KLAMKIN'S INEQUALITY AND ITS APPLICATION

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Abstract. In this paper we consider a very useful inequality that Murray Klamkin[1)] proved in 1975 (Uldmkin, 1975). The inequality has many applications, proving new inequalities included. A proof and some applications are proposed.

Keywords: Klamkin's inequality; triangle sides; scalar product; law of cosines; application; examples

Theorem 1. (Klamkin's inequality). Let $x,\ y$ and z be real numbers such that $x+y+z>0$. Then for any point P in the plane of the triangle ΔABC the following inequality holds true:

$$x|PA|^2+y|PB|^2+z|PC|^2 \geq \frac{yza^2+zxb^2+xyc^2}{x+y+z}\ , \qquad (1)$$

where a,b,c are the lengts of the sides of the triangle ΔABC.

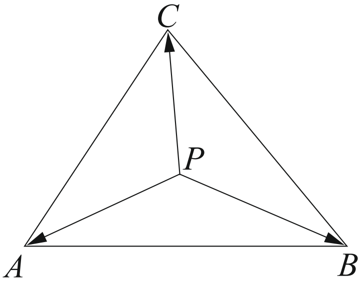
The equality in (1) holds if and only if the point P satisfies the equality

$$\overrightarrow{AP}=\frac{y}{x+y+z}\overrightarrow{AB}+\frac{z}{x+y+z}\overrightarrow{AC}\,. \qquad (2)$$

Proof: Observe the vector $x\overrightarrow{PA}+y\overrightarrow{PB}+z\overrightarrow{PC}$. Evidently, the next inequality

$$\left(x\overrightarrow{PA}+y\overrightarrow{PB}+z\overrightarrow{PC}\right)^2 \geq 0 \qquad (3)$$

holds true.



Because of the properties of the scalar product, this inequality (3) has the form

$$x^2|PA|^2+y^2|PB|^2+z^2|PC|^2+2xy\overrightarrow{PA}\cdot\overrightarrow{PB}+2yz\overrightarrow{PB}\cdot\overrightarrow{PC}+2zx\overrightarrow{PC}\cdot\overrightarrow{PA}\geq 0 \quad . \tag{4}$$

By using the law of cosines, we get the equalities

$$\left.\begin{aligned} 2\overrightarrow{PA}\cdot\overrightarrow{PB}&=|PA|^2+|PB|^2-c^2,\\ 2\overrightarrow{PB}\cdot\overrightarrow{PC}&=|PB|^2+|PC|^2-a^2,\\ 2\overrightarrow{PC}\cdot\overrightarrow{PA}&=|PC|^2+|PA|^2-b^2.\end{aligned}\right\} \tag{5}$$

Now from (4) and (5), we obtain:

$$\left(x^2+xy+xz\right)|PA|^2+\left(y^2+yx+yz\right)|PB|^2+\left(z^2+zx+zy\right)|PC|^2-yza^2-zxb^2-xyc^2\geq 0 \,. \tag{6}$$

Finally, if we divide the inequality (6) by $x+y+z>0$, we get the inequality (1).

Evidently, the equality holds if and only if $x\overrightarrow{PA}+y\overrightarrow{PB}+z\overrightarrow{PC}=\vec{0}$, i.e. $x\overrightarrow{PA}+y\left(\overrightarrow{PA}+\overrightarrow{AB}\right)+z\left(\overrightarrow{PA}+\overrightarrow{AC}\right)=\vec{0}$, and from here we get (2) after arrangment.

In the sequel we propose several examples of application of the inequality (1).

Example 1. For any point P in the plane of the triangle ΔABC the following inequality holds true:

$$|PA|^2+|PB|^2+|PC|^2\geq\frac{a^2+b^2+c^2}{3} \,. \tag{7}$$

Solution: This inequality follows directly from (1) when $x=y=z=1$.

Accounting for (2), the equality holds in (7) if and only if

$$\overrightarrow{AP}=\frac{1}{3}\overrightarrow{AB}+\frac{1}{3}\overrightarrow{AC}=\frac{2}{3}\overrightarrow{AA_1}\,,$$

where A_1 is the middpoint of the side BC. It means that the point divides the median AA_1 in ratio 2 : 1 computed from the vertex of the triangle, i.e. the point M is the centroid of the triangle.

Example 2. In every triangle ΔABC the following inequality holds true:

$$a^2+b^2+c^2\leq 9R^2 \,. \tag{8}$$

Solution: Put $P\equiv O$ in inequality (1), where the point O is the circumcenter of the triangle ΔABC. Since $|OA|=|OB|=|OC|=R$, now it follows from (7) that

$$3R^2 \geq \frac{a^2+b^2+c^2}{3}, \text{ i.e.}$$

$$a^2+b^2+c^2 \leq 9R^2, \text{ q.e.d.}$$

The equality holds in (8) if and only if $a=b=c$, i.e. for equilateral triangle.

Example 3. For anyone point P in the plane of the triangle ΔABC the followng inequality holds true:

$$a|PA|^2+b|PB|^2+c|PC|^2 \geq abc. \tag{9}$$

Solution: The proof follows directly from (1) when $x=a, y=b, z=c$.

Because of (2), the equality in (9) holds if and only if

$$\overrightarrow{AP}=\frac{b}{a+b+c}\overrightarrow{AB}+\frac{c}{a+b+c}=\frac{bc}{a+b+c}\left(\frac{\overrightarrow{AB}}{c}+\frac{\overrightarrow{AC}}{b}\right).$$

It follows now that the vector $\overrightarrow{AP}$ is collinear with the angular bisector. Analogously, it follows that the vectors $\overrightarrow{BP}$ and $\overrightarrow{CP}$ are collinear with the corresponding angular bisectors. Therefore, $P \equiv I$, where I is the incenter.

Example 4. (Euler's inequality) In every triangle ΔABC the following inequality holds true:

$$R \geq 2r. \tag{10}$$

Solution: Let $P \equiv O$, where O is the circumcenter, i.e. $|PA|=|OA|=R$, $|PB|=|OB|=R$ and $|PC|=|OC|=R$. Now it follows from (9) that

$$R^2(a+b+c) \geq abc$$

$$\Rightarrow R^2 \geq \frac{abc}{a+b+c},$$

and from here using the formulas $abc=4RF=4Rrs$ and $a+b+c=2s$ we obtain:

$$R^2 \geq \frac{4Rrs}{2s}, \text{ i.e.}$$

$$R \geq 2r, \text{ q.e.d.}$$

The equality in (10) holds for $a=b=c$, i.e. for the equilateral triangle.

Example 5. For any point P in the plane of the triangle ΔABC the followng inequality holds true:

$$\sin 2\alpha |PA|^2 + \sin 2\beta |PB|^2 + \sin 2\gamma |PC|^2 \ge 2F \ . \qquad (11)$$

Solution: Put $x=\sin 2\alpha,\ y=\sin 2\beta,\ z=\sin 2\gamma$ in (1). The right hand side of the inequality (1) takes the form:

$$\frac{a^2 \sin 2\beta \sin 2\gamma + b^2 \sin 2\gamma \sin 2\alpha + c^2 \sin 2\alpha \sin 2\beta}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma},$$

Applying the law of sines formulas $\sin 2\alpha = 2\sin\alpha\cos\alpha$, $\sin 2\beta = 2\sin\beta\cos\beta$, $\sin 2\gamma = 2\sin\gamma\cos\gamma$

$$\frac{16R^2 \sin\alpha\sin\beta\sin\gamma\left(\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma\right)}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}$$

and the identities

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$$

and

$$\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta\sin\gamma = \sin\alpha\sin\beta\sin\gamma \ ,$$

the right hand side of the inequality (1) takes the form

$$4R^2 \sin\alpha\sin\beta\sin\gamma \ .$$

Finally, observe that

$$4R^2 \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{abc}{2R} = 2F \ .$$

Consequently, the inequality (11) is true because it follows from the inequality (1).

The equality in (11) holds if and only if $P \equiv O$, which is left from for the reader to prove it.

Example 6. Let P be an arbitrary point in the interior of the trinagle ΔABC. Prove the inequality

$$\frac{|PA|^2}{c}\left(\frac{1}{a}+\frac{1}{b}\right) + \frac{|PB|^2}{a}\left(\frac{1}{b}+\frac{1}{c}\right) + \frac{|PC|^2}{b}\left(\frac{1}{c}+\frac{1}{a}\right) \ge 2 \ . \qquad (12)$$

Solution: This inequality is evidently equivalent to the inequality

$$(a+b)|PA|^2 + (b+c)|PB|^2 + (a+c)|PC|^2 \ge 2abc \ . \qquad (13)$$

We will now use the inequality (1).

If $x=a,\ y=b,\ z=c$ and $x=b,\ y=c,\ z=a$, we obtain two inequalities:

$$a|PA|^2 + b|PB|^2 + c|PC|^2 \ge abc$$

and

$$b|PA|^2 + c|PB|^2 + a|PC|^2 \ge abc \ .$$

Summing the two inequalites, we obtain the following inequality

$$|PA|^2(a+b)+|PB|^2(b+c)+|PC|^2(a+c)\geq 2abc\,,$$

Which in fact is the inequality (13), thas proving (12).
The equality holds in (12) if and only if is $a=b=c$, i.e. for equilateral triangle.

NOTES

1. Murray Klamkin (1921 – 2004) is a Canadian mathematician, born in USA

REFERENCES

Arslanagić, Š. (2005). *Matematika za nadarene*. Sarajevo: Bosanska riječ.

Bottema, O., R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović &P. M. Vasić. (1969). *Geometric Inequalities*. Groningen: (The Netherlands)Wolters-Noordhoff Publishing.

Grozdev, S. (2007). *For High Achievements in Mathematics*: *The Bulgaria Experience* (*Theory and Practice*). Sofia: ADE (ISBN 978-954-92139-1-1).

Klamkin, M. S. (1975). Geometric Inequalities via the Polar moment of Inertia. *Math. Mag*., *48*, 44 – 46.

Sergeeva, T., M. Shabanova & S. Grozdev (2014). *Foundations of Dynamic Geometry*. Moscow: ASOU (in Russian).