XIV International Zhautykov Olympiad in Mathematics
Almaty, 2020

January 10, 9.00-13.30
First day
(Each problem is worth 7 points)

1. A positive integer n does not divide $2^a 3^b + 1$ for any positive integers a and b. Prove that n does not divide $2^c + 3^d$ for any positive integers c and d.

Solution. Assume the contrary: n divides $2^c + 3^d$. Clearly n is not divisible by 3; therefore n divides $3^k - 1$ for some k. Choosing s so that $ks > d$ we see that n divides $3^{ks-d}(2^c + 3^d) = 2^c 3^{ks-d} + 3^{ks}$. Then n also divides $2^c 3^{ks-d} + 1 = 2^c 3^{ks-d} + 3^{ks} - (3^{ks} - 1)$, a contradiction.

2. In a set of 20 elements there are $2k+1$ different subsets of 7 elements such that each of these subsets intersects exactly k other subsets. Find the maximum k for which this is possible.

The answer is $k = 2$.

Solution. Let M be the set of residues mod20. An example is given by the sets $A_i = \{4i+1, 4i+2, 4i+3, 4i+4, 4i+5, 4i+6, 4i+7\} \subset M$, $i = 0, 1, 2, 3, 4$.

Let $k \geq 2$. Obviously among any three 7-element subsets there are two intersecting subsets.

Let A be any of the $2k+1$ subsets. It intersects k other subsets $B_1, \ldots, B_k$. The remaining subsets $C_1, \ldots, C_k$ do not intersect A and are therefore pairwise intersecting. Since each C_i intersects k other subsets, it intersects exactly one B_j. This B_j can not be the same for all C_i because B_j can not intersect $k+1$ subsets.

Thus there are two different C_i intersecting different B_j; let C_1 intersect B_1 and C_2 intersect B_2. All the subsets that do not intersect C_1 must intersect each other; there is A among them, therefore they are A and all B_i, $i \neq 1$. Hence every B_j and B_j, $i \neq 1$, $j \neq 1$, intersect. Applying the same argument to C_2 we see that any B_i and B_j, $i \neq 2$, $j \neq 2$, intersect. We see that the family $A, B_1, \ldots, B_k$ contains only one pair, B_1 and B_2, of non-untersecting subsets, while B_1 intersects C_1 and B_2 intersects C_2. For each i this list contains k subsets intersecting B_i. It follows that no C_i with $i > 2$ intersects any B_j, that is, there are no such C_i, and $k \leq 2$.

3. A convex hexagon $ABCDEF$ is inscribed in a circle. Prove the inequality

$$AC \cdot BD \cdot CE \cdot DF \cdot AE \cdot BF \geq 27 AB \cdot BC \cdot CD \cdot DE \cdot EF \cdot FA.$$

Solution. Let

$$d_1 = AB \cdot BC \cdot CD \cdot DE \cdot EF \cdot FA, d_2 = AC \cdot BD \cdot CE \cdot DF \cdot AE \cdot BF, d_3 = AD \cdot BE \cdot CF.$$

Applying Ptolemy's theorem to quadrilaterals $ABCD$, $BCDE$, $CDEF$, $DEFA$, $EFAB$, $FABC$, we obtain six equations $AC \cdot BD - AB \cdot CD = BC \cdot AD$, ..., $FB \cdot AC - FA \cdot BC = AB \cdot FC$. Putting these equations in the well-known inequality

$$\sqrt[6]{(a_1 - b_1)(a_2 - b_2) \cdot \ldots \cdot (a_6 - b_6)} \leq \sqrt[6]{a_1 a_2 \ldots a_6} - \sqrt[6]{b_1 b_2 \ldots b_6} \quad (a_i \geq b_i > 0, i = 1, \ldots, 6),$$

we get

$$\sqrt[3]{d_3}\sqrt[6]{d_1} \leq \sqrt[3]{d_2} - \sqrt[3]{d_1}. \tag{1}$$

Applying Ptolemy's theorem to quadrilaterals $ACDF$, $ABDE$ и $BCEF$, we obtain three equations $AD \cdot CF = AC \cdot DF + AF \cdot CD$, $AD \cdot BE = BD \cdot AE + AB \cdot DE$, $BE \cdot CF = BF \cdot CE + BC \cdot EF$. Putting these equations in the well-known inequality

$$\sqrt[3]{(a_1 + b_1)(a_2 + b_2)(a_3 + b_3)} \geq \sqrt[3]{a_1 a_2 a_3} + \sqrt[3]{b_1 b_2 b_3} \quad (a_i > 0, b_i > 0, i = 1, 2, 3),$$

we get

$$\sqrt[3]{d_3^2} \geq \sqrt[3]{d_2} + \sqrt[3]{d_1}. \tag{2}$$

It follows from (1) and (2) that $(\sqrt[3]{d_2} - \sqrt[3]{d_1})^2 \geq \sqrt[3]{d_1}(\sqrt[3]{d_2} + \sqrt[3]{d_1})$, that is, $\sqrt[3]{d_2} \geq 3\sqrt[3]{d_1}$ and $d_2 \geq 27 d_1$, q.e.d.

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Solutions of the second day

№4. In a scalene triangle ABC I is the incenter and CN is the bisector of angle C. The line CN meets the circumcircle of ABC again at M. The line ℓ is parallel to AB and touches the incircle of ABC. The point R on ℓ is such that $CI \perp IR$. The circumcircle of MNR meets the line IR again at S. Prove that $AS = BS$.

Solution. In this solution we make use of directed angles. *A directed angle* $\angle(n, m)$ between lines n and m is the angle of counterclockwise rotation transforming n into a line parallel to m.

Let d be the tangent to the circumcircle of $\triangle ABC$ containing N and different from AB. Then $\angle(\ell, CI) = \angle(NB, NI) = \angle(NI, d)$. Since $CI \perp IR$, the line d contains R because of symmetry with respect to IR.

Let T be the common point of MS and ℓ. We have $\angle(MN, MS) = \angle(RN, RS) = \angle(RS, RT)$, that is, R, T, I, M are concyclic. Therefore $\angle(RT, MT) = \angle(RI, MI) = 90°$. It follows that $MS \perp AB$. But M belongs to the perpendicular bisector of AB, and so does S. Thus $AS = BS$, q.e.d.

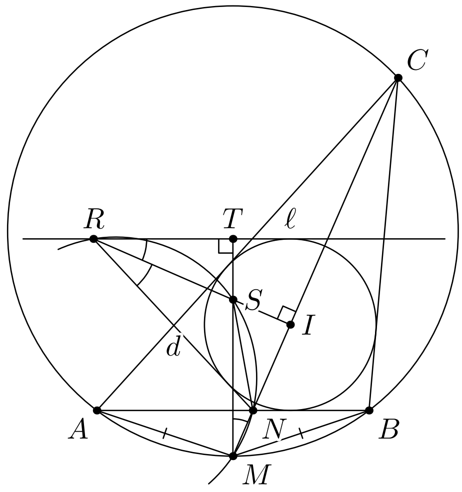

Рис. 1

№5. Find all the functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that $f(4x+3y) = f(3x+y) + f(x+2y)$ for all integers x and y.

Answer: $f(x) = \frac{ax}{5}$ for x divisible by 5 and $f(x) = bx$ for x not divisible by 5, where a and b are arbitrary integers.

Solution. Putting $x = 0$ in the original equation

$$f(4x+3y) = f(3x+y) + f(x+2y) \qquad (1),$$

we get

$$f(3y) = f(y) + f(2y). \qquad (2)$$

Next, (1) for $y = -2x$ gives us $f(-2x) = f(x) + f(-3x) = f(x) + f(-x) + f(-2x)$ (in view of (2)). It follows that

$$f(-x) = -f(x). \qquad (3)$$

Now, let $x = 2z - v$, $y = 3v - z$ in (1). Then

$$f(5z+5v) = f(5z) + f(5v) \qquad (4)$$

for all $z, v \in \mathbb{Z}$. It follows immediately that $f(5t) = tf(5)$ for $t \in \mathbb{Z}$, or $f(x) = \frac{ax}{5}$ for any x divisible by 5, where $f(5) = a$.

Further, we claim that

$$f(x) = bx, \qquad (5)$$

where $b = f(1)$, for all x not divisible by 5. In view of (3) it suffices to prove the claim for $x > 0$. We use induction in k where $x = 5k + r$, $k \in \mathbb{Z}$, $0 < r < 5$. For $x = 1$ (5) is obvious. Putting $x = 1$, $y = -1$ in (1) gives $f(1) = f(2) + f(-1)$ whence $f(2) = f(1) - f(-1) = 2f(1) = 2b$. Then $f(3) = f(1) + f(2) = 3b$ by (2). Finally, (1) with $x = 1$, $y = 0$ gives $f(4) = f(3) + f(1) = 3b + b = 4b$. Thus the induction base is verified.

Now suppose (5) is true for $x < 5k$. We have $f(5k+1) = f(4(2k-2) + 3(3-k)) = f(3(2k-2) + (3-k)) + f((2k-2) + 2(3-k)) = f(5k-3) + f(4) = (5k-3)b + 4b = (5k+1)b$; $f(5k+2) = f(4(2k-1) + 3(2-k)) = f(3(2k-1) + (2-k)) + f((2k-1) + 2(2-k)) = f(5k-1) + f(3) = (5k-1)b + 3b = (5k+2)b$; $f(5k+3) = f(4 \cdot 2k + 3(1-k)) = f(3 \cdot 2k + (1-k)) + f(2k + 2(1-k)) = f(5k+1) + f(2) = (5k+1)b + 2b = (5k+3)b$; $f(5k+4) = f(4(2k+1) + 3(-k)) = f(3(2k+1) + (-k)) + f((2k+1) + 2(-k)) = f(5k+3) + f(1) = (5k+3)b + b = (5k+4)b$. Thus (5) is proved.

It remains to check that the function $f(x) = \frac{ax}{5}$ for x divisible by 5, $f(x) = bx$ for x not divisible by 5 satisfies (1). It is sufficient to note that 5 either divides all the numbers $4x + 3y$, $3x + y$, $x + 2y$ or does not divide any of these numbers (since $3x + y = 5(x + y) - 2(x + 2y) = 2(4x + 3y) - 5(x + y)$).

№6. Some squares of a $n \times n$ table ($n > 2$) are black, the rest are white. In every white square we write the number of all the black squares having at least one common vertex with it. Find the maximum possible sum of all these numbers.

The answer is $3n^2 - 5n + 2$.

Solution. The sum attains this value when all squares in even rows are black and the rest are white. It remains to prove that this is the maximum value.

The sum in question is the number of pairs of differently coloured squares sharing at least one vertex. There are two kinds of such pairs: sharing a side and sharing only one vertex. Let us count the number of these pairs in another way.

We start with zeroes in all the vertices. Then for each pair of the second kind we add 1 to the (only) common vertex of this pair, and for each pair of the first kind we add $\frac{1}{2}$ to each of the two common vertices of its squares. For each pair the sum of all the numbers increases by 1, therefore in the end it is equal to the number of pairs.

Simple casework shows that

(i) 3 is written in an internal vertex if and only if this vertex belongs to two black squares sharing a side and two white squares sharing a side;

(ii) the numbers in all the other internal vertices do not exceed 2;

(iii) a border vertex is marked with $\frac{1}{2}$ if it belongs to two squares of different colours, and 0 otherwise;

(iv) all the corners are marked with 0.

Note: we have already proved that the sum in question does not exceed $3 \times (n-1)^2 + \frac{1}{2}(4n - 4) = 3n^2 - 4n + 1$. This estimate is valuable in itself.

Now we prove that the numbers in all the vertices can not be maximum possible simultaneously. To be more precise we need some definitions.

Definition. The number in a vertex is *maximum* if the vertex is internal and the number is 3, or the vertex is on the border and the number is $\frac{1}{2}$.

Definition. *A path* – is a sequence of vertices such that every two consecutive vertices are one square side away.

Lemma. In each colouring of the table every path that starts on a horizontal side, ends on a vertical side and does not pass through corners, contains a number which is not maximum.

Proof. Assume the contrary. Then if the colour of any square containing the initial vertex is chosen, the colours of all the other squares containing the vertices of the path is uniquely defined, and the number in the last vertex is 0.

Now we can prove that the sum of the numbers in any colouring does not exceed the sum of all the maximum numbers minus quarter of the number of all border vertices (not including corners). Consider the squares 1×1, 2×2, ..., $(N-1) \times (N-1)$ with a vertex in the lower left corner of the table. The right side and the upper side of such square form a path satisfying the conditions of the Lemma. Similar set of $N - 1$ paths is produced by the squares 1×1, 2×2, ..., $(N-1) \times (N-1)$ with a vertex in the upper right corner of the table. Each border vertex is covered by one of these $2n - 2$ paths, and each internal vertex by two.

In any colouring of the table each of these paths contains a number which is not maximum. If this number is on the border, it is smaller than the maximum by (at least) $\frac{1}{2}$ and does not belong to any other path. If this number is in an internal vertex, it belongs to two paths and is smaller than the maximum at least by 1. Thus the contribution of each path in the sum in question is less than the maximum possible at least by $\frac{1}{2}$, q.e.d.

An interesting question: is it possible to count all the colourings with maximum sum using this argument?