)^{a_k} = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k} = n,$$

which completes our proof.

Remark: We can prove Fact 6 differently (without using Fact 5). We observe numbers $1 \cdot 2 \cdot \ldots \cdot n$ and $f(1) \cdot f(2) \cdot \ldots \cdot f(n)$, and their unique factorizations. They coincide for infinitely many positive integers n. For fixed primes p, q, if we take sufficiently great n, we can use well-known formula for $\nu_p(n!)$ to prove that $\nu_p(n!) > \nu_q(n!)$ for all $q > p$ (here positive integer n depends on p, q).

3^{RD} European Mathematical Cup

6^{th} December 2014–14^{th} December 2014

Senior Category

MLADI NADARENI MATEMATIČARI
Marin Getaldić

Problems and Solutions

Problem 1. Prove that there are infinitely many positive integers which can't be expressed as $a^{d(a)} + b^{d(b)}$ where a and b are positive integers.

For positive integer a expression $d(a)$ denotes the number of positive divisors of a. *(Borna Vukorepa)*

Solution. We will show that $a^{d(a)}$ is a square of an integer for every positive integer a.

If a is a square of an integer, **any its power is also a square of an integer.**

If a is not a perfect square, number of it's positive divisors is even. We can prove this by pairing divisiors of a as d and $\frac{a}{d}$. A divisor d won't be paired with itself because that would imply $a = d^2$. This proves that $d(a)$ is even and hence $a^{d(a)}$ is a perfect square for every positive integer a.

The expression in the problem is hence a sum of two squares. Every number of the form $4t+3$ can't be written as a sum of two squares because 0 and 1 are the only quadratic residues modulo 4, so it is impossible for a sum of two squares to give remainder 3 modulo 4.

Problem 2. Jeck and Lisa are playing a game on an $m \times n$ board, with $m, n > 2$. Lisa starts by putting a knight onto the board. Then in turn Jeck and Lisa put a new piece onto the board according to the following rules:

1. Jeck puts a queen on an empty square that is two squares horizontally and one square vertically, or alternatively one square horizontally and two squares vertically, away from Lisa's last knight.
2. Lisa puts a knight on an empty square that is on the same, row, column or diagonal as Jeck's last queen.

The one who is unable to put a piece on the board loses the game. For which pairs (m, n) does Lisa have a winning strategy?

(Stijn Cambie)

Solution. We shall show that Lisa has a winning strategy if and only if m and n are both odd.

Lisa's winning strategy

Suppose the game is played on an $m \times n$ board with m and n both odd. Then Lisa puts her first knight in a corner and partitions the remaining squares of the board into 'dominoes'. In each turn Jeck has to put a queen in one of these dominoes and Lisa puts a knight on the other square of the domino. As the board is finite, Jeck can't keep finding new dominoes and so Lisa will win.

Jeck's winning strategy

Suppose the game is played on an $m \times n$ board with m or n even. We shall show that Jeck is able to partition the board into pairs of squares that are two squares horizontally and one square vertically, or alternatively one square horizontally and two squares vertically, away from each other. In each turn Lisa has to put a knight in one of these and Jeck puts a queen on the other square of the pair. As the board is finite, Lisa can't keep finding new pairs and so Jeck will win. Now we prove that Jeck can make the required partition.

Case 1. Suppose $4|m$ or $4|n$. We know that any $k \times 4l$ board ($k \geq 2$) can be divided into 2×4 and 3×4 boards (firstly divide $k \times 4l$ board in l boards of dimensions $k \times 4$; after that every $k \times 4$ board divide in $\frac{k}{2}$ boards of dimensions 2×4, or in $\frac{k-3}{2}$ boards of dimensions 2×4 and one 3×4 board, dependently on parity of k). The following diagrams show that every 2×4 and every 3×4 board allows a required partition.

1	2	3	4
3	4	1	2

1	2	3	4
3	5	1	6
2	6	4	5

1 point.

Case 2. Suppose $m, n \equiv 1, 2 \pmod 4$. Any $(5+4k)\times(6+4l)$ board can be divided into a 5×6 board, a $4k\times 6$ board, a $5\times 4l$ board and a $4k\times 4l$ board. The following diagram shows that a 5×6 board allows a required partition.

1	2	14	13	12	11
3	4	12	11	14	15
2	1	13	15	7	8
4	3	5	6	9	10
5	6	9	10	8	7

According to case 1 a $4k\times 6$ board, a $5\times 4l$ board and a $4k\times 4l$ board also allow a partition.

Case 3. Suppose $m, n \equiv 2, 3 \pmod 4$. Any $(3+4k)\times(6+4l)$ board can be divided into a 3×6 board, a $4k\times 6$ board, a $3\times 4l$ board and a $4k\times 4l$ board. The following diagram shows that a 3×6 board allows a required partition.

1	2	3	4	7	8
3	4	1	6	9	5
2	6	9	5	8	7

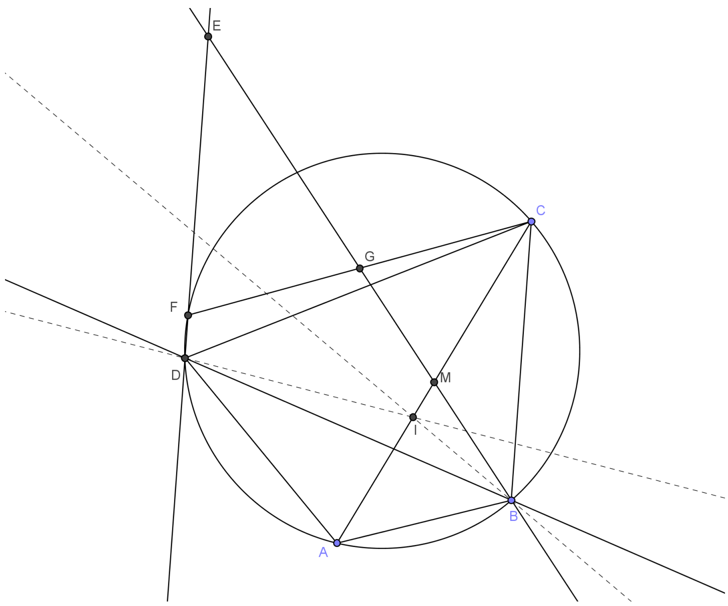
According to case 1 a $4k\times 6$ board, a $3\times 4l$ board and a $4k\times 4l$ board also allow a partition.

Case 4. Suppose $m, n \equiv 2 \pmod 4$ .Any $(6+4k)\times(6+4l)$ board can be divided into a 6×6 board, a $4k\times 6$ board, a $6\times 4l$ board and a $4k\times 4l$ board. The 6×6 board can be partitioned in two 3×6 boards, which were already solved. According to case 1 a $4k\times 6$ board, a $6\times 4l$ board and a $4k\times 4l$ board also allow a partition.

Problem 3. Let $ABCD$ be a cyclic quadrilateral with the intersection of internal angle bisectors of $\angle ABC$ and $\angle ADC$ lying on the diagonal AC. Let M be the midpoint of AC. The line parallel to BC that passes through D intersects the line BM in E and the circumcircle of $ABCD$ at F where $F\neq D$. Prove that $BCEF$ is a parallelogram.

(Steve Dinh)

Solution. We prove the problem in reverse as this is much more natural in this problem.



We note that if $BCEF$ is a parallelogram then its diagonals are bisecting each other so the point $G\equiv BE\cap CF$ should be the midpoint of CF.

If G is the midpoint of CF then $\triangle GBC$ and $\triangle GEF$ are congruent as $CG = GF$ and $FE||BC$ gives $\angle GEF = \angle GBC$ and $\angle GFE = \angle GCB$. Hence this implies $BG = GE$ and in particular $BCEF$ is a paralelogram as its diagonals bisect each other. Hence G being midpoint of CF is equivalent to our problem.

As M is the midpoint of AC by the midline theorem applied to triangle ACF we have G is the midpoint of CG if and only if $MG||AF$. Hence we only need to prove $BM||AF$.

Now we further notice that, using $FD||BC$, this is equivalent to $\angle AFD = \angle MBC$.

We further see that $\angle AFD = \angle ABD$ as they are angles over the same chord. So our claim is equivalent to $\angle ABD = \angle MBC$.

We add that here depending on the relative position of F on the circles we might have $\pi - \angle AFD = \angle MBC$ but then $\pi - \angle AFD = \angle ABD$ so the final conclusion still holds.

We know that $\angle BDA = \angle BCM$ as they are angles over the same chord. Now this gives us that our claim is equivalent to the claim $\triangle BCM \sim \triangle BDA$.

The same angle equality shows that this is equivalent to $\frac{BC}{CM} = \frac{AD}{BD}$. Using the fact M is the midpoint of AC we have $CM = \frac{AC}{2}$ so our claim is equivalent to $2AD \cdot BC = BD \cdot AC$.

We further have by the angle bisector theorem applied to $\triangle ABC$ and $\triangle CDA$:

$$\frac{AB}{BC} = \frac{AI}{CI} = \frac{AD}{CD}$$

So using this our claim is equivalent to $AB \cdot CD + AD \cdot BC = BD \cdot AC$ which we can recognise to be the Ptolomeys theorem for cyclic quadrilaterals.

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following holds:

$$f\left(x^2\right) + f\left(2y^2\right) = (f(x+y) + f(y))(f(x-y) + f(y)).$$

(Matija Bucić)

Solution. Let $P(x,y)$ be the assertion $f\left(x^2\right) + f\left(2y^2\right) = (f(x+y) + f(y))\left(f(x-y) + f(y)\right)$.

$P(0,x)$ gives us

$$f(0) + f\left(2x^2\right) = 2f(x)\left(f(x) + f(-x)\right) \tag{1}$$

and $P(0,-x)$ gives us

$$f(0) + f\left(2x^2\right) = 2f(-x)\left(f(x) + f(-x)\right). \tag{2}$$

By combining (1) and (2) we get

$$f(x)^2 = f(-x)^2. \tag{3}$$

$P(0,0)$ gives us $2f(0) = 4f(0)^2$, thus we have two cases:

1. $f(0) = \frac{1}{2}$.

   $P(x,0)$ gives us

   $$f(x^2) = \left(f(x) + \frac{1}{2}\right)^2 - \frac{1}{2}, \tag{4}$$

   while $P(-x,0)$ gives us

   $$f(x^2) = \left(f(-x) + \frac{1}{2}\right)^2 - \frac{1}{2}. \tag{5}$$

   Combining (4) and (5) and using (3) we get

   $$f(x) = f(-x). \tag{6}$$

   The assertion $P\left(x^2, x^2\right)$ can be written as

   $$f\left(x^4\right) + f\left(2x^4\right) = \left(f\left(2x^2\right) + f\left(x^2\right)\right)\left(\frac{1}{2} + f\left(x^2\right)\right). \tag{7}$$

   For an arbitrary $x \in \mathbb{R}$, let us denote $a = f(x)$. Using (4) we get:

   $$f\left(x^2\right) = \left(a + \frac{1}{2}\right)^2 - \frac{1}{2},$$
   $$f\left(x^4\right) = \left(f\left(x^2\right) + \frac{1}{2}\right)^2 - \frac{1}{2} = \left(a + \frac{1}{2}\right)^4 - \frac{1}{2}.$$

   Using (1) and (6) we get:

   $$f\left(2x^2\right) = 4f(x)^2 - \frac{1}{2} = 4a^2 - \frac{1}{2},$$
   $$f(2x^4) = 4f(x^2)^2 - \frac{1}{2} = 4\left(\left(a + \frac{1}{2}\right)^2 - \frac{1}{2}\right)^2 - \frac{1}{2}.$$

   Plugging the last 4 equations in (7) we get:

   $$\left(a + \frac{1}{2}\right)^4 + 4\left(\left(a + \frac{1}{2}\right)^2 - \frac{1}{2}\right)^2 - 1 = \left(4a^2 - 1 + \left(a + \frac{1}{2}\right)^2\right)\left(a + \frac{1}{2}\right)^2,$$

   which is equivalent to

   $$\left(a + \frac{1}{2}\right)^2 (4a - 2) = 0.$$

Therefore $a = \pm\frac{1}{2}$ and $f(x) = \pm\frac{1}{2}$. Now if we use (6) in (1) we get

$$f(0) + f\left(2x^2\right) = 4(f(x))^2 = 1$$

so $f\left(2x^2\right) = \frac{1}{2}$ for every x, now using (6) we conclude $f(x) = \frac{1}{2}$ for all x which is easily checked to be a solution.

2. $f(0) = 0$.
We immediately see using $P(x, 0)$ that

$$f(x^2) = f(x)^2. \tag{8}$$

By comparing $P(x, y)$ and $P(x, -y)$ and using (3) we get:

$$\left(f(y) - f(-y)\right)\left(f(x+y) + f(x-y)\right) = 0.$$

If there exists $c \in \mathbb{R}$ such that $f(c) \neq f(-c)$ we have for all x

$$f(x+c) = -f(x-c),$$

Plugging in $x + c$ in x here gives us:

$$f(x+2c) = -f(x). \tag{9}$$

Specially, $f(2c) = 0$. Now, $P(2c - y, y)$:

$$\begin{aligned} f((2c-y)^2) + f(2y^2) &= (f(2c) + f(y))(f(2c-2y) + f(y)), \\ (-f(-y))^2 + f(2y^2) &= f(y)f(2c-2y) + f(y)^2, \\ f(2y^2) &= f(y)f(2c-2y) = -f(y)f(-2y) \end{aligned} \tag{10}$$

Let $S(x)$ denote the statement $(x \neq 0) \wedge (f(x) = f(-x) \neq 0)$. If there is no $d \in \mathbb{R}$ such that $S(d)$ then $f(x) = -f(-x)$ for all $x \in \mathbb{R}$. $P(0, x)$ gives us

$$f(2x^2) = 2f(x)(f(x) + f(-x)) = 0,$$

which gives us another solution $f(x) = 0$. Now, let us assume that there exists $d \in \mathbb{R}$ such that $S(d)$ holds. Obviously, $S(-d)$ holds, as well. $P(0, d)$ gives us

$$f(2d^2) = 4f(d)^2$$

and (10) gives us

$$\begin{aligned} f(2d^2) &= -f(d)f(-2d) \\ f(-2d) &= -4f(d) \\ f(2d) &= -4f(-d) = -4f(d) = f(-2d) \end{aligned}$$

Therefore, $S(2d)$ also holds. Inductively, we deduce that $S(2^n d)$ holds for every $n \in \mathbb{N}$. Also, $f(2^n d) = (-4)^n f(d)$, which means that f is unbounded.
$P(x, c)$, using the fact $f(x^2) = f(x)^2$:

$$f(x)^2 + f(2c^2) = f(x+c)f(x-c) + f(c)(f(x+c) + f(x-c)) + f(c)^2,$$

and since $f(x+c) = -f(x-c)$ and $f(2c^2) = 0$ (this follows from $P(0, c)$) we have

$$f(x)^2 + f(x+c)^2 = f(c)^2,$$

which implies that f is bounded and that is contradiction. Therefore, there is no $c \in \mathbb{R}$ such that $f(c) = -f(c)$ and therefore

$$f(x) = f(-x), \quad \text{for all } x \in \mathbb{R}. \tag{11}$$

$P(0, x)$:

$$f(2x^2) = 4f(x)^2 = 4f(x^2).$$

Therefore, using (11)

$$f(2x) = 4f(x), \quad \text{for all } x \in \mathbb{R}. \tag{12}$$

$P(x, y)$ can now be written as follows:

$$f(x)^2 + 3f(y)^2 = f(y)(f(x+y) + f(x-y)) + f(x+y)f(x-y),$$

and similarly, $P(y, x)$ can be written as

$$f(y)^2 + 3f(x)^2 = f(x)(f(x+y) + f(x-y)) + f(x+y)f(x-y).$$

subtracting the previous two equalities

$$(f(x) - f(y))(2f(x) + 2f(y) - f(x+y) - f(x-y)) = 0. \tag{13}$$

Assume that for some $x, y \in \mathbb{R}$ $f(x) = f(y) = a$. Let $f(x+y) = b$ and $f(x-y) = c$.

Now we have:

$$4a^2 = bc + ab + ac \tag{14}$$

$P(x+y, x-y)$:

$$f(x+y)^2 + 4f(x-y)^2 = (f(2x) + f(x-y))(f(2y) + f(x-y)),$$

i.e.

$$b^2 + 4c^2 = (4a+c)^2 \tag{15}$$

If we plug in $x \to x+y$, $y \to x-y$ in (13) we get

$$(f(x+y) - f(x-y))(2f(x+y) + 2f(x-y) - f(2x) - f(2y)) = 0$$

i.e.

$$(b-c)(2b + 2c - 8a) = 0.$$

If $b = c$ (15) gives us

$$5b^2 = (4a+b)^2,$$
$$b^2 = 4a^2 + 2ab$$

while (14) gives us

$$4a^2 = b^2 + 2ab$$

Thus, $ab = 0$ and $a = b = c = 0$ which implies $2a + 2a - b - c = 0$. On the other hand, if $b \neq c$ we also have $2a + 2a - b - c = 0$
Therefore, $f(x) = f(y)$ implies $2f(x) + 2f(y) = f(x+y) + f(x-y)$ while $f(x) \neq f(y)$, using (13) also implies $2f(x) + 2f(y) = f(x+y) + f(x-y)$.
Therefore, for all x, y:

$$2f(x) + 2f(y) = f(x+y) + f(x-y) \tag{16}$$

Now we have:

$$\begin{aligned} f(x)^2 + 3f(y)^2 &= f(x+y)f(x-y) + f(y)(f(x+y) + f(x-y)) \\ &= f(x+y)f(x-y) + f(y)(2f(x) + 2f(y)), \\ (f(x) - f(y))^2 &= f(x+y)f(x-y). \end{aligned} \tag{17}$$

Combining (16) i (17) gives us

$$(f(x+y) - f(x) - f(y))^2 = 4f(x)f(y). \tag{18}$$

Let $g : \mathbb{R} \to \mathbb{R}_0^+$ be the function such that $f(x) = g(x)^2$. Equations (8), (11) and (12) imply that $g(x^2) = g(x)^2$, $g(-x) = g(x)$ and $g(2x) = 2g(x)$, respectively.
Equation (16) can be written as

$$f(x+y) - f(x) - f(y) = -(f(x-y) - f(x) - f(y)).$$

If $f(x+y) - f(x) - f(y) \geqslant 0$, from (18) we conclude that $g(x+y) = g(x) + g(y)$. Otherwise, $f(x-y) - f(x) - f(y) \geqslant 0$ and equation (18) can be rewritten as

$$(f(x-y) - f(x) - f(y))^2 = 4f(x)f(y).$$

From the last equation we can conclude that $g(x-y) = g(x) + g(y)$.
Therefore

$$g(x+y) = g(x) + g(y) \quad \text{or} \quad g(x-y) = g(x) + g(y) \tag{19}$$

and thus one of the following two equations hold:

$$g(x^2 + y^2) + 2g(xy) = g(x^2 + y^2 + 2xy) = g(x+y)^2 \tag{20}$$

or

$$g(x^2 + y^2) + 2g(xy) = g(x^2 + y^2 - 2xy) = g(x-y)^2 \tag{21}$$

From (18) we conclude:

$$g(x+y) = g(x) + g(y) \quad \text{or} \quad g(x+y) = |g(x) - g(y)|. \tag{22}$$

By putting $-y$ instead of y in (22) and using $g(-y) = g(y)$ we get:

$$g(x-y) = g(x) + g(y) \quad \text{or} \quad g(x-y) = |g(x) - g(y)|. \tag{23}$$

Equations (22) and (23) imply that each of $g(x-y)^2$ and $g(x+y)^2$ can be written as either $(g(x) + g(y))^2$ or $(g(x) - g(y))^2$. Thus, no matter whether (20) or (21) holds, one of the following equations must hold:

$$g(x^2 + y^2) + 2g(xy) = (g(x) + g(y))^2 \tag{24}$$

or

$$g(x^2+y^2)+2g(xy)=(g(x)-g(y))^2 \tag{25}$$

Without loss of generality we may assume that $g(x) \geqslant g(y)$. If $g(x^2+y^2)=g(x)^2+g(y)^2$ then equations (24) i (25) imply that $g(xy)=g(x)g(y)$ or $g(xy)=-g(x)g(y)$ and because g is non-negative we conclude that $g(xy)=g(x)g(y)$. Otherwise, $g(x^2+y^2)=|g(x^2)-g(y^2)|=g(x)^2-g(y)^2$ and we have

$$g(x)^2+g(y)^2 \pm 2g(x)g(y)=g(x)^2-g(y)^2+2g(xy)$$

and

$$g(y)^2 \pm g(x)g(y)=g(xy).$$

However, since $g(x) \geqslant g(y)$ and $g(xy) \geqslant 0$ we get

$$g(y)^2+g(x)g(y)=g(xy).$$

Therefore, we conclude that

$$g(xy)=g(y)^2+g(x)g(y) \quad (\text{for } g(y) \leqslant g(x)) \quad \text{or} \quad g(xy)=g(x)g(y). \tag{26}$$

Thus,

$$g(xy) \geqslant g(x)g(y). \tag{27}$$

If for some a,b it holds that $g(a^2+b^2) \neq g(a)^2+g(b)^2$ we may assume that $g(a)>g(b)$ and we have $g(a^2+b^2)=g(a)^2-g(b)^2$, and

$$g(ab)=g(b)^2+g(a)g(b).$$

Let us denote $a'=2a$ and $b'=\frac{1}{2}b$. We have $g(a')=2g(a)$ and $g(b')=\frac{1}{2}g(b)$. Therefore, $g(a')>g(a)>g(b)>g(b')$. Note that $g(a'b')=g(ab)$ and $g(a')g(b')=g(a)g(b)$. From (26) we conclude that either $g(a'b')=g(a')g(b')$ or $g(a'b')=g(b')^2+g(a')g(b')$. Each of these two cases is only possible when $g(b)=0$. However, this implies that $g(a^2+b^2)=g(a^2)-g(b^2)=g(a^2)+g(b^2)$ which is a contradiction.

Therefore, there are no a,b such that $g(a^2+b^2) \neq g(a)^2+g(b)^2$ and for all $x,y \geqslant 0$ $g(x+y)=g(x)+g(y)$ which, together with the fact that g is non-negative, means that g satisfies a Cauchy functional equation whose only solution is $g(x)=g(1)x$. Since $g(1)=g(1)^2$ we get that $g(1)=1$ and $f(x)=x^2$ for all x.

Therefore there are 3 solutions which are given by

- $f(x)=0 \ \ \forall x \in \mathbb{R}$,
- $f(x)=\frac{1}{2} \ \ \forall x \in \mathbb{R}$ and
- $f(x)=x^2 \ \ \forall x \in \mathbb{R}$.