43rd International Mathematical Olympiad

19-30 July 2002

United Kingdom

Short-listed Problems and Solutions

Short-listed Problems and Solutions

The Problem Selection Committee received a total of 130 problem proposals, from 41 countries. In this booklet, a short-list of 27 problems is presented, classified under Number Theory (N1-N6), Geometry (G1-G8), Algebra (A1-A6) and Combinatorics (C1-C7). Within each section, the problems are arranged in ascending order of estimated difficulty, although of course it is very hard to judge this accurately.

The Problem Selection Committee would like to thank the following countries for submitting problems:

Albania	Greece	Poland
Armenia	Holland	Puerto Rico
Australia	Hong Kong	Romania
Brazil	India	Russia
Bulgaria	Iran	Slovakia
Canada	Ireland	Slovenia
Colombia	Israel	Sweden
Croatia	Japan	Taiwan (ROC)
Czech Republic	Korea	Thailand
Estonia	Lithuania	Ukraine
Finland	Luxembourg	United States
France	Moldova	Uzbekistan
Georgia	Mongolia	Yugoslavia
Germany	New Zealand	

Members of the Problem Selection Committee:

Béla Bollobás	Imre Leader (Chairman)
Christopher Bradley	Adam McBride
Robin Chapman	David Monk
Dan Crisan	Peter Shiu
Tony Gardiner	Vin de Silva

Contents

Typeset by Bill Richardson.

N1. What is the smallest positive integer t such that there exist integers $x_1, x_2, \ldots, x_t$ with

$$x_1^3 + x_2^3 + \ldots + x_t^3 = 2002^{2002} ?$$

Solution. The answer is $t = 4$.

We first show that 2002^{2002} is not a sum of three cubes by considering numbers modulo 9. Thus, from $2002 \equiv 4 \pmod 9$, $4^3 \equiv 1 \pmod 9$ and $2002 = 667 \times 3 + 1$ we find that

$$2002^{2002} \equiv 4^{2002} \equiv 4 \pmod 9,$$

whereas, from $x^3 \equiv 0, \pm 1 \pmod 9$ for any integer x, we see that $x_1^3 + x_2^3 + x_3^3 \not\equiv 4 \pmod 9$.

It remains to show that 2002^{2002} is a sum of four cubes. Starting with

$$2002 = 10^3 + 10^3 + 1^3 + 1^3$$

and using $2002 = 667 \times 3 + 1$ once again, we find that

$$\begin{aligned} 2002^{2002} &= 2002 \times \left(2002^{667}\right)^3 \\ &= \left(10 \times 2002^{667}\right)^3 + \left(10 \times 2002^{667}\right)^3 + \left(2002^{667}\right)^3 + \left(2002^{667}\right)^3. \end{aligned}$$

Comments

1. This is an easy question. The only subtle point is that, to show that 2002^{2002} is not the sum of three cubes, we need to consider a non-prime modulus. Indeed, to restrict the number of cubes mod n we would like $\phi(n)$ to be a multiple of 3 (so that Fermat-Euler is helping us), but taking n to be 7 or 13 or 19 does not help: there are too many cubes. So we try a composite n with $\phi(n)$ a multiple of 3, and the first such is $n = 9$.

2. The proposer's original version of the problem only asked for a proof that three cubes is impossible and five cubes is possible. It is a fortunate feature of the number 2002^{2002} that we are able to settle the case of four cubes.

N2. Let $n \geqslant 2$ be a positive integer, with divisors $1 = d_1 < d_2 < \ldots < d_k = n$. Prove that $d_1 d_2 + d_2 d_3 + \ldots + d_{k-1} d_k$ is always less than n^2, and determine when it is a divisor of n^2.

Solution. Note that if d is a divisor of n then so is n / d, so that the sum

$$s = \sum_{1 \leqslant i < k} d_i\, d_{i+1} = n^2 \sum_{1 \leqslant i < k} \frac{1}{d_i\, d_{i+1}} \leqslant n^2 \sum_{1 \leqslant i < k} \left(\frac{1}{d_i} - \frac{1}{d_{i+1}} \right) < \frac{n^2}{d_1} = n^2.$$

Note also that $d_2 = p$, $d_{k-1} = n / p$, $d_k = n$, where p is the least prime divisor of n.

If $n = p$ then $k = 2$ and $s = p$, which divides n^2.

If n is composite then $k > 2$, and $s > d_{k-1} d_k = n^2 / p$. If such an s were a divisor of n^2 then also n^2 / s would be a divisor of n^2. But $1 < n^2 / s < p$, which is impossible because p is the least prime divisor of n^2.

Hence, the given sum is a divisor of n^2 if and only if n is prime.

Comments

1. The problem is perhaps not quite as easy as the short solution here appears to suggest. Even having done the first part, it is very easy to get stuck on the second part.

2. It would be possible to delete from the question the fact that the given expression is always less than n^2. But, in our opinion, the form as given above is natural and inviting to a reader.

N3. Let $p_1, p_2, \ldots, p_n$ be distinct primes greater than 3. Show that $2^{p_1 p_2 \cdots p_n} + 1$ has at least 4^n divisors.

Comment

1. The natural strategy for this problem is to use induction on the number of primes involved, hoping that the number of divisors increases by a factor of 4 for each new prime in the expression. By the usual properties of the divisor function $d(m)$, it would be enough to show that $2^{p_1 p_2 \cdots p_n} + 1$ contains at least two new prime factors not contained in $2^{p_1 p_2 \cdots p_{n-1}} + 1$. Unfortunately this does not seem to be easy. Instead, we will show in an elementary way that there is at least *one* new prime at each step. To finish the proof, we will need the following additional observation: if $k > m$ then $d(km) \geqslant 2d(m)$, which follows from the simple fact that if a divides m then both a and ka divide km.

Solution. We claim first that if u and v are coprime odd numbers then the highest common factor of $2^u + 1$ and $2^v + 1$ is 3. Certainly 3 divides $2^u + 1$ and $2^v + 1$, because u and v are odd. Suppose now that some $t > 3$ divides $2^u + 1$ and $2^v + 1$. Then we have $2^u \equiv -1 \pmod t$ and $2^v \equiv -1 \pmod t$. But if any 2^x is $-1 \bmod t$ then the set of all such x is the set of all odd multiples of $r/2$, where r is the order of $2 \bmod t$. It follows that $r/2$ divides both u and v, which is impossible as $r > 2$.

Note also that the factorisation

$$2^{uv} + 1 = \left(2^u + 1\right)\left(2^{u(v-1)} - 2^{u(v-2)} + \ldots + 2^{2u} - 2^u + 1\right)$$

shows that $2^{uv} + 1$ is divisible by $2^u + 1$ and $2^v + 1$, and so is also divisible by $\left(2^u + 1\right)\left(2^v + 1\right)/3$.

Let us now prove the desired result by induction on n. It is certainly true when $n = 1$ (for example, because $2^{p_1} + 1$ is a multiple of 3 and is at least 27), so we assume that $2^{p_1 \cdots p_{n-1}} + 1$ has at least 4^{n-1} divisors and consider $2^{p_1 \cdots p_n} + 1$. Setting $u = p_1 \ldots p_{n-1}$ and $v = p_n$ in the above, we see that $2^u + 1$ and $\left(2^v + 1\right)/3$ are coprime, whence $m = \left(2^u + 1\right)\left(2^v + 1\right)/3$ has at least $2 \times 4^{n-1}$ divisors.

Now, we know that m divides $2^{uv} + 1$. Moreover, from $uv > 2(u + v)$ when $u, v \geqslant 5$, we see that $2^{uv} + 1 > m^2$. By the fact mentioned in the comment above, it follows that $d\left(2^{uv} + 1\right) \geqslant 2d(m) \geqslant 4^n$, as required.

Further comment

2. From a more advanced point of view, $f\left(p_1 p_2 \ldots p_n\right)$ is the product of cyclotomic polynomials at 2, that is the product of $\Phi_{2m}(2)$ over $m \mid p_1 \ldots p_n$. It turns out that $\Phi_r(2)$ and $\Phi_s(2)$ are coprime unless r/s is a prime power (this is not an easy fact), from which it follows that $f\left(p_1 p_2 \ldots p_n\right)$ has at least 2^{n-1} prime divisors. Hence $d\left(f\left(p_1 p_2 \ldots p_n\right)\right) \geqslant 2^{2^{n-1}}$, which is much more than 4^n when n is large.

N4. Is there a positive integer m such that the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{m}{a+b+c}$$

has infinitely many solutions in positive integers $a,\ b,\ c$?

Solution. If $a = b = c = 1$ then $m = 12$, and we proceed to show that, for this fixed value of m, there are infinitely many solutions in positive integers a, b, c. Write

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} - \frac{12}{a+b+c} = \frac{p(a,\ b,\ c)}{abc(a+b+c)},$$

where $p(a,\ b,\ c) = a^2(b+c) + b^2(c+a) + c^2(a+b) + a + b + c - 9abc$. Suppose that $(x,\ a,\ b)$ is a solution with $x < a < b$, that is $p(x,\ a,\ b) = 0$. Then, regarding this as a quadratic equation in x, we see that $y = (ab+1)/x > b$ is also a solution, except that we need to establish that such a value y is integral.
Let $a_0 = a_1 = a_2 = 1$, and define

$$a_{n+2} = \frac{a_n a_{n+1} + 1}{a_{n-1}}, \qquad \text{for each } n \geqslant 1.$$

We now prove the following assertions simultaneously by induction:

$$\text{(i) } a_{n-1} \mid a_n a_{n+1} + 1, \qquad \text{(ii) } a_n \mid a_{n-1} + a_{n+1}, \qquad \text{(iii) } a_{n+1} \mid a_{n-1} a_n + 1.$$

The three assertions are true when $n = 1$ from the initial values for a_0, a_1, a_2, and we suppose that they are true when $n = k$. Thus (i) implies that a_{k-1} and a_k are coprime and that a_{k-1} divides $\left(a_k a_{k+1} + 1\right)a_{k+1} + a_{k-1}$, whereas (ii) gives $a_k \mid a_k a_{k+1}^2 + a_{k+1} + a_{k-1}$, so that together $a_k a_{k-1} \mid a_k a_{k+1}^2 + a_{k+1} + a_{k-1}$, that is $a_k \mid a_{k+1}\left(a_k a_{k+1} + 1\right)/a_{k-1} + 1 = a_{k+1}a_{k+2} + 1$, which is (i) when $n = k + 1$.
Similarly (i) also implies that a_{k-1} and a_{k+1} are coprime, and that $a_{k-1} \mid a_k a_{k+1} + 1 + a_k a_{k-1}$, whereas (iii) gives $a_{k+1} \mid a_k a_{k-1} + 1 + a_k a_{k+1}$, so that together $a_{k-1}a_{k+1} \mid a_k\left(a_{k-1} + a_{k+1}\right) + 1$, that is $a_{k+1} \mid a_k + \left(a_k a_{k+1} + 1\right)/a_{k-1} = a_k + a_{k+2}$, which is (ii) when $n = k + 1$.
Finally, the definition of a_{k+2} together with (i) implies $a_{k+2} \mid a_k a_{k+1} + 1$, which is (iii) when $n = k + 1$.

Therefore $\left(a_n\right)$ is a sequence of integers, strictly increasing from $n \geqslant 2$, and $p\left(a_n, a_{n+1}, a_{n+2}\right) = 0$ for all n. In other words, $\left(a_n,\ a_{n+1},\ a_{n+2}\right)$ is a solution to the given equation, with

$$\left(a_n\right) = \left(1,\ 1,\ 1,\ 2,\ 3,\ 7,\ 11,\ 26,\ 41,\ 97,\ 154,\ \ldots\right).$$

Comments

1. Another method is to define $\left(c_n\right)$ by $c_0 = 2$, $c_1 = 3$, and $c_{2n} = 3c_{2n-1} - c_{2n-2}$ and $c_{2n+1} = 2c_{2n} - c_{2n-1}$, and use induction to show that the triples $\left(c_n,\ c_{n+1},\ c_{n+2}\right)$ are solutions.

2. One may also apply Pell's equation to show that there are infinitely many solutions for $m = 12$. Indeed, let $p(a,\ b,\ c)$ be as above. With an eye on eliminating a variable in p by a substitution of the form $a + c = rb$ with a suitable r, we find that $p(1,\ 1, r-1) = 2(r-2)(r-3)$, showing that $r = 2,\ 3$ are suitable candidates. We therefore consider

$$p(a,\ b,\ 2b - a) = 3b\left(3a^2 - 6ab + 2b^2 + 1\right) = 3b\left(3(a-b)^2 - b^2 + 1\right)$$

and recall the well-known result that there are infinitely many solutions to the Pell equation $x^2 = 3y^2 + 1$. Thus there are infinitely many positive integers $a < b$ satisfying $p(a, b, 2b - a) = 0$.

3. In fact, using a little more theory on quadratic forms, it can be shown that if the equation is soluble for a given value of m then there are infinitely many solutions for that value of m.

4. There is nothing special about $m = 12$: there are infinitely many possible values of m. Indeed, the given equation may be rewritten as $m = (a + b + c)(1 + ab + bc + ca)/abc$, which becomes $m = (1 + b + c) + (1 + b + c)^2/bc$ on setting $a = 1$. One can define a sequence (b_n) with the property that $b_n b_{n+1}$ divides $(1 + b_n + b_{n+1})^2$; take, for example, $b_1 = 4$, $b_2 = 5$, set $b_{n+2} = 3b_{n+1} - b_n - 2$, and induction then shows that $(b_{n+1} + b_n + 1)^2 = 5b_n b_{n+1}$. The corresponding value for m is then $b_n + b_{n+1} + 6$. We have one solution for this value of m, so by the remark above there are infinitely many solutions for this value of m.

N5. Let $m, n \geqslant 2$ be positive integers, and let $a_1, a_2, \ldots, a_n$ be integers, none of which is a multiple of m^{n-1}. Show that there exist integers $e_1, e_2, \ldots, e_n$, not all zero, with $|e_i| < m$ for all i, such that $e_1a_1 + e_2a_2 + \ldots + e_na_n$ is a multiple of m^n.

Solution. Write N for m^n. Let B be the set of all n-tuples $b = (b_1, b_2, \ldots, b_n)$, where each b_i is an integer with $0 \leqslant i < m$. For $b \in B$, write $f(b)$ for $b_1a_1 + b_2a_2 + \ldots + b_na_n$. If some distinct $b, b' \in B$ have $f(b) \equiv f(b') \pmod N$ then we are done: setting $e_i = b_i - b_i'$ we have $e_1a_1 + \ldots + e_na_n \equiv 0 \pmod N$. So we are done unless no two $f(b)$ are congruent mod N. Since $|B| = N$, this implies that, mod N, the numbers $f(b)$ for $b \in B$ are precisely the numbers $0, 1, \ldots, N-1$ (in some order). We wish to show that this is impossible.

Consider the polynomial $\sum_{b \in B} X^{f(b)}$. On the one hand, it factorises as

$$\prod_{i=1}^{n} \left(1 + X^{a_i} + X^{2a_i} + \ldots + X^{(m-1)a_i}\right),$$

but on the other hand it is equal to $1 + X + X^2 + \ldots + X^{N-1}$ whenever $X^N = 1$. But now set $X = \exp(2\pi i / N)$, a primitive N-th root of unity. Then

$$1 + X + X^2 + \ldots + X^{N-1} = \frac{1 - X^N}{1 - X} = 0,$$

but for each i we have

$$1 + X^{a_i} + X^{2a_i} + \ldots + X^{(m-1)a_i} = \frac{1 - X^{ma_i}}{1 - X},$$

which is non-zero because ma_i is not a multiple of N. This is a contradiction.

Comments

1. The proof begins with a standard pigeonhole argument. The exceptional case (with each congruence class mod N hit exactly once) is quickly identified, and looks at first glance at though it should be easily attackable. However, it is actually rather challenging. The use of the polynomial and N-th roots of unity is probably the most natural approach. We do not know of any bare-hands or essentially different proof.

2. The condition that no a_i is a multiple of m^{n-1} cannot be removed, as may be seen by taking $a_i = m^{i-1}$ for each i.

N6. Find all pairs of positive integers $m,\ n \geqslant 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

Solution. Suppose m, n is such a pair. Clearly $n < m$.

Step 1. We claim that $f(x) = x^m + x - 1$ is exactly divisible by $g(x) = x^n + x^2 - 1$ in $\mathbf{Z}[x]$. Indeed, since $g(x)$ is monic, the division algorithm gives

$$f(x)/g(x) = q(x) + r(x)/g(x)$$

where $\deg(r) < \deg(g)$. The remainder term $r(x)/g(x)$ tends to zero as $x \to \infty$; on the other hand it is an integer at infinitely many integers a. Thus $r(a)/g(a) = 0$ infinitely often, and so $r \equiv 0$. The claim follows; and in particular, we note that $f(a)/g(a)$ is an integer for all integers a.

Step 2. Both $f(x)$ and $g(x)$ have a unique root in the interval $(0, 1)$, since both functions are increasing in $[0, 1]$ and span the range $[-1,\ 1]$. Moreover it is the same root, since g divides f; call it α.

Step 3. We can use α to show that $m < 2n$. Certainly $\alpha > \phi$, where $\phi = 0.618\ldots$ is the positive root of $h(x) = x^2 + x - 1$. This is because f is increasing in $(0, 1)$ and $f(\phi) < h(\phi) = 0 = f(\alpha)$. On the other hand, if $m \geqslant 2n$ then $1 - \alpha = \alpha^m \leqslant (\alpha^n)^2 = (1 - \alpha^2)^2$, and the outer terms rearrange to give $\alpha(\alpha - 1)(\alpha^2 + \alpha - 1) \geqslant 0$, which requires $\alpha \leqslant \phi$, a contradiction.

Step 4. We show that the only solution with $m < 2n$ is $(m, n) = (5,\ 3)$. This is pure number theory, at last. Suppose we have a solution. We consider the value $a = 2$, and write $d = g(2) = 2^n + 3$, so that $-2^m \equiv 1 \pmod{d}$. Let $m = n + k$ where $1 \leqslant k < n$, so that

$$-2^m \equiv \left(d - 2^n\right)2^k \equiv 3 \times 2^k \pmod{d},$$

which shows that $-2^m \not\equiv 1 \pmod{d}$ when $1 \leqslant k \leqslant n - 2$. When $k = n - 1$, that is $m = 2n - 1$, the least positive residue (mod d) for -2^m is given by $3 \times 2^{n-1} - d = 2^{n-1} - 3$, which takes the value 1 only when $n = 3$, giving $m = 5$. Finally, the identity $a^5 + a - 1 = \left(a^3 + a^2 - 1\right)\left(a^2 - a + 1\right)$ shows that $(m, n) = (5,\ 3)$ is indeed a solution.

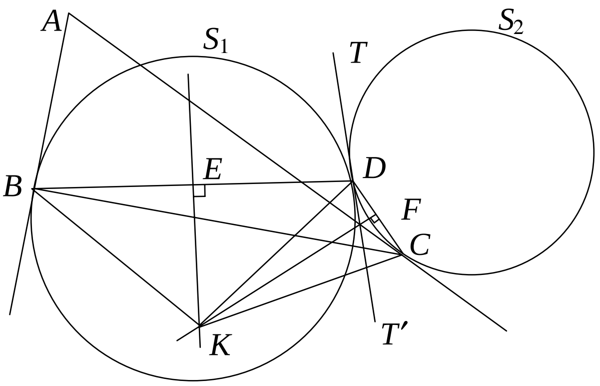
Comment

1. Although the above solution is entirely elementary, several separate good ideas seem to be needed to crack the problem. Step 1 is the natural way to begin, and Step 4 has several variations. Perhaps the most important—and most difficult—idea is the use of the common root α (in Steps 2 and 3) to obtain the quantitative bound $m < 2n$. All solutions we have seen make use of this idea in some form.

G1. Let B be a point on a circle S_1, and let A be a point distinct from B on the tangent at B to S_1. Let C be a point not on S_1 such that the line segment AC meets S_1 at two distinct points. Let S_2 be the circle touching AC at C and touching S_1 at a point D on the opposite side of AC from B. Prove that the circumcentre of triangle BCD lies on the circumcircle of triangle ABC.

Comments

1. In both solutions that follow, the key idea is to work with the perpendicular bisectors of BD and CD.
2. There does not appear to be a straightforward coordinate solution.



Solution 1. Let E and F be the midpoints of BD and CD respectively, K be the circumcentre of triangle BCD and let TDT' be the common tangent to the two circles. Then EK is perpendicular to BD and bisects the angles between the tangents BA, DT to S_1 at B, D. Hence K is equidistant from BA and DT. Similarly, KF is perpendicular to CD and K is equidistant from AC and DT. Hence K is the centre of a circle touching BA, AC and DT. Accordingly, AK is a bisector of $\angle BAC$. But K is also on the perpendicular bisector of BC and it is known that this line meets the bisectors of $\angle BAC$ on the circumcircle of ABC.

Solution 2. We use the same notation as in the first solution.
Since the tangents at the ends of a chord are equally inclined to that chord, we have $\angle TDB = \angle ABD$ and $\angle T'DC = \angle DCA$. Hence

$$\begin{aligned}\angle BDC &= 180^\circ - \angle ABD + \angle DCA \\ &= 180^\circ - (\angle ABC - \angle DBC) + (\angle DCB - \angle ACB) \\ &= (180^\circ - \angle ABC - \angle ACB) + (\angle DBC + \angle DCB) \\ &= \angle BAC + 180^\circ - \angle BDC.\end{aligned}$$

Thus

$$2\angle BDC = 180^\circ + \angle BAC.$$

Finally

$$\begin{aligned}\angle BKC &= \angle BKD + \angle DKC \\ &= 2(\angle EKD + \angle DKF) = 2\angle EKF \\ &= 2(180^\circ - \angle BDC) = 180^\circ - \angle BAC,\end{aligned}$$

so that K lies on circle ABC.

G2. Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E respectively. Prove that

$$AB + AC \geqslant 4DE.$$

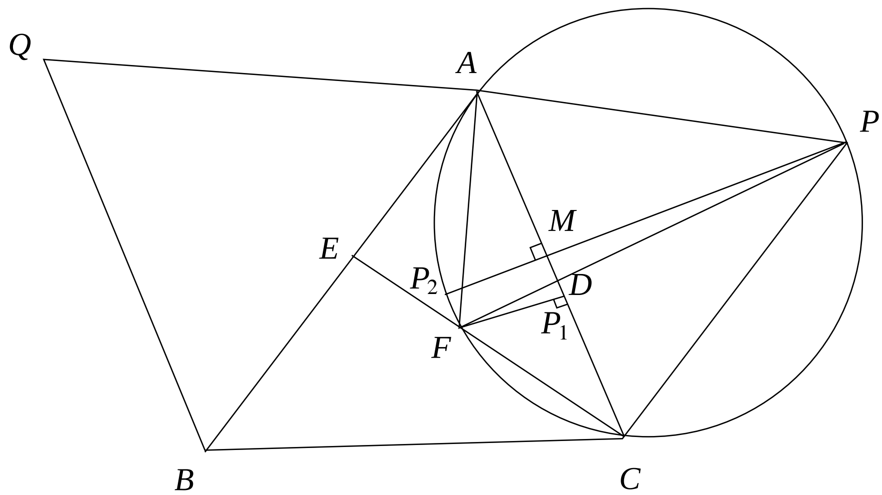
Comments

1. We present two solutions, a geometrical one and an algebraic one, both of which use standard procedures and are of moderate difficulty.
2. Though the geometrical solution uses known properties of the Fermat point, these are very easy to deduce directly.
3. A complex variable solution is also possible because of the 120° angles, but it is comparable with the other methods in length and difficulty.
4. Ptolemy's inequality applied to the quadrilateral $ADFE$ does not seem to produce the required result.

Solution 1. We need the following lemma:

Lemma. A triangle DEF is given. Points P and Q lie on FD, FE respectively, so that $PF \geqslant \lambda DF$ and $QF \geqslant \lambda EF$, where $\lambda > 0$. If $\angle PFQ \geqslant 90°$ then $PQ \geqslant \lambda DE$.

Proof: Let $\angle PFQ = \theta$. Since $\theta \geqslant 90°$, we have $-\cos\theta \geqslant 0$. Now, by the cosine law, we have $PQ^2 = PF^2 + QF^2 - 2\cos\theta(PF)(QF) \geqslant (\lambda DF)^2 + (\lambda EF)^2 - 2\cos\theta(\lambda DF)(\lambda EF) = (\lambda DE)^2$ from which $PQ \geqslant \lambda DE$, as required.



We now start the main proof. Note that $\angle AFE = \angle BFE = \angle CFD = \angle AFD = 60°$. Now let the lines BF, CF meet the circumcircles of triangles CFA, AFB at the points P, Q respectively. Then it is easy to see that both triangles CPA and AQB are equilateral. We now use the lemma with $\lambda = 4$ and $\theta = 120°$. To see how, let P_1 be the foot of the perpendicular from F to the line AC and suppose the perpendicular bisector of AC meets the circumcircle CFA at P and P_2. Let M be the midpoint of AC. Then $PD/DF = PM/FP_1 \geqslant PM/MP_2 = 3$ so $PF \geqslant 4DF$. Similarly we have $QF \geqslant 4EF$. Since $\angle DFE = 120°$, the lemma applies and so $PQ \geqslant 4DE$. Finally, using the triangle inequality, $AB + AC = AQ + AP \geqslant PQ \geqslant 4DE$.

Further comment

5. An alternative argument may be used to prove $PF \geqslant 4DF$. Since the area $[CFA] = [AFD] + [CFD]$ we have $(CF)(AF) = (CF)(DF) + (AF)(DF)$, from which
$$DF = \frac{(CF)(AF)}{(CF) + (AF)}.$$
But it is easily shown, by Ptolemy's theorem for the cyclic quadrilateral $AFCP$ for example, that $CF + AF = PF$, so $PF/DF = \{(CF) + (AF)\}^2 / \{(CF)(AF)\} \geqslant 4$.

Solution 2. Let x, y, z denote the lengths of AF, BF, CF respectively. Then, from (*), we have $DF = xz/(x + z)$ and similarly $EF = xy/(x + y)$. Applying the cosine law to triangles ABF, ACF, DEF the given inequality becomes
$$\sqrt{x^2 + xy + y^2} + \sqrt{x^2 + xz + z^2} \geqslant 4\sqrt{\left(\frac{xy}{x + y}\right)^2 + \left(\frac{xz}{x + z}\right)^2 + \left(\frac{xy}{x + y}\right)\left(\frac{xz}{x + z}\right)}$$
Since $(x + y)/4 \geqslant xy/(x + y)$ and $(x + z)/4 \geqslant xz/(x + z)$ it is sufficient to prove
$$\sqrt{x^2 + xy + y^2} + \sqrt{x^2 + xz + z^2} \geqslant \sqrt{(x + y)^2 + (x + z)^2 + (x + y)(x + z)}.$$
It is easy to check that the square of the left-hand side minus the square of the right-hand side comes to
$$2\sqrt{(x^2 + xy + y^2)(x^2 + xz + z^2)} - \left(x^2 + 2(y + z)x + yz\right).$$
It is sufficient, therefore to show that the square of the first term is greater than or equal to the square of the second term. But a short calculation shows that the difference between these two squares is equal to $3\left(x^2 - yz\right)^2 \geqslant 0$.

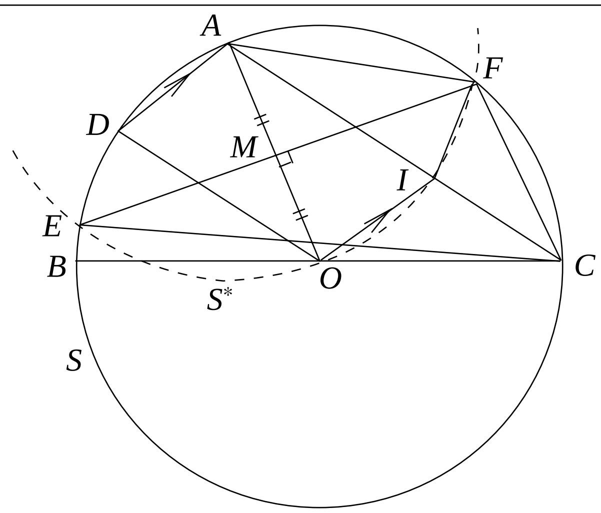
Further comment

6. It is easy to show that equality holds if and only if triangle ABC is equilateral, but there seems no interest in making this part of the question.

G3. The circle S has centre O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incentre of the triangle CEF.

Comments

1. The condition $\angle AOB < 120°$ ensures that I is internal to triangle CEF.
2. Besides the two solutions given, other proofs using circle and triangle properties are possible; a coordinate method would appear to be lengthy.



Solution 1. A is the midpoint of arc EAF, so CA bisects $\angle ECF$. Now, since $OA = OC$, $\angle AOD = \frac{1}{2}\angle AOB = \angle OAC$ so OD is parallel to IA and $ODAI$ is a parallelogram. Hence $AI = OD = OE = AF$ since $OEAF$ (with diagonals bisecting each other at right angles) is a rhombus. Thus

$$\angle IFE = \angle IFA - \angle EFA = \angle AIF - \angle ECA$$
$$= \angle AIF - \angle ICF = \angle IFC.$$

Therefore, IF bisects angle EFC and I is the incentre of triangle CEF.

Solution 2. As in the first solution, $ODAI$ is a parallelogram. Thus both O and I lie on the image S^* of the circle S under the half-turn about the midpoint M of EF. Let I_0 be the incentre of the triangle CEF. Since A is the midpoint of the arc EF of S which does not contain C, both I and I_0 lie on the side CA, which is the internal bisector of $\angle ECF$. Note that

$$AO = OE = EA = AF = FO,$$

implying that AEO and AFO are congruent equilateral triangles. It follows that $\angle EOF = 120°$. Since I_0 is the incentre and O the circumcentre of CEF we have

$$\angle EI_0F = 90° + \tfrac{1}{2}\angle ECF = 90° + \tfrac{1}{4}\angle EOF = 120°.$$

It follows that I_0, as well as I, lies on S^*. Since S^* has a unique intersection with the side AC, we conclude that $I = I_0$.

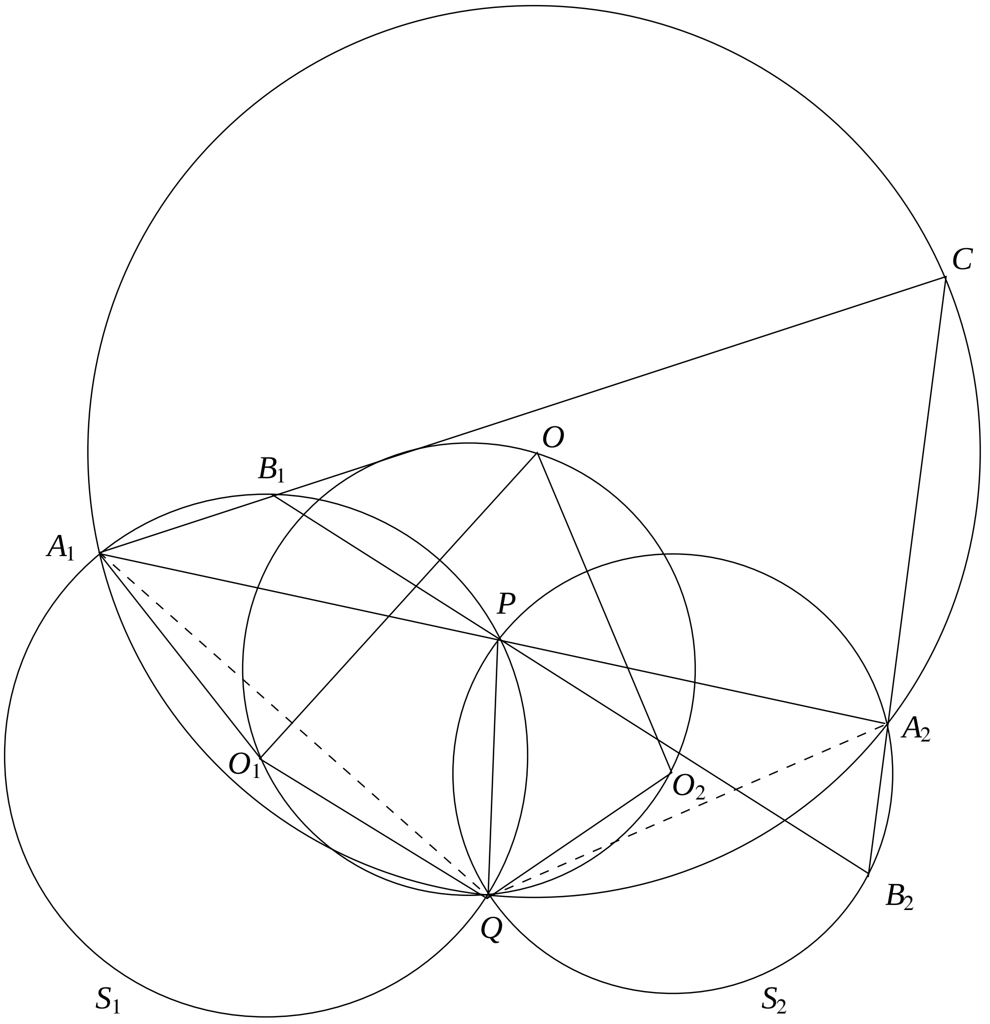
G4. Circles S_1 and S_2 intersect at points P and Q. Distinct points A_1 and B_1 (not at P or Q) are selected on S_1. The lines A_1P and B_1P meet S_2 again at A_2 and B_2 respectively, and the lines A_1B_1 and A_2B_2 meet at C. Prove that, as A_1 and B_1 vary, the circumcentres of triangles A_1A_2C all lie on one fixed circle.

Comments

1. The solution establishes the essential fact that the circle to be identified passes through Q and the centres O_1, O_2 of S_1, S_2 respectively. A solver must appreciate this before composing a solution. The motivation may arise from considering certain special or limiting cases. For example, when A_1P is tangent to S_2 at P then A_2 coincides with P and C coincides with B_1. The circumcircle of triangle A_1A_2C is then S_1 and its circumcentre O coincides with O_1. Also if B_1 is close to Q, so are B_2 and C, indicating that Q lies on the circle to be identified.

2. Although the solution given is short and the problem is by no means hard, it is not as straightforward as the solution may at first sight suggest (see above comment).

3. An analytic solution is possible, but the best we could manage took three full sheets of writing!



Solution.

Step 1. The points A_1, C, A_2, Q are concyclic.

Proof: We prove this by showing that the opposite angles of the quadrilateral add up to 180°. We have $\angle A_1CA_2 + \angle A_1QA_2 = \angle A_1CA_2 + \angle A_1QP + \angle PQA_2 = \angle B_1CB_2 + \angle CB_1B_2 + \angle CB_2B_1 = 180°$. Here we have made use of the circle property that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle and also that angles in the same segment are equal.

Step 2. Let O be the circumcentre of triangle A_1A_2C. Then the points O, O_1, Q, O_2 are concyclic.

Proof: We again prove that opposite angles of the quadrilateral add up to 180°. From Step 1 we have $OQ = OA_1$. Also $O_1Q = O_1A_1$. Hence $\angle OO_1Q = \frac{1}{2}\angle A_1O_1Q = 180° - \angle A_1PQ$. Similarly $\angle OO_2Q = 180° - \angle A_2PQ$. Here we have used the property that the angle at the centre is twice the angle at the circumference and the angle properties of a cyclic quadrilateral. Hence $\angle OO_1Q + \angle OO_2Q = 180° - \angle A_1PQ + 180° - \angle A_2PQ = 180°$. Thus, the centres of the circumcircles of all possible triangles A_1A_2C (and similarly for triangles B_1B_2C) lie on a fixed circle through O_1, O_2 and Q.

Further comment

4. There are some additional features about this configuration which may arise in alternative proofs. For example, if the tangents at A_1, A_2 meet at C' then A_1, A_2, C, C' are concyclic. Since it is easy to prove that C', A_1, A_2, Q are concyclic, we have an alternative proof of Step 1.

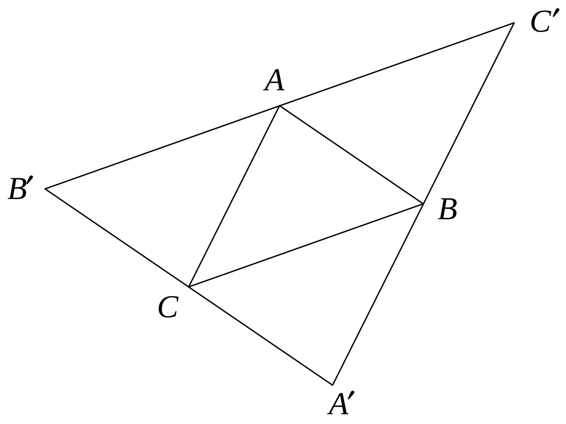
G5. For any set S of five points in the plane, no three of which are collinear, let $M(S)$ and $m(S)$ denote the greatest and smallest areas, respectively, of triangles determined by three points from S. What is the minimum possible value of $M(S)/m(S)$?

Solution.

When the five points are arranged at the vertices of a regular pentagon, it is easy to check that $M(S)/m(S)$ equals the golden ratio, $\tau = (1+\sqrt{5})/2$. We claim that this is best possible.

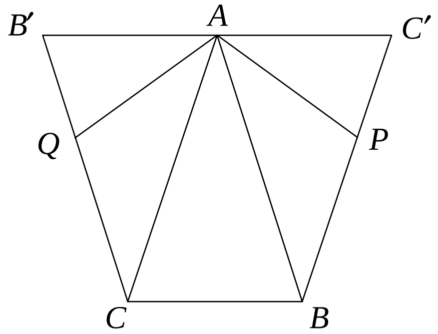
Let S be an arbitrary configuration, and label the points A, B, C, D and E, so that $\triangle ABC$ has maximal area $M(S)$. In the following five steps, we prove the claim by showing that some triangle has area $M(S)/\tau$ or smaller.

Step 1. Construct a larger triangle $\triangle A'B'C'$ with parallel sides to $\triangle ABC$ so that A, B and C lie at the midpoints of the edges $B'C'$, $C'A'$ and $A'B'$, respectively. The point D must then lie on the same side of $B'C'$ as BC otherwise $\triangle DBC$ would have greater area than $\triangle ABC$. Arguing similarly with the other edges and with the vertex E, it follows that both D and E necessarily lie within $\triangle A'B'C'$ (perhaps on its boundary).



Step 2. We can assume more. Of the three triangles $\triangle A'BC$, $\triangle AB'C$ and $\triangle ABC'$ at least one of them contains neither D nor E. Rearranging the labels A, B and C if necessary, we can assume that D and E are contained inside the quadrilateral $BCB'C'$.

Step 3. Note that if an affine linear transformation of the plane is applied to the configuration S, the ratio $M(S)/m(S)$ remains unchanged (since all areas change by the same factor). We can therefore make the convenient assumption that A, B and C are vertices of a regular pentagon $APBCQ$; if this is not already true, then a suitable affine linear transformation can be found carrying A, B and C to the required positions. Since $\angle ABP = \angle BAC = 36°$, it follows that P lies on BC'. Similarly, Q lies on CB'.



Step 4. If either D or E lies in the pentagon $APBCQ$, then we are done. We argue for D as follows: Note that $\triangle APB$ has area $M(S)/\tau$. If D lies in $\triangle APB$, then $\triangle DPB$ has area at most $M(S)/\tau$. Likewise we are done if D lies in $\triangle AQC$. Finally if D is contained in $\triangle ABC$, then one of $\triangle DAB$, $\triangle DBC$ or $\triangle DCA$ has area at most $M(S)/3 < M(S)/\tau$. Similarly for E.

Step 5. What remains is the case where D and E are contained in the union of the triangles $\triangle APC'$ and $\triangle AQB'$. Then $|AE|, |AD| \leqslant |AP| = |AQ|$, and on the other hand the angle $\theta = \angle EAD$ satisfies one of $0 < \theta \leqslant 36°$ (if E and D lie in the same triangle) or $108° \leqslant \theta < 180°$ (if they lie in different triangles). Either way, we have
$\text{Area}(\triangle ADE) = \frac{1}{2}|AD||AE| \sin\theta \leqslant \frac{1}{2}|AP||AQ| \sin 108° = \text{Area}(\triangle APQ) = M(S)/\tau$.
This completes the proof that the minimum value of $M(S)/m(S)$ is τ.

Comments

1. The difficulty is in knowing where to begin. The winning configuration (a regular pentagon) is certainly eminently guessable, but what next? It is natural to look at a largest (or smallest) triangle and work from there. After that, naive case-checking or diagram-chasing doesn't seem to work very well. The crucial observation is in Step 3, when we note that $\triangle ABC$ can be identified with part of a regular pentagon. *Now* the case-checking and diagram-chasing becomes comparatively clean, since the known geometry of the pentagon can be used as a reference.

2. Without something like Step 3 the problem is forbidding. It is still possible, but quite difficult, to find a clean argument – most attempts are likely to be messy and/or incomplete.

3. Reading the above proof carefully, it is easy to show that the minimum is attained precisely when S is an affine linear transformation of the vertices of a regular pentagon.

4. The proof above in no way generalises when the number n of points is greater than 5. It would be extremely interesting if a contestant were to find a proof that did work for some other values of n. For general n, the answer is unknown, and not even known asymptotically; this is related to the famous Heilbronn problem on the smallest triangle formed from n points in the unit square.

G6. Let $n \geqslant 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centres $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leqslant i < j \leqslant n} \frac{1}{O_i O_j} \leqslant \frac{(n-1)\pi}{4}.$$

Comments

1. We present a solution, which, though fairly short, requires considerable ingenuity to devise. The question seems medium to hard in difficulty.
2. The last part of the solution is a double-counting argument, and doubtless there are many equivalent formulations possible.

Solution. We use the following Lemma.

Lemma. Let Ω be a circle of radius ρ and PR, QS two chords intersecting at X, so that $\angle PXQ = \angle RXS = 2\alpha$. Then arc PQ + arc $RS = 4\alpha\rho$. (See Diagram 1.)

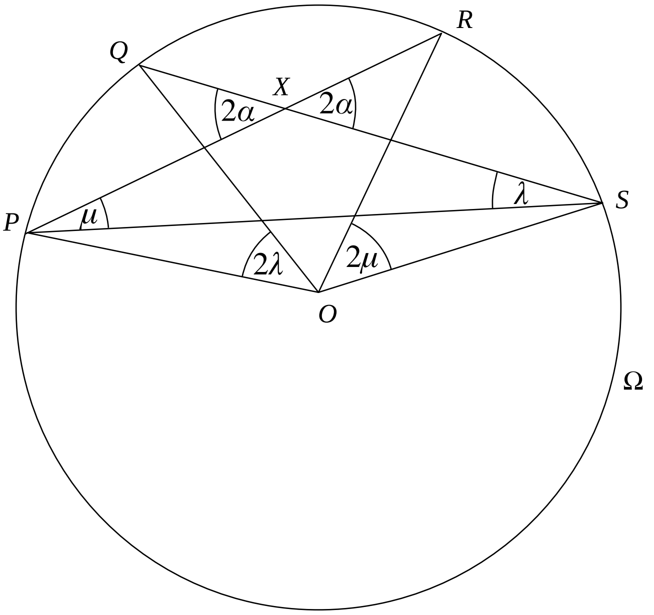


Diagram 1

Proof: Let O be the centre of Ω. Let $\angle POQ = 2\lambda$ and $\angle ROS = 2\mu$; then $\angle QSP = \lambda$ and $\angle RPS = \mu$, since the angle at the centre is twice the angle at the circumference. Then $\angle RXS = 2\alpha = \lambda + \mu$ and arc PQ + arc $RS = 2\lambda\rho + 2\mu\rho = 4\alpha\rho$.

We now start the main proof.

Surround all the given circles with a large circle Ω of radius ρ. Consider two circles C_i, C_j, with centres O_i, O_j respectively. From the given condition C_i and C_j do not intersect. Let 2α be the angle between their two internal common tangents PR, QS (see Diagram 2). We have $O_iO_j = 2 \operatorname{cosec} \alpha$, so $\alpha \geqslant \sin \alpha = 2 / O_iO_j$.

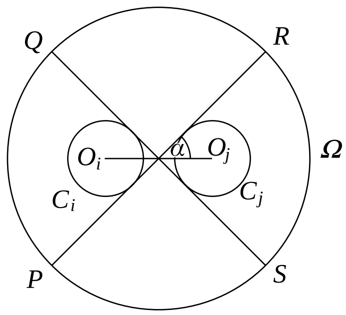


Diagram 2

Now, from the lemma, arc PQ + arc RS = $4\alpha\rho \geqslant 8\rho O_iO_j$, so that

$$\frac{1}{O_iO_j} \leqslant \frac{\text{arc}\,PQ + \text{arc}\,RS}{8\rho}.$$

We now wish to consider the sum of all these arc lengths as i, j range over all pairs, and we claim that any point of Ω is covered by such arcs at most $(n - 1)$ times. To see this, let T be any point of Ω and TU a half-line tangent to Ω, as in Diagram 3. Consider this half-line as it is rotated about T as shown. At some stage it will intersect a pair of circles for the first time. Relabel these circles C_1 and C_2. The half-line can never intersect three circles, so at some further stage intersection with one of these circles, say C_1, is lost and the half-line will never meet C_1 again during its transit. Continuing in this way and relabelling the circles conveniently, the maximum number of times the half-line can intersect pairs of circles is $(n - 1)$, namely when it intersects C_1 and C_2, C_2 and C_3, …, C_{n-1} and C_n. As T was arbitrary, it follows that the sum of all the arc lengths is less than or equal to $2(n - 1)\pi\rho$, and hence

$$\sum_{1 \leqslant i < j \leqslant n} \frac{1}{O_iO_j} \leqslant \frac{(n - 1)\pi}{4}.$$

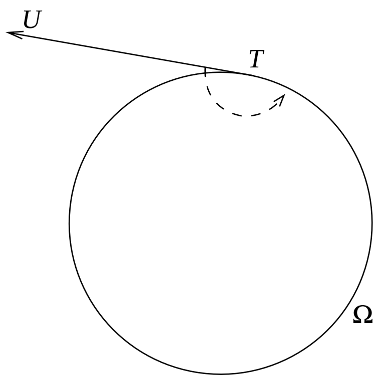


Diagram 3

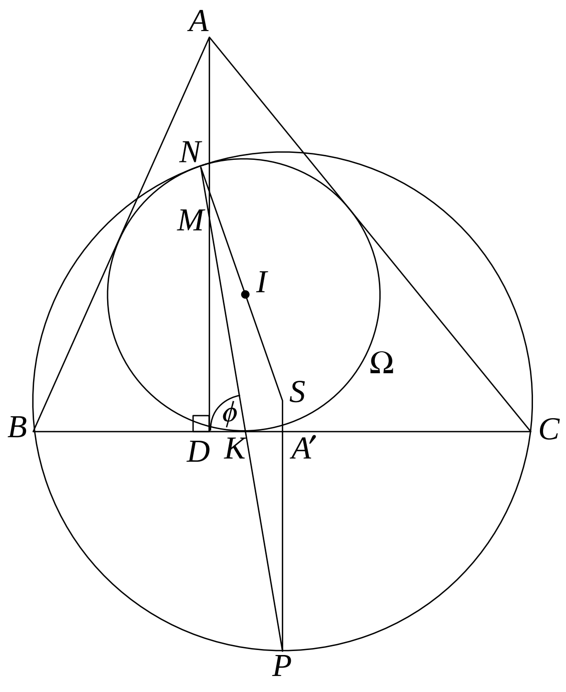
Further comment

3. If the lemma proves elusive, a solver could construct a proof in which Ω is sufficiently large for the intersection points to be close to its centre, thus removing any need for the lemma.

G7. The incircle Ω of the acute-angled triangle ABC is tangent to BC at K. Let AD be an altitude of triangle ABC and let M be the midpoint of AD. If N is the other common point of Ω and KM, prove that Ω and the circumcircle of triangle BCN are tangent at N.

Comments

1. We give two solutions, both of which involve a mixture of pure geometry and computation. The problem is difficult, but not excessively so.
2. In the first solution, the point P is defined as the point of intersection of NK and the perpendicular bisector of BC, and is shown to lie on the circumcircle of triangle BCN by proving $(NK)(KP) = (BK)(KC)$.
3. In the second solution, the point P is defined as the intersection (other than N) of NK and the circumcircle of triangle BCN, and is shown to lie on the perpendicular bisector of BC by proving that NK bisects $\angle BNC$.
4. In the two solutions, we perform some manipulations that only make sense when AB is not equal to AC. This is why we start by dealing with the (trivial) case when $AB = AC$. It would be possible to add the words 'non-isosceles' in the statement of the problem, but we feel that this would detract from its elegance, especially as the result does still hold in the isosceles case.



Solution 1. We may assume that $AB \neq AC$, as if $AB = AC$ then the result is trivial (as the distance between the centres of the two internally tangent circles is equal to the difference of their radii). By symmetry, we may assume that $AB < AC$.

Let the perpendicular bisector of the side BC intersect NK and BC at P and A' respectively. It is sufficient to prove that N, the incentre I of triangle ABC and S, the circumcentre of triangle BCN, are collinear. Since IK and SP are parallel, both being perpendicular to BC, it is sufficient to prove that P lies on the circumcircle of triangle BCN; for once we know $SP = SN$ then $\angle PNS = \angle NPS = \angle NKI = \angle PNI$, and NIS is a straight line. To establish what is wanted we show $(NK)(KP) = (BK)(KC)$.

Using the standard notation for triangle ABC (with $s = (a + b + c)/2$), we have $BK = s - b$ and $KC = s - c$, so $(BK)(KC) = (s - b)(s - c)$. By the cosine law for triangle ABC, we have $\cos B = \left(c^2 + a^2 - b^2\right)/2ca$ and then $BD = c\cos B = \left(c^2 + a^2 - b^2\right)/2a$. Now $KA' = BA' - BK = \frac{1}{2}(b - c)$ and $DK = BK - BD = (b - c)(s - a)/a$. Let $\angle MKD = \phi$; then

$$\tan\phi = \frac{MD}{DK} = \frac{\frac{1}{2}(AD)\,a}{(b - c)(s - a)} = \frac{[ABC]}{(b - c)(s - a)},$$

where $[ABC]$ is the area of ABC. Now $\angle NIK = 2\phi$, so $NK = 2r\sin\phi$, where r is the inradius of ABC. Finally, from triangle $A'KP$ we have $KP = KA'\sec\phi$ and hence

$$(NK)(KP) = 2r(KA')\tan\phi = \frac{r[ABC]}{(s - a)} = \frac{[ABC]^2}{s(s - a)} = (s - b)(s - c) = (BK)(KC).$$

Here we have used the well-known expressions for area: $[ABC] = rs = \sqrt{s(s-a)(s-b)(s-c)}$.

Solution 2. As in Solution 1, we may assume $AB < AC$, and it is sufficient to show that NIS is a straight line. But now we define P to be the intersection (other than N) of NK with the circumcircle of triangle BNC. Now $SP = SN$ implies $\angle SPN = \angle SNP$ and $IN = IK$ implies $\angle IKN = \angle INK$, and NIS is a straight line if and only if all these angles are equal, which is when IK and SP are parallel. Since IK is perpendicular to BC this means that SP must be also, and hence it is sufficient to show that P is the midpoint of the arc BC. To establish this, we show that NKP bisects $\angle BNC$ for which it is sufficient to show that $BN/CN = BK/CK$. Again let $\angle MKD = \phi$. Now, by the cosine rule,

$$BN^2 = NK^2 + BK^2 - 2(NK)(BK)\cos\phi$$

and

$$CN^2 = NK^2 + CK^2 + 2(NK)(CK)\cos\phi.$$

So it is sufficient to show

$$\frac{BK^2}{CK^2} = \frac{NK^2 + BK^2 - 2(NK)(BK)\cos\phi}{NK^2 + CK^2 + 2(NK)(CK)\cos\phi}.$$

or $(CK - BK)NK = 2(BK)(CK)\cos\phi$.
Now $NK = 2r\sin\phi$, so it is sufficient to prove $2r(CK - BK)\tan\phi = 2(BK)(CK)$. But $\tan\phi = MD/DK = \frac{1}{2}AD/DK = \frac{1}{2}c\sin B/(s - b - c\cos B)$, since M is the midpoint of AD. Now $BK = r\cot\frac{1}{2}B$ and $CK = r\cot\frac{1}{2}C$, so it is sufficient to prove

$$\left(\cot\tfrac{1}{2}C - \cot\tfrac{1}{2}B\right)(c\sin B)/(a + c - b - 2c\cos B) = \cot\tfrac{1}{2}B\cot\tfrac{1}{2}C.$$

Using the sine rule and $a = c\cos B + b\cos C$ this reduces to proving that

$$\sin C\sin B\left(\cot\tfrac{1}{2}C - \cot\tfrac{1}{2}B\right) = \cot\tfrac{1}{2}B\cot\tfrac{1}{2}C(\sin C - \sin B + \sin B\cos C - \sin C\cos B). \qquad (*)$$

Putting (*) into half-angles, and cancelling $\sin\frac{1}{2}(B - C)$ this resolves to

$$\sin B\sin C = 4\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C,$$

which is true.

Further comment

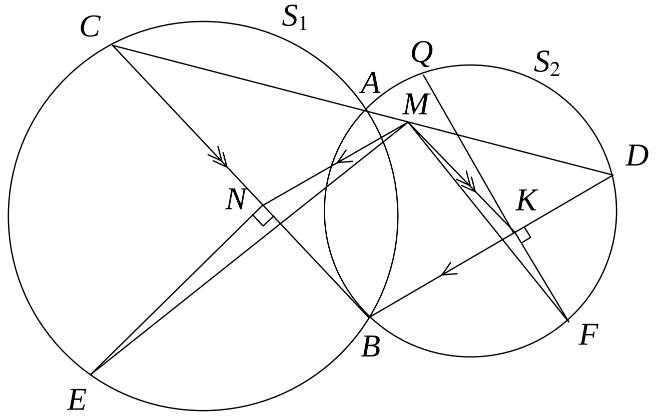
5. Slight changes in the text are necessary in Solution 2 when $AB > AC$, but the solution is essentially the same.

G8. Let S_1 and S_2 be circles meeting at the points A and B. A line through A meets S_1 at C and S_2 at D. Points M, N, K lie on the line segments CD, BC, BD respectively, with MN parallel to BD and MK parallel to BC. Let E and F be points on those arcs BC of S_1 and BD of S_2 respectively that do not contain A. Given that EN is perpendicular to BC and FK is perpendicular to BD prove that $\angle EMF = 90°$.

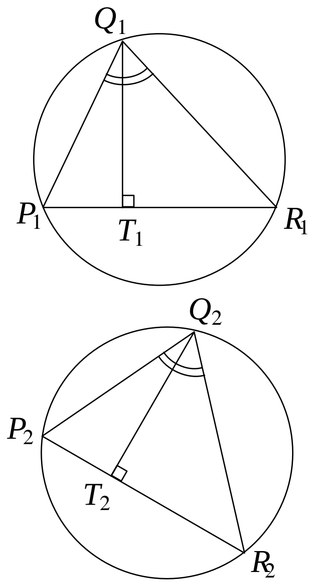
Comments

1. In the solution, the lemma looks elaborate but merely formalizes the 'obvious' similarity of two figures involving circular arcs. This seems worth making explicit as it appears to be the key to the problem.

2. A coordinate approach would be impracticable.



Solution.



Lemma. If $P_1Q_1R_1$ and $P_2Q_2R_2$ are circular arcs with $\angle P_1Q_1R_1 = \angle P_2Q_2R_2$ and T_1, T_2 are the feet of the perpendiculars from Q_1, Q_2 to P_1R_1, P_2R_2 respectively, then if $P_1T_1 / T_1R_1 = P_2T_2 / T_2R_2$ then the triangles $P_1Q_1R_1$, $P_2Q_2R_2$ are similar.

Proof: If Q_2' is the unique point on arc $P_2Q_2R_2$ making triangles $P_1Q_1R_1$, $P_2Q_2'R_2$ equiangular and therefore similar, and if $Q_2'T_2'$ is perpendicular to P_2R_2, then $P_2T_2' / T_2'R_2 = P_1T_1 / T_1R_1 = P_2T_2 / T_2R_2$, so $T_2' = T_2$ and $Q_2' = Q_2$.

Turning now to the problem we have

$$BN / NC = DM / MC \qquad \text{since } MN \,||\, DB$$

$$= DK / KB \qquad \text{since } MK \,||\, CB.$$

Let FK produced meet S_2 again at Q. Then $\angle BQD = \angle BAD = \angle BEC$. By the Lemma, triangles BEC, DQB are similar. Hence $\angle EBC = \angle QDB = \angle QFB$ and the right-angled triangles BNE, FKB are similar.

Now $\angle MNB = \angle MKB$ since $MKBN$ is a parallelogram, so $\angle ENM = \angle MKF$. Also $\dfrac{MN}{KF} = \dfrac{BK}{KF} = \dfrac{EN}{NB} = \dfrac{EN}{MK}$. Therefore triangles ENM, MKF are similar and $\angle NME = \angle KFM$. Since lines MN, KF are perpendicular, so are EM and FM.

Further comment

3. From the similarity of the right-angled triangles BNE, FKB it follows easily that $\angle EAF = 90°$ as well.

A1. Find all functions f from the reals to the reals such that

$$f(f(x)+y) = 2x + f(f(y)-x)$$

for all real x, y.

Solution. For each real c, the function given by $f(x) = x + c$ is a solution for the given functional equation, since it makes both sides equal $x + y + 2c$. We claim that these are the only solutions.

Our strategy is to derive an equation of the form $f(X) = X + c$, where X is an expression whose values are guaranteed to run over all real numbers.

We claim first that f is surjective. Indeed, set $y = -f(x)$ in the functional equation. This gives

$$f(0) = 2x + f(f(-f(x)) - x)$$

or

$$f(0) - 2x = f(f(-f(x)) - x).$$

As all real numbers have the form $f(0) - 2x$, for each real y there is a z with $y = f(z)$, as claimed.

In particular there is an a with $f(a) = 0$. Set $x = a$ in the functional equation. This gives

$$f(y) = 2a + f(f(y) - a),$$

or equivalently

$$f(y) - a = f(f(y) - a) + a.$$

As f is surjective, for each real x there is a real y with $x = f(y) - a$. Hence

$$x = f(x) + a$$

for all x, that is $f(x) = x - a$.

Comment

1. This is an easy problem. It seems to be crucial to note that f is surjective.

A2. Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers, for which there exists a real number c with $0 \leqslant a_i \leqslant c$ for all i, such that

$$|a_i - a_j| \geqslant \frac{1}{i+j} \text{ for all } i, j \text{ with } i \neq j.$$

Prove that $c \geqslant 1$.

Solution. Fix $n \geqslant 2$, and let $\sigma(1), \sigma(2), \ldots, \sigma(n)$ be the permutation of $1, 2, \ldots, n$ which orders the first n elements of the sequence:

$$0 \leqslant a_{\sigma(1)} < a_{\sigma(2)} < \ldots < a_{\sigma(n)} \leqslant c.$$

Then

$$\begin{aligned}
c &\geqslant a_{\sigma(n)} - a_{\sigma(1)} \\
&= \left(a_{\sigma(n)} - a_{\sigma(n-1)}\right) + \left(a_{\sigma(n-1)} - a_{\sigma(n-2)}\right) + \ldots + \left(a_{\sigma(2)} - a_{\sigma(1)}\right) \\
&\geqslant \frac{1}{\sigma(n) + \sigma(n-1)} + \frac{1}{\sigma(n-1) + \sigma(n-2)} + \ldots + \frac{1}{\sigma(2) + \sigma(1)}. \qquad (*)
\end{aligned}$$

Now, using the Cauchy-Schwarz inequality, we obtain

$$\left(\frac{1}{\sigma(n)+\sigma(n-1)} + \ldots + \frac{1}{\sigma(2)+\sigma(1)}\right)\left((\sigma(n)+\sigma(n-1)) + \ldots + (\sigma(2)+\sigma(1))\right) \geqslant (n-1)^2$$

so

$$\begin{aligned}
\frac{1}{\sigma(n)+\sigma(n-1)} + \ldots + \frac{1}{\sigma(2)+\sigma(1)} &\geqslant \frac{(n-1)^2}{2(\sigma(1)+\ldots+\sigma(n)) - \sigma(1) - \sigma(n)} \\
&= \frac{(n-1)^2}{n(n+1) - \sigma(1) - \sigma(n)} \\
&\geqslant \frac{(n-1)^2}{n^2+n-3} \\
&\geqslant \frac{n-1}{n+3}.
\end{aligned}$$

From (*) it follows that the inequality

$$c \geqslant \frac{n-1}{n+3} = 1 - \frac{4}{n+3}$$

holds for all $n \geqslant 2$. Thus we must have $c \geqslant 1$.

Comments

1. What makes the question challenging is: how do we bring in the value of c? Which bits of the data should we use? The key step is to realise that to make use of c all we need are the distances between *adjacent* terms. Having got equation (*), the rest is then easy, and there are several ways to finish off the proof.

2. We do not know what the smallest value of c actually is.

3. The solution relies on finding a lower bound for the quantity

$$\frac{1}{\sigma(1)+\sigma(2)}+\frac{1}{\sigma(2)+\sigma(3)}+\ \dots\ +\frac{1}{\sigma(n-1)+\sigma(n)}$$

where σ is an arbitrary permutation of $(1,\ 2,\ \dots,\ n)$. An alternative would be to take this as the heart of the question, and ask for the exact minimum, thus:

A2'. What is the minimum value of

$$\frac{1}{\sigma(1)+\sigma(2)}+\frac{1}{\sigma(2)+\sigma(3)}+\ \dots\ +\frac{1}{\sigma(n-1)+\sigma(n)}$$

as σ ranges over all permutations of $\{1, 2, \dots, n\}$?

The optimal permutation turns out to be the one given by $\sigma(1) = 1$, $\sigma(n) = 2$, $\sigma(2) = 3$, $\sigma(n-1) = 4$, and so on. To prove this, we use induction, but it is vital to prove a *stronger* statement: that if we look at permutations of a general sequence $x_1, x_2, \dots, x_n$ instead of just $1, 2, \dots, n$ (where say $x_1 < x_2 < \dots < x_n$), then the optimal permutation is again $\sigma(x_1) = x_1$, $\sigma(x_n) = x_2$, $\sigma(x_2) = x_3$, $\sigma(x_{n-1}) = x_4$, and so on. The proof is more difficult, and more interesting, than that of A2. The only drawback is that A2' lacks the 'how on earth can we make use of the information?' puzzle that contestants face with A2.

A3. Let P be a cubic polynomial given by $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are integers and $a \neq 0$. Suppose that $xP(x) = yP(y)$ for infinitely many pairs x, y of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

Comment

1. The main ideas in the solution are that $x + y$ is bounded for all solutions x, y (consider the shape of the quartic $xP(x)$), and that P is then symmetric about one particular value of $(x + y)/2$ which is taken infinitely often.

Solution. Let x, y be distinct integers satisfying $xP(x) = yP(y)$ so that

$$x\left(ax^3 + bx^2 + cx + d\right) = y\left(ay^3 + by^2 + cy + d\right)$$

$$\text{i.e. } a\left(x^4 - y^4\right) + b\left(x^3 - y^3\right) + c\left(x^2 - y^2\right) + d(x - y) = 0.$$

Dividing by $x - y (\neq 0)$ leads to

$$a\left(x^3 + x^2y + xy^2 + y^3\right) + b\left(x^2 + xy + y^2\right) + c(x + y) + d = 0. \tag{1}$$

It is convenient to write

$$s = x + y, \qquad t = xy. \tag{2}$$

Since

$$x^3 + x^2y + xy^2 + y^3 = (x + y)\left(x^2 + y^2\right) = s\left(s^2 - 2t\right)$$

and

$$x^2 + xy + y^2 = s^2 - t,$$

(1) can be written in the form

$$as\left(s^2 - 2t\right) + b\left(s^2 - t\right) + cs + d = 0$$

or equivalently as

$$P(s) = (2as + b)t. \tag{3}$$

We claim that the integer s can take only finitely many values. Indeed, consider the right-hand side of (3). Since $s^2 - 4t = (x - y)^2 \geqslant 0$, we have $|t| < s^2/4$ so that
$|(2as + b)t| \leqslant |(2as + b)(s^2/4)|$.
Equation (3) therefore leads to

$$\left| as^3 + bs^2 + cs + d \right| \leqslant \left| \frac{a}{2}s^3 + \frac{b}{4}s^2 \right|$$

which can only be true for finitely many values of the integer s, as required.

Write $Q(x)$ for $xP(x)$. The equation $xP(x) = yP(y)$ becomes $Q(x) = Q(y)$, which holds for infinitely many pairs of distinct integers x, y. Equivalently, $Q(r) = Q(s - r)$ holds for infinitely many pairs of integers s, r with s as in (2). But s can only take finitely many values. Hence for (at least) one integer s, the equation $Q(r) = Q(s - r)$ must hold for infinitely many integers. But then $Q(x)$ and $Q(s - x)$ are polynomials of degree 4 which are equal for infinitely many integer values of x. They must therefore be equal for all real numbers x.

To finish the proof, we consider two cases.

Case 1: $s \neq 0$. We have $xP(x) = (s - x)P(s - x)$ for all real numbers x. Take $x = s$ to get $sP(s) = 0$ so that $P(s) = 0$ as $s \neq 0$. Hence $x = s$ is an integer root of $P(x) = 0$.

Case 2: $s = 0$. We now have $Q(x) = Q(-x)$ for all real numbers x so that Q is an even function. As $Q(x)$ is divisible by x, it must be divisible by x^2 i.e. $Q(x) = xP(x) = x^2R(x)$ for some polynomial $R(x)$. Hence $P(x) = xR(x)$ so that $P(0) = 0$. Again, the equation $P(x) = 0$ has an integer root, namely $x = 0$.

Further comment

2. By examining further Cases 1 and 2 above, it is not too hard to show that polynomials P satisfying the conditions of the problem have the general form

$$P(x) = (x - k)\left(ax^2 - akx + m\right)$$

where a, k and m are integers.

A4. Find all functions f from the reals to the reals such that

$$\left(f(x)+f(z)\right)\left(f(y)+f(t)\right)=f(xy-zt)+f(xt+yz)$$

for all real x, y, z, t.

Solution. We are given that

$$f(xy-zt)+f(xt+yz)=\left(f(x)+f(z)\right)\left(f(y)+f(t)\right) \qquad (*)$$

for all real x, y, z, t. The equation (*) has the solutions $f(x)=0$ for all x, $f(x)=1/2$ for all x and $f(x)=x^2$ for all x. These make both sides of (*) equal to 0, to 1 and to $\left(x^2+z^2\right)\left(y^2+t^2\right)$ respectively. We claim that there are no other solutions.

Suppose (*) holds. Then setting $x=y=z=0$ gives $2f(0)=2f(0)\left(f(0)+f(t)\right)$. In particular $2f(0)=4f(0)^2$ and so $f(0)=0$ or $f(0)=1/2$. If $f(0)=1/2$ we get $f(0)+f(t)=1$ and so f is identically 1/2.

Suppose then that $f(0)=0$. Then setting $z=t=0$ in (*) gives $f(xy)=f(x)f(y)$, that is f is multiplicative. In particular $f(1)=f(1)^2$ and so $f(1)=0$ or 1. If $f(1)=0$ then $f(x)=f(x)f(1)=0$ for all x.

So we may suppose that $f(0)=0$ and $f(1)=1$. Setting $x=0$ and $y=t=1$, (*) gives

$$f(-z)+f(z)=2f(z)$$

and so $f(-z)=f(z)$ for each z, that is f is an even function. So it suffices to show that $f(x)=x^2$ for positive x. Note that $f\left(x^2\right)=f(x)^2\geqslant 0$; as f is an even function, $f(y)\geqslant 0$ for all y.

Now put $t=x$ and $z=y$ in (*) to get

$$f\left(x^2+y^2\right)=\left(f(x)+f(y)\right)^2.$$

This shows that $f\left(x^2+y^2\right)\geqslant f(x)^2=f\left(x^2\right)$. Hence if $u\geqslant v\geqslant 0$ then $f(u)\geqslant f(v)$, that is f is an increasing function on the positive reals.

Set $y=z=t=1$ in (*) to yield

$$f(x-1)+f(x+1)=2\left(f(x)+1\right).$$

By induction on n, it readily follows from this that $f(n)=n^2$ for all non-negative integers n. As f is even, $f(n)=n^2$ for all integers n, and further, as f is multiplicative, $f(a)=a^2$ for all rationals a. Suppose that $f(x)\neq x^2$ for some positive x. If $f(x)<x^2$ take a rational a with $x>a>\sqrt{f(x)}$. Then $f(a)=a^2>f(x)$, but $f(a)\leqslant f(x)$ as f is increasing. This is a contradiction. A similar argument shows that $f(x)>x^2$ is impossible. Thus $f(x)=x^2$ for all positive x, and since f is even, $f(x)=x^2$ for all real x.

Comments

1. This is a medium difficulty problem, requiring no really clever ideas, but a willingness to experiment with the functional equation to squeeze out diverse consequences. One also needs the passage from knowing f on the rationals to knowing it on the reals: here the key point is that we know that f is increasing (on the positive reals).

2. This problem is clearly inspired by the famous identity

   $$\left(x^2+z^2\right)\left(y^2+t^2\right)=(xy-zt)^2+(xt+yz)^2$$

   used to study sums of two squares in number theory.

3. In its original form, the problem had the variable t set equal to 1, thus:

   A4'. Find all functions f from the reals to the reals such that

$$\big(f(x) + f(z)\big)\big(f(y) + 1\big) = f(xy - z) + f(x + yz)$$

   for all real x, y, z.

   This has the 'sums of two squares' hidden, so that it may be less clear that $f(x) = x^2$ is a solution. We feel that this version is less elegant and attractive than the form we have given. It is quite easy to transform either problem into the other.

A5. Let n be a positive integer that is not a perfect cube. Define real numbers a, b, c by

$$a = \sqrt[3]{n}, \qquad b = \frac{1}{a - [a]}, \qquad c = \frac{1}{b - [b]},$$

where $[x]$ denotes the integer part of x.
Prove that there are infinitely many such integers n with the property that there exist integers r, s, t, not all zero, such that $ra + sb + tc = 0$.

Solution. Note first that it is sufficient to find *rational* numbers r, s, t not all zero such that $ra + sb + tc = 0$.
Let $m = [a]$ and $k = n - m^3$. Then

$$1 \leqslant k \leqslant \left((m+1)^3 - 1\right) - m^3 = 3m(m+1).$$

From the factorisation $a^3 - m^3 = (a - m)\left(a^2 + am + m^2\right)$ we have

$$b = \frac{1}{a - m} = \frac{a^2 + am + m^2}{k}.$$

Since $a < m + 1$, the numerator is less than

$$(m+1)^2 + (m+1)m + m^2 = 3m^2 + 3m + 1.$$

To simplify the calculation, we shall assume that $[b] = 1$. This is true provided that

$$3m^2 + 3m + 1 < 2k. \tag{$*$}$$

Now

$$b - [b] = b - 1 = \frac{a^2 + am + m^2 - k}{k}.$$

Factorise the numerator in the form $a^2 + am + m^2 - k = (a - x)(a - y)$, so that $x + y = -m$ and $xy = m^2 - k$. Note that x, y are real since the discriminant $(x - y)^2$ is $m^2 - 4\left(m^2 - k\right) = 4k - 3m^2 > 0$ by (*). Then

$$c = \frac{1}{b - 1} = \frac{k}{(a - x)(a - y)} = \frac{k\left(a^2 + ax + x^2\right)\left(a^2 + ay + y^2\right)}{\left(a^3 - x^3\right)\left(a^3 - y^3\right)}.$$

(Note that $a^2 + ax + x^2$ and $a^2 + ay + y^2$ are strictly positive.)
Since

$$x^3 + y^3 = (x + y)\left[(x + y)^2 - 3xy\right] = -m\left[m^2 - 3\left(m^2 - k\right)\right] = m\left(2m^2 - 3k\right)$$

and

$$x^3y^3 = (xy)^3 = \left(m^2 - k\right)^3$$

are integers, so is $l = \left(a^3 - x^3\right)\left(a^3 - y^3\right) = n^2 - \left(x^3 + y^3\right)n + x^3y^3$. Then

$$\begin{aligned}
c &= \frac{k}{l}\left(a^2 + ax + x^2\right)\left(a^2 + ay + y^2\right) \\
&= \frac{k}{l}\left(a^4 + (x + y)a^3 + \left(x^2 + xy + y^2\right)a^2 + xy(x + y)a + x^2y^2\right) \\
&= \frac{k}{l}\left(ka^2 + \left(m\left(k^2 - m\right) + n\right)a + \left(m^2 - k\right)^2 - nm\right).
\end{aligned}$$

To make $ra + sb + tc = 0$, choose s and t so that the coefficients of a^2 and a^0 vanish i.e.

$$\frac{s}{k} + \frac{tk^2}{l} = 0 \qquad \text{and} \qquad \frac{sm^2}{k} + \frac{tk}{l}\left((m^2 - k)^2 - nm\right) = 0.$$

The first equation gives $s = -\dfrac{tk^3}{l}$ and the second then becomes

$$-\frac{tk^2m^2}{l} + \frac{tk}{l}\left((m^2 - k)^2 - mn\right) = 0$$

$$\text{i.e.} \frac{tk}{l}\left[(m^2 - k)^2 - mn - km^2\right] = 0.$$

The bracket simplifies as

$$\begin{aligned} m^4 - 2km^2 + k^2 - mn - km^2 &= m(m^3 - n) - 3km^2 + k^2 \\ &= -mk - 3km^2 + k^2 \\ &= k(k - 3m^2 - m). \end{aligned}$$

Choose $k = 3m^2 + m$, which satisfies (*) and also satisfies $1 \leqslant k \leqslant 3m(m + 1)$. We have therefore shown that for integers n of the form $m^3 + 3m^2 + m$ we can obtain non-zero rational values of s, t so that $sb + tc$ is a rational multiple of a. In view of the opening comment, this completes the proof.

Comment

1. One of the key ideas is to force b to have integer part 1: this greatly simplifies what is to come. But there are still more ideas needed: pure brute-force calculation would be doomed to failure.

A6. Let A be a non-empty set of positive integers. Suppose that there are positive integers $b_1, \dots b_n$ and $c_1, \dots, c_n$ such that

(i) for each i the set $b_iA + c_i = \{b_ia + c_i : a \in A\}$ is a subset of A, and

(ii) the sets $b_iA + c_i$ and $b_jA + c_j$ are disjoint whenever $i \neq j$.

Prove that

$$\frac{1}{b_1} + \dots + \frac{1}{b_n} \leqslant 1.$$

Comment

1. In the following proof, the key idea, after trying an example with the b_i equal, is to weight the number of times each b_i appears (this is the use of the p_i below). After that, the calculations are quite easy, and there are several ways to accomplish them. However, this key idea is rather non-trivial, so we feel this is a hard problem.

Solution. For a contradiction, assume that

$$\frac{1}{b_1} + \dots + \frac{1}{b_n} > 1.$$

So certainly $n \geqslant 2$. Note also that A is infinite.

For each i, define $f_i(x) = b_ix + c_i$. Each function f_i maps A to itself. By condition (ii), if $f_i(a) = f_j(a')$ for $a, a' \in A$ then $i = j$ and so $a = a'$. By iterating this argument, it follows that if we have $a, a' \in A$ and

$$f_{i_1}\left(f_{i_2}\left(\dots f_{i_r}(a) \dots\right)\right) = f_{j_1}\left(f_{j_2}\left(\dots f_{j_r}(a') \dots\right)\right)$$

then each $i_k = j_k$ (and $a = a'$).

The idea of our proof is to take a suitable $a \in A$ and generate a family of numbers of the form $f_{i_1}\left(f_{i_2}\left(\dots f_{i_r}(a) \dots\right)\right)$ by choosing $i_1, \dots, i_r$ appropriately. These iterates are all distinct, and we will get an upper estimate on their size, but a lower bound on their number. Before embarking on the main argument, let us consider the case when $b_1 = b_2 = b_3 = 2$ as an illustration. If we choose $a \in A$ large enough, then for any $i_k \in \{1,2,3\}$ we have $f_{i_1}\left(f_{i_2}\left(\dots f_{i_r}(a) \dots\right)\right) \leqslant (2.01)^r a$. But there are 3^r such numbers, all distinct. Taking r large enough we see this is impossible.

This argument is trickier to generalize when the b_i are not all equal. To do so we will look at all sequences of the functions f_i of length N where each f_i appears in a proportion dependent only on i. Take positive rational numbers $p_1, \dots p_n$ (to be chosen later) with $p_1 + \dots + p_n = 1$ and an integer N which is a multiple of N_0, the least common multiple of the denominators of the p_i. Hence each p_iN is an integer. Let $a \in A$ and consider the set $\Phi(N)$ of the numbers of the form $f_{i_1}\left(f_{i_2}\left(\dots f_{i_N}(a) \dots\right)\right)$ where, for each i, exactly p_iN of the i_k equal i. As these are all distinct, $\Phi(N)$ has

$$|\Phi(N)| = \frac{N!}{(p_1N)! \dots (p_nN)!}$$

elements.

Choose $d_i > b_i$ with $1/d_1 + \dots + 1/d_n > 1$. There is a number K such that $K \leqslant f_i(x) \leqslant d_ix$ for all i whenever $x \geqslant K$. As A is infinite, we may choose $a \in A$ with $a \geqslant K$. Then

$$f_{i_1}\left(f_{i_2}\left(\dots f_{i_N}(a) \dots\right)\right) \leqslant d_{i_1} \dots d_{i_N}a = d_1^{p_1N} \dots d_n^{p_nN}a.$$

To obtain a contradiction we need an estimate for $|\Phi(N)|$. We can use Stirling's formula, or similar, to get a precise asymptotic formula for $|\Phi(N)|$, but we can obtain a weaker but still adequate lower bound in a completely elementary fashion. Note that

$$\frac{|\Phi(N + N_0)|}{|\Phi(N)|}$$

$$= \frac{(N + N_0)(N + N_0 - 1)\ldots(N + 1)}{[(p_1N + p_1N_0)(p_1N + p_1N_0 - 1)\ldots(p_1N + 1)]\ldots[(p_nN + p_nN_0)\ldots(p_nN + 1)]}$$

$$= \frac{(1 + N_0/N)(1 + (N_0 - 1)/N)\ldots(1 + 1/N)}{[(p_1 + p_1N_0/N)(p_1 + (p_1N_0 - 1)/N)\ldots(p_1 + 1/N)]\ldots[(p_n + p_nN_0/N)\ldots(p_n + 1/N)]}.$$

If we choose any $q > p_1^{p_1}\ldots p_n^{p_n}$ then $|\Phi(N + N_0)|/|\Phi(N)| > 1/q^{N_0}$ for large enough N. It follows that there is a constant U with $|\Phi(N)| > U/q^N$ for all N divisible by N_0. But, as the size of the elements of $\Phi(N)$ is at most $a(d_1^{p_1}\ldots d_n^{p_n})^N$, we have $|\Phi(N)| \leqslant a(d_1^{p_1}\ldots d_n^{p_n})^N$.
It is now clear how to choose $p_1, \ldots, p_n$ and q. We take $p_1, \ldots, p_n$ proportional to $1/d_1, \ldots, 1/d_n$, and q with $p_1^{p_1}\ldots p_n^{p_n} < q < 1/(d_1^{p_1}\ldots d_n^{p_n})$. Then our bounds on $|\Phi(N)|$ are contradictory for large N.

Further comment

2. Another approach is to consider the functions $N(x) = |\{a \in A : a \leqslant x\}|$ and $N_i(x) = |\{a \in b_iA + c_i : a \leqslant x\}|$. One gets

$$N(x) \geqslant \sum_{i=1}^{n} N_i(x) \geqslant \sum_{i=1}^{n} N\left(\frac{x}{b_i} - u\right)$$

for a suitable number u. By ingenious manipulation of this inequality, a contradiction can be obtained. However these manipulations seem less natural than the approach through iterating applications of the f_i.

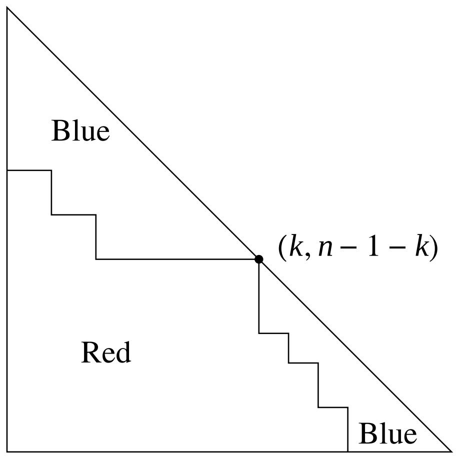
C1. Let n be a positive integer. Each point (x, y) in the plane, where x and y are non-negative integers with $x + y < n$, is coloured red or blue, subject to the following condition: if a point (x, y) is red, then so are all points (x', y') with $x' \leqslant x$ and $y' \leqslant y$. Let A be the number of ways to choose n blue points with distinct x-coordinates, and let B be the number of ways to choose n blue points with distinct y-coordinates. Prove that $A = B$.

Comment

1. This is an easy question, with an interesting variety of approaches. We give three different solutions: one is by induction on n, one is by induction on the number of red points, and one is a direct bijection.

Solution 1. Let the number of blue points with x-coordinate i be a_i, and let the number of blue points with y-coordinate i be b_i. Our task is to show that $a_0 a_1 \dots a_{n-1} = b_0 b_1 \dots b_{n-1}$, and to accomplish this we will show that $a_0, a_1, \dots, a_{n-1}$ is a permutation of $b_0, b_1, \dots, b_{n-1}$.

We prove this result by induction on n. The case $n = 1$ is trivial, so we pass to the induction step: we may assume the result for all smaller values of n.



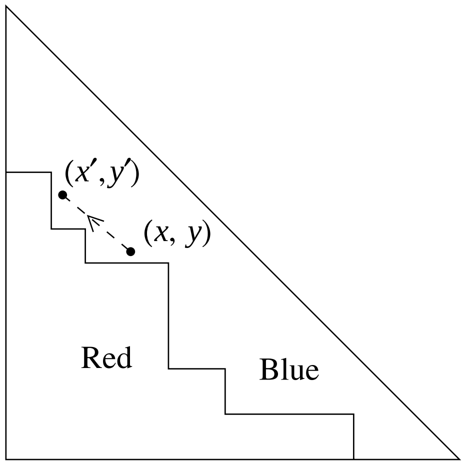
Consider first the case when every point (x, y) with $x + y = n + 1$ is blue. Ignoring these points, we have a configuration for $n - 1$, with blue columns of sizes $a_0 - 1, a_1 - 1, \dots, a_{n-2} - 1$ and blue rows of sizes $b_0 - 1, b_1 - 1, \dots, b_{n-2} - 1$. It follows by the induction hypothesis that $a_0 - 1, a_1 - 1, \dots, a_{n-2} - 1$ is a permutation of $b_0 - 1, b_1 - 1, \dots, b_{n-2} - 1$, and since $a_{n-1} = b_{n-1} = 1$ we are done.

Now suppose instead that some point $(k, n - 1 - k)$ is red. Then the entire rectangle of all points (x, y) with $x \leqslant k$ and $y \leqslant n - 1 - k$ is red. Thus, considering just the points (x, y) with $x < k$, the induction hypothesis tells us that $a_0, a_1, \dots, a_{k-1}$ is a permutation of $b_{n-k}, b_{n-k+1}, \dots, b_{n-1}$, and similarly we have that $a_{k+1}, a_{k+2}, \dots, a_{n-1}$ is a permutation of $b_0, b_1, \dots, b_{n-2-k}$. Since $a_k = b_{n-1-k} = 0$, we are done.

Solution 2. As above, we wish to show that $a_0, a_1, \dots, a_{n-1}$ is a permutation of $b_0, b_1, \dots, b_{n-1}$. We prove this by induction on the number of red points: the result is trivial when there are no red points. Choose a red point (x, y) with $x + y$ maximal. Then $a_x = b_y = n - 1 - x - y$. If we change this red point to blue, then we have a configuration with fewer red points, with all blue rows and columns unchanged except that the values of a_x and b_y decrease by 1. So from the induction hypothesis we have that $a_0, a_1, \dots, a_{n-1}$, with a_x replaced by $a_x - 1$, is a permutation of $b_0, b_1, \dots, b_{n-1}$, with b_y replaced by $b_y - 1$. Since $a_x = b_y$, it follows that $a_0, a_1, \dots, a_{n-1}$ is a permutation of $b_0, b_1, \dots, b_{n-1}$, as required.

Solution 3. We give an explicit bijection between $a_0, a_1, \ldots, a_{n-1}$ and $b_0, b_1, \ldots, b_{n-1}$. If $a_x = 0$ then also $b_x = 0$, and we let a_x correspond with b_x. If $a_x > 0$, let (x, y) be the bottom blue point in column x. Now, among the points (x, y), $(x - 1, y + 1)$, $(x - 2, y + 2)$, …, there must be at least one that is the leftmost blue point of a row: let the first one be (x', y'). Then we let a_x correspond with $b_{y'}$.

This is clearly reversible: if $b_y > 0$ then we let (x, y) be the leftmost blue point in row y, choose the first point among (x, y), $(x + 1, y - 1)$, $(x + 2, y - 2)$, … that is the bottom blue point of a row, say (x', y'), and let b_y correspond to $a_{x'}$.

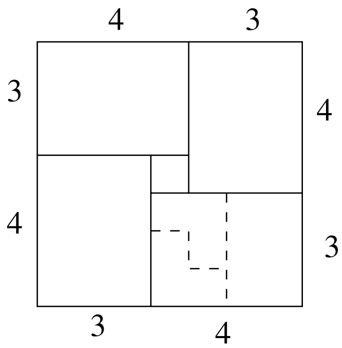


C2. For n an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A *tromino* is an L-shape formed by three connected unit squares. For which values of n is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?

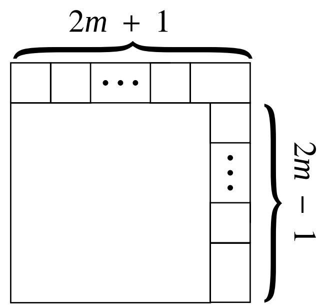
Solution. Write $n = 2m + 1$. The key observation for the second part of the problem, which also helps in eliminating cleanly the case $n = 5$ for the first part, is the following. Consider the black squares at an even height above the bottom row: there are $(m + 1)^2$ of them, and no two are covered by any one tromino. So we always need at least $(m + 1)^2$ trominos to cover.

This disposes of the cases $n = 3$ and $n = 5$ (and $n = 1$), as in each of these cases we have that $3(m + 1)^2$ is greater than n^2, so that the black squares cannot be covered by trominos.

It remains only to show that when $n \geqslant 7$ we can cover the black squares with exactly $(m + 1)^2$ trominos. For $n = 7$, the numbers make this just about possible, as $3(m + 1)^2$ is 48. There are several ways to achieve this. One simple way is to note that we can make a 2×3 rectangle from trominos, and two of these together make a 4×3 rectangle. If we lay four of these around the 7×7 board, we have covered every square except the central one. But now take a tromino that is adjacent to the central square: by the way the 4×3 rectangles have been built up, it may be moved to uncover a white square and cover the central black square instead.



The case $n = 7$



From $(2m - 1) \times (2m - 1)$
to $(2m + 1) \times (2m + 1)$

In general, having covered the black squares on a $(2m - 1) \times (2m - 1)$ board with m^2 trominos, let us form a $(2m + 1) \times (2m + 1)$ board by surrounding it with a $(2m - 1) \times 2$ rectangle and a $(2m + 1) \times 2$ rectangle. Now, we may break up the $(2m - 1) \times 2$ rectangle into 2×2 squares ($m - 2$ of them) and one 2×3 rectangle, so that its black squares may be covered by $(m - 2) + 2$ trominos, and similarly the black squares of the $(2m + 1) \times 2$ rectangle may be covered by $(m - 1) + 2$ trominos. This gives a total of $m^2 + m + (m + 1)$ trominos, as required.

Comment

1. Covering problems are not uncommon, but this problem seems to be rather unusual. What is nice is that when one first thinks about the problem it seems very messy, but when one has made the key observation above it all becomes very clean and elegant!

C3. Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called *full* if it satisfies the following condition: for each positive integer $k \geqslant 2$, if the number k appears in the sequence then so does the number $k - 1$, and moreover the first occurrence of $k - 1$ comes before the last occurrence of k. For each n, how many full sequences are there?

Solution. We claim there are $n!$ full sequences. To do this, we will construct a bijection with the set of permutations of $\{1, 2, \dots, n\}$.

Let $a_1, \dots, a_n$ be a full sequence, and let $r = \max(a_1, \dots, a_n)$. Then all the numbers from 1 to r occur in $a_1, \dots, a_n$. Let $S_i = \{k : a_k = i\}$ for $1 \leqslant i \leqslant r$. Then all the S_i are non-empty, and they partition the set $\{1, 2, \dots, n\}$. The condition that the sequence is full means that $\min S_{k-1} < \max S_k$ for $2 \leqslant k \leqslant r$. Now we write down a permutation $b_1, \dots, b_n$ of $\{1, 2, \dots, n\}$ by writing down the elements of S_1 in descending order, then the elements of S_2 in descending order and so on. This gives a map from full sequences to permutations of $\{1, 2, \dots, n\}$.

Note also that this map is reversible. Indeed, given a permutation $b_1, \dots, b_n$ of $\{1, 2, \dots, n\}$ let $S_1 = \{b_1, \dots, b_{k_1}\}$ where $b_1 > \dots > b_{k_1} < b_{k_1+1}$, let $S_2 = \{b_{k_1+1}, \dots, b_{k_2}\}$ where $b_{k_1+1} > \dots > b_{k_2} < b_{k_2+1}$ and so on. Then let $a_j = i$ whenever $i \in S_j$.

It follows that the full sequences are in bijection with the set of permutations of $\{1, 2, \dots, n\}$, as required.

Comment

1. It is easy to guess, from some small examples, that the answer is $n!$, but finding a bijection is not easy. An alternative proof goes by induction on n: given a full sequence of length n, we form a sequence of length $n - 1$ by removing from it the first occurrence of its highest number. It is easy to check that this sequence of length $n - 1$ is full. One can then check that each full sequence of length $n - 1$ arises in this way from exactly n full sequences of length n.

C4. Let T be the set of ordered triples (x, y, z), where x, y, z are integers with $0 \leqslant x, y, z \leqslant 9$. Players A and B play the following guessing game. Player A chooses a triple (x, y, z) in T, and Player B has to discover A's triple in as few moves as possible. A *move* consists of the following: B gives A a triple (a, b, c) in T, and A replies by giving B the number $|x + y - a - b| + |y + z - b - c| + |z + x - c - a|$. Find the minimum number of moves that B needs to be sure of determining A's triple.

Solution. It is easy to see that two moves cannot be enough. Indeed, each answer is an even integer between 0 and 54 inclusive, so that there are 28 possibilities for each answer. Thus with two moves the number of possible outcomes is at most 28^2, which is less than the required number of outcomes, namely 1000.

We now set out to show that three moves are enough, by providing an explicit strategy. The first move should be $(0, 0, 0)$. The reply is $2(x + y + z)$, so that we now know the value of $s = x + y + z$. Clearly $0 \leqslant s \leqslant 27$, but to reduce the number of cases, in the algorithm below we may assume that $s \leqslant 13$. Indeed, if $s \geqslant 14$ then we perform the algorithm below, but always 'reflecting', i.e. asking $(9 - a, 9 - b, 9 - c)$ instead of (a, b, c) – we will recover the reflection of (x, y, z) at the end.

Case 1: $s \leqslant 9$. This is the easy case. The second move should be $(9, 0, 0)$. We learn $y + z + (9 - x - y) + (9 - x - z) = 18 - 2x$, so we now know the value of x. And similarly asking $(0, 9, 0)$ tells us the value of y, so we are done (as $z = s - x - y$).

Case 2: $9 < s \leqslant 13$. The second move should be $(9, s - 9, 0)$. We learn $z + |9 - x - z| + |9 - x|$, which is say $2k$, where $k = z$ if $x + z \geqslant 9$ and $k = 9 - x$ if $x + z < 9$. Note that whichever value k takes we do have $z \leqslant k \leqslant s$.

Case 2a: $s - k \leqslant 9$. The third move should be $(s - k, 0, k)$. We learn $y + |k - y - z| + |z - k|$. Since $k \leqslant y + z$ (if $k = z$ then this is obvious, while if $k = 9 - x$ then $k - y - z = 9 - s$), this is just $y + (y + z - k) + (k - z) = 2y$. Thus we know y, and hence $x + z$. So we know whether k is z or $9 - x$, and we are done.

Case 2b: $s - k > 9$. The third move should be $(9, s - k - 9, k)$. We learn $|s - k - x - y| + |s - 9 - y - z| + |9 + k - x - z|$, which is $(k - z) + (9 - x) + (9 + k - x - z) = 18 + 2k - 2(x + z)$. So we know $x + z$, and so we know whether k is z or $9 - x$. In either case, we know one of x and z, and from $x + z$ we may deduce the other one, and we are done.

Comments

1. Case 1 is the natural, simple case to deal with first. And then, in Case 2, the key idea is to use s itself in the triple we ask. Similarly, in Case 2a the key idea is to somehow include k in the triple we ask, while Case 2b is just a more complicated version of Case 2a. One could view the second move in Case 2 as the obvious modification of the second move in Case 1 when s is 'out of range', and similarly the third move in Case 2b is the obvious modification of the third move in Case 2a.

2. As two moves only just fail, it is very natural to guess that three moves is the right answer. But finding an actual strategy seems complicated. This question requires clarity of thought, but no specialist knowledge at all.

3. There is nothing special about the fact that there are 10 possible values for each digit. Changing 10 to a larger number, such as 2002, would not change the solution.

C5. Let $r \geqslant 2$ be a fixed positive integer, and let F be an infinite family of sets, each of size r, no two of which are disjoint. Prove that there exists a set of size $r - 1$ that meets each set in F.

Solution. We will show the following: if A is a set of size less than r that is contained in infinitely many sets of F, then either A meets all sets in F (in which case we are done) or else there is an $x \notin A$ such that $A \cup \{x\}$ is itself contained in infinitely many sets of F. Since there certainly exists such a set A (for example, the empty set), we may then iterate this result r times and we will be done (as a set of size r clearly cannot be contained in infinitely many sets of F!).
To prove the result, suppose that some set $R = \{x_1, x_2, \ldots, x_r\}$ in F is disjoint from A. Of the infinitely many sets in F that contain A, each must meet R, and so some x_i is a member of infinitely many of them. So we may take $x = x_i$.

Comments

1. Although the above proof is very short, it does seem to require a creative insight, namely the clever auxiliary result we prove.

2. An alternative proof is to note that, if the result is false, then for each set R in F, and each point $x \in R$, the set $R - \{x\}$ is not suitable, so that there must be a set S in F with $R \cap S = \{x\}$. In other words, each point x belonging to some set in F is the intersection of two sets in F. This implies that there is no finite set Y with the property that any two sets in F must meet at some point belonging to Y. Although this is highly implausible, it does seem tricky to prove impossible. The simplest way is probably to prove a stronger result: that if F and G are families of r-sets, with each member of F meeting each member of G, then there exists a finite set Y such that each member of F meets each member of G at some point of Y. This goes easily by induction on r.

   As in the proof above, this proof requires a key creative insight, namely the generalisation from one family to two. This does suggest that the problem may be quite hard.

C6. Let n be an even positive integer. Show that there is a permutation $x_1, x_2, \ldots, x_n$ of $1, 2, \ldots, n$ such that for every $1 \leqslant i \leqslant n$ the number x_{i+1} is one of $2x_i$, $2x_i - 1$, $2x_i - n$, $2x_i - n - 1$ (where we take $x_{n+1} = x_1$).

Solution. Write $n = 2m$. We shall define a directed graph G with vertices labelled $1, \ldots, m$ and edges labelled $1, \ldots, 2m$. The edges issuing from vertex i are labelled $2i - 1$ and $2i$, and those entering it are labelled i and $i + m$.

The underlying graph of G is connected: by induction on j there is a path from 1 to j since if $j > 1$ then $j = 2k - 1$ or $2k$ with $1 \leqslant k < j$ and there is an edge from k to j. Also, the indegree and outdegree of each vertex is the same (namely 2). The directed graph G thus has an Euler circuit. Let x_i be the label of the i-th edge in such a circuit. If edge x_i enters and edge x_{i+1} leaves vertex j then $x_i \equiv j \pmod m$ and $x_{i+1} = 2j - 1$ or $2j$. Hence $2x_i \equiv 2j \pmod{2m = n}$ and so $x_{i+1} \equiv 2x_i - 1$ or $2x_i \pmod n$ as is required.

Comments

1. The problem requires one to prove the existence of a Hamilton cycle in a certain graph. There is no obvious way to do this. The above solution avoids this difficulty by constructing a directed graph on a smaller vertex set whose edges are labelled $1, 2, \ldots, n$ in such a way that an Euler circuit (a closed walk that traverses each edge exactly once) corresponds to the desired permutation.

2. The above proof uses the simple fact that if all indegrees and outdegrees are equal (and the underlying graph is connected) then there is an Euler circuit. This is very easy to prove, for example by considering a closed walk of maximal length. Indeed, the proposer's solution does essentially the same thing with bare hands.

3. Although the above solution is short, it is definitely hard to think of. There is a slight similarity with the construction of de Bruijn sequences, but this is only apparent with hindsight once one has written down the proof.

C7. Among a group of 120 people, some pairs are friends. A *weak quartet* is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets?

Solution. We proceed in three steps. First, we will show that, for a maximum number of weak quartets, our graph (thinking of the friends as defining a graph on 120 vertices) breaks up as a disjoint union of complete graphs. Then we show that these complete graphs have sizes that are as equal as possible (ie. differ by at most 1 from each other). And lastly we will find which of these is the best.

For the first step, we would like to show that any two adjacent vertices have the same neighbours (apart from themselves, of course). For a graph G on our 120 vertices, write $Q(G)$ for the number of weak quartets in G. For adjacent vertices x, y of G, let G' be the graph formed from G by 'copying' y to x: in other words, for each $z \neq x, y$, we add the edge xz if yz is an edge and we remove the edge xz if yz is not an edge. Similarly, let G'' be the graph formed from G by copying x to y.

We claim that $Q(G) \leqslant \frac{1}{2}\big(Q(G') + Q(G'')\big)$. Indeed, let us compare the weak quartets in G with those in G' and G''. The number of weak quartets containing neither x nor y is the same in G, G' and G'', while the number containing both x and y is at least as great in G' and G'' as it is in G. The number containing y but not x in G' is at least twice what it is in G, while the number containing x but not y in G'' is at least twice what it is in G. This establishes our claim.

It follows that, for an extremal G, we must have $Q(G) = Q(G') = Q(G'')$. So we may repeat this copying operation pair by pair, to obtain a graph in which any two adjacent vertices have the same common neighbours. Indeed, if x and y are two adjacent vertices then we copy y to x; if there another vertex z adjacent to x (and so also to y) then we copy z to x and then to y, and so on. This completes the first step.

Our aim now is to show that the sizes of the complete graphs in G may be taken to be as equal as possible. There are various ways to do this: one way is as follows. Let the complete graphs in G have sizes $a_1, a_2, \ldots, a_n$, where, just for convenience in what is to follow, we allow $a_i = 0$. Then we have

$$Q(G) = \sum_{i=1}^{n} \binom{a_i}{2} \sum_{j<k,\ j,k \neq i} a_j a_k.$$

(Here as usual we just take $\binom{a}{2}$ to mean $a(a-1)/2$, to cover the case when $a < 2$.) Now, let us consider two of the a_i, say a and b, and let us see how $Q(G)$ varies as we change the values of a and b (keeping the other values, and the sum $a+b$, fixed). We have

$$Q(G) = A\left(\binom{a}{2} + \binom{b}{2}\right) + B(a+b) + C\left(\binom{a}{2}b + \binom{b}{2}a\right),$$

where A, B, C do not depend on a or b. If we swap a and b, we get the same expression, which tells us that the expression is a quadratic, symmetric about $s/2$, where s is the fixed sum $a+b$. (The expression may appear to be cubic, but it is easy to see that there is no cubic term, either by direct calculation or because no cubic can be symmetric!)

This tells us that the maximum when $0 \leqslant a \leqslant s$ occurs either at $a = s/2$ or at $a = 0$ (and $a = s$), and that the maximum value for integer a occurs when $a = 0$ or when $a = b$ or when $a = b \pm 1$. Repeating for each pair of the a_i, we have completed the second step.

Our final step is some calculation. Writing n for the number of (non-empty) complete graphs, we see that

$$Q(G) = n\binom{120/n}{2}\binom{n-1}{2}(120/n)^2$$

whenever n divides 120. It is easy to check that, for $n \leqslant 6$, the maximum occurs at $n = 5$, with value $15.23.24^3$. Moreover, because of the fact that the maximum over all real a_i in the previous paragraph occurred when all the non-zero a_i were equal, we also know that the maximum possible value of $Q(G)$ is at most the maximum value of the expression

$$n\binom{120/n}{2}\binom{n-1}{2}(120/n)^2$$

as n varies from 3 to 120. But this function is at most $120^4(n-1)(n-2)/4n^3$, which is a decreasing function of n for $n \geqslant 6$ and is at most $15.23.24^3$ for $n = 7$. This completes the third step: the maximum value is $15.23.24^3$.

Comments

1. The rough strategy outlined at the start of the proof is not too hard to think of. However, the actual extremal configuration (with 5 complete graphs of size 24) is far from obvious. In addition, fitting in the detail to implement the outline strategy presents a number of challenges. The hardest of these is the first step, to show that we have a disjoint union of complete graphs.
2. The number 120 has been chosen to make the numerical calculations as simple as possible.