

# 7<sup>th</sup> Iranian Geometry Olympiad



Contest problems with solutions

# 7<sup>th</sup> Iranian Geometry Olympiad Contest problems with solutions.

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With special thanks to Matin Yousefi and Alireza Danaei.

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# Elementary Level

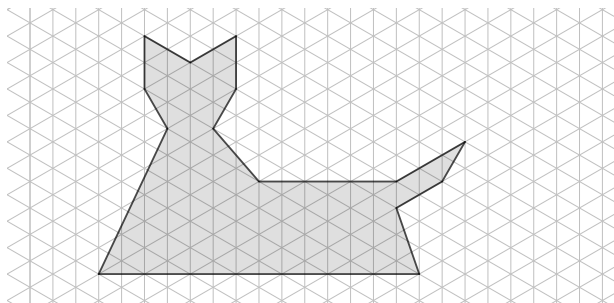


# Problems

**Problem 1.** By a *fold* of a polygon-shaped paper, we mean drawing a segment on the paper and folding the paper along that. Suppose that a paper with the following figure is given. We cut the paper along the boundary of the shaded region to get a polygon-shaped paper.

Start with this shaded polygon and make a rectangle-shaped paper from it with at most 5 number of folds. Describe your solution by introducing the folding lines and drawing the shape after each fold on your solution sheet.

(Note that the folding lines do not have to coincide with the grid lines of the shape.)



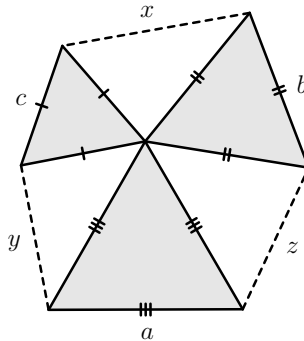
(! p.5)

**Problem 2.** A parallelogram  $ABCD$  is given ( $AB \neq BC$ ). Points  $E$  and  $G$  are chosen on the line  $CD$  such that  $AC$  is the angle bisector of both angles  $\angle EAD$  and  $\angle BAG$ . The line  $BC$  intersects  $AE$  and  $AG$  at  $F$  and  $H$ , respectively. Prove that the line  $FG$  passes through the midpoint of  $HE$ .

(! p.8)

**Problem 3.** According to the figure, three equilateral triangles with side

lengths  $a, b, c$  have one common vertex and do not have any other common point. The lengths  $x, y$  and  $z$  are defined as in the figure. Prove that  $3(x + y + z) > 2(a + b + c)$ .



(! p.9)

**Problem 4.** Let  $P$  be an arbitrary point in the interior of triangle  $ABC$ . Lines  $BP$  and  $CP$  intersect  $AC$  and  $AB$  at  $E$  and  $F$ , respectively. Let  $K$  and  $L$  be the midpoints of the segments  $BF$  and  $CE$ , respectively. Let the lines through  $L$  and  $K$  parallel to  $CF$  and  $BE$  intersect  $BC$  at  $S$  and  $T$ , respectively; moreover, denote by  $M$  and  $N$  the reflection of  $S$  and  $T$  over the points  $L$  and  $K$ , respectively. Prove that as  $P$  moves in the interior of triangle  $ABC$ , line  $MN$  passes through a fixed point.

(! p.10)

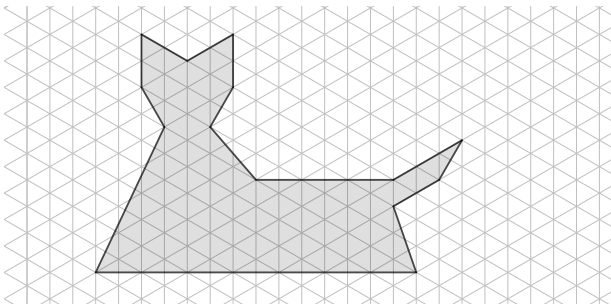
**Problem 5.** We say two vertices of a simple polygon are *visible* from each other if either they are adjacent, or the segment joining them is completely inside the polygon (except two endpoints that lie on the boundary). Find all positive integers  $n$  such that there exists a simple polygon with  $n$  vertices in which every vertex is visible from exactly 4 other vertices.

(A simple polygon is a polygon without hole that does not intersect itself.)

(! p.11)

# Solutions

**Problem 1.** By a *fold* of a polygon-shaped paper, we mean drawing a segment on the paper and folding the paper along that. Suppose that a paper with the following figure is given. We cut the paper along the boundary of the shaded region to get a polygon-shaped paper. Start with this shaded polygon and make a rectangle-shaped paper from it with at most 5 number of folds. Describe your solution by introducing the folding lines and drawing the shape after each fold on your solution sheet. (Note that the folding lines do not have to coincide with the grid lines of the shape.)



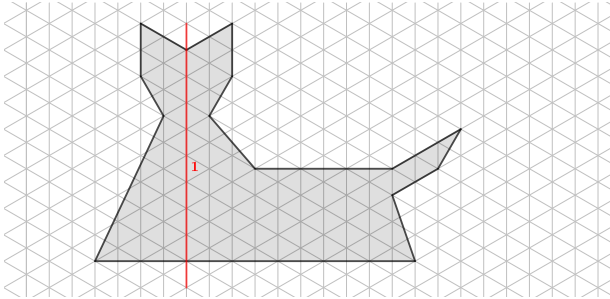
*Proposed by Mahdi Etesamifard*

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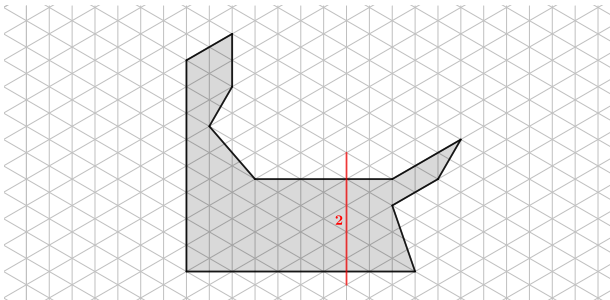
**Solution.** There are different ways of folding to get a rectangle. For instance, a solution can be given with only 4 number of folds, as following

First fold:

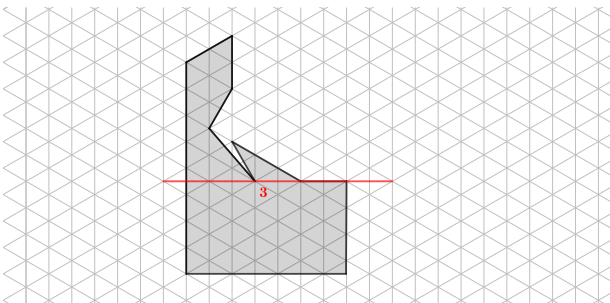




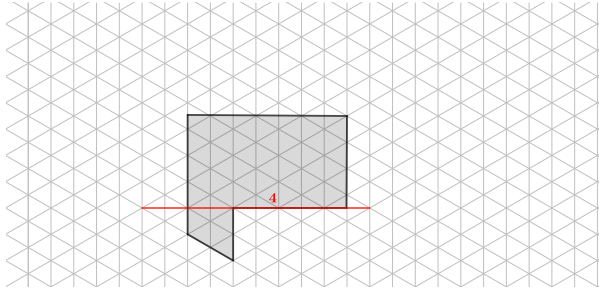
Second fold:



Third fold:



Fourth fold:

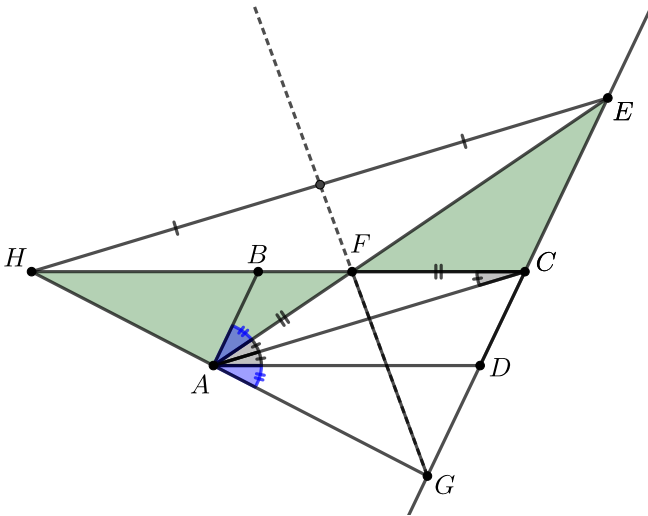


**Comment.** that by moving the folding lines slightly in 3rd and 4th folds (to down and up respectively), all the folding segments will be internal.

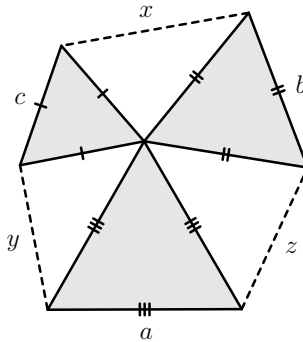
**Problem 2.** A parallelogram  $ABCD$  is given ( $AB \neq BC$ ). Points  $E$  and  $G$  are chosen on the line  $CD$  such that  $AC$  is the angle bisector of both angles  $\angle EAD$  and  $\angle BAG$ . The line  $BC$  intersects  $AE$  and  $AG$  at  $F$  and  $H$ , respectively. Prove that the line  $FG$  passes through the midpoint of  $HE$ .

*Proposed by Mahdi Etesamifard*

**Solution.** Since  $AD$  and  $BC$  are parallel, we deduce that  $\angle FCA = \angle DAC = \angle FAC$ . So  $FA = FC$ . Similarly,  $GA = GC$ . So triangles  $\triangle GAF$  and  $\triangle GCF$  have a common side and two equal sides and are congruent. Resulting  $\angle GAF = \angle GCF$  which leads to  $\angle HAF = \angle ECF$  and  $\angle AFH = \angle CFE$ . Therefore triangles  $\triangle AFH$  and  $\triangle CFE$  are congruent as well and we get  $FE = FH$ . Similarly,  $GE = GH$ . So both points  $F$  and  $G$  lie on perpendicular bisector of segment  $HE$ . Resulting that  $FG$  is the perpendicular bisector of segment  $HE$ .



**Problem 3.** According to the figure, three equilateral triangles with side lengths  $a, b, c$  have one common vertex and do not have any other common point. The lengths  $x, y$  and  $z$  are defined as in the figure. Prove that  $3(x + y + z) > 2(a + b + c)$ .



*Proposed by Mahdi Etesamifard*

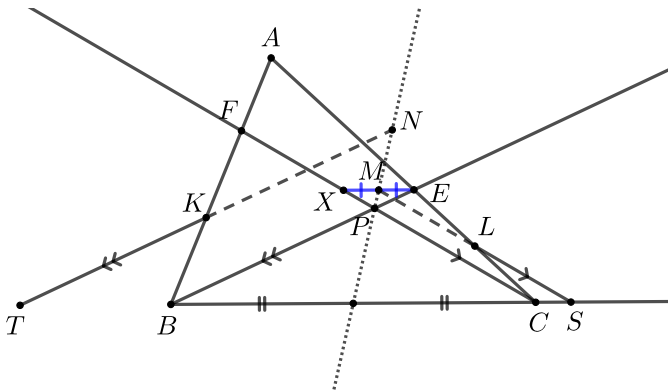
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**Solution.** Consider the three white triangles in the picture, rotating each triangle  $60^\circ$  degrees, clock-wise, will make a side of it coincide with another side of another triangle. So we can rotate one of them and glue it to the next, then rotating the glued figure a broken path will be formed between two points with distances like  $2a$  which has length  $x + y + z$ . Thus  $x + y + z > 2a$  summing up all three possible inequalities proves the desired.

**Problem 4.** Let  $P$  be an arbitrary point in the interior of triangle  $ABC$ . Lines  $BP$  and  $CP$  intersect  $AC$  and  $AB$  at  $E$  and  $F$ , respectively. Let  $K$  and  $L$  be the midpoints of the segments  $BF$  and  $CE$ , respectively. Let the lines through  $L$  and  $K$  parallel to  $CF$  and  $BE$  intersect  $BC$  at  $S$  and  $T$ , respectively; moreover, denote by  $M$  and  $N$  the reflection of  $S$  and  $T$  over the points  $L$  and  $K$ , respectively. Prove that as  $P$  moves in the interior of triangle  $ABC$ , line  $MN$  passes through a fixed point.

*Proposed by Ali Zamani*

**Solution.** Since in quadrilateral  $EMCS$ , diagonals bisect each other, this quadrilateral is a parallelogram. So,  $EM \parallel BC$ . Let  $X$  be the intersection of  $EM$  and  $CF$ . Note that  $ML \parallel CX$  and  $L$  is the midpoint of  $CE$ , resulting that  $M$  is the midpoint of  $EX$  as well. Since  $EX \parallel BC$ , using parallel lines, one can find that  $MP$  passes through the midpoint of  $BC$ . Similarly,  $NP$  passes through the midpoint of  $BC$ . Hence proved.



**Problem 5.** We say two vertices of a simple polygon are *visible* from each other if either they are adjacent, or the segment joining them is completely inside the polygon (except two endpoints that lie on the boundary). Find all positive integers  $n$  such that there exists a simple polygon with  $n$  vertices in which every vertex is visible from exactly 4 other vertices.

(A simple polygon is a polygon without hole that does not intersect itself.)

*Proposed by Morteza Sagha an*

**Solution.**

First we prove there is no such polygon for  $n > 6$ . Let  $A_1, A_2, \dots, A_n$  be the vertices.

**Lemma 1.** *Let  $A_i$  be visible from  $A_{i-1}, A_j, A_k, A_{i+1}$  in clockwise order (note that the first and the last one are the edge-neighbors). Then  $A_{i-1}, A_j$  can see each other,  $A_j, A_k$  can see each other and  $A_k, A_{i+1}$  can see each other.*

*Proof.* One can consider the triangulation of the three parts of polygon separated by  $A_i A_j$  and  $A_i A_k$ . □

**Lemma 2.** *Using the same naming as Lemma 1,  $A_j A_k$  is a side.*

*Proof.* Assume that  $A_j A_k$  is an internal diagonal. By Lemma 1,  $A_j$  can see  $A_{j-1}$ . But  $A_j A_i$  and  $A_j A_k$  are internal diagonals. So  $A_j A_{i-1}$  is a side. So there is only one vertex between  $A_i, A_j$  on the perimeter of polygon. Similarly, there is only one vertex between  $A_j, A_k$  and only one vertex between  $A_k, A_i$  on the perimeter of polygon. This contradicts  $n > 6$ . So  $A_j A_k$  is a side and  $k = j - 1$ . □

Now let  $i$  be such that  $A_{i-1}, A_{i+1}$  are visible from each other. We know that such  $i$  exists, for instance you can take an ear triangle in the triangulation of the polygon. By Lemma 2,  $A_{i-1}$  can see  $A_{i+2}$ ,  $A_{i+1}$  can see  $A_{i-2}$  and  $A_{i-2}$  can see  $A_{i+2}$ . So we found the four vertices visible from  $A_{i-1}, A_{i+1}$ . If  $A_i$  can see a vertex, then it is visible by either  $A_{i-1}$  or  $A_{i+1}$  (by Lemma 1). So  $A_i$  should see  $A_{i-2}, A_{i+2}$  and this means  $A_{i-2} A_{i+2}$  is a side (by Lemma 2). Any convex pentagon is an example.

The only remaining case is  $n = 6$  which means in Lemma 2 there are vertices  $A_i, A_j, A_k$  such that  $A_i A_j, A_j A_k, A_k A_i$  are internal diagonals. Let them be  $A_2, A_4, A_6$  in the hexagon. So  $A_3$  is not visible from  $A_6$ , meaning that one of the angles  $A_2, A_4$  is larger than  $180^\circ$ . But then  $A_3$  cannot see either  $A_1$  or  $A_5$ . Which contradicts the fact that  $A_3$  is visible from 4 other vertices. So  $n = 6$  is also not possible and the only possible  $n$  is 5.



# Intermediate Level





# Problems

**Problem 1.** A trapezoid  $ABCD$  is given where  $AB$  and  $CD$  are parallel. Let  $M$  be the midpoint of the segment  $AB$ . Point  $N$  is located on the segment  $CD$  such that  $\angle ADN = \frac{1}{2} \angle MNC$  and  $\angle BCN = \frac{1}{2} \angle MND$ . Prove that  $N$  is the midpoint of the segment  $CD$ .

(! p.17)

**Problem 2.** Let  $ABC$  be an isosceles triangle ( $AB = AC$ ) with its circumcenter  $O$ . Point  $N$  is the midpoint of the segment  $BC$  and point  $M$  is the reflection of the point  $N$  with respect to the side  $AC$ . Suppose that  $T$  is a point so that  $ANBT$  is a rectangle. Prove that  $\angle OMT = \frac{1}{2} \angle BAC$ .

(! p.18)

**Problem 3.** In acute-angled triangle  $ABC$  ( $AC > AB$ ), point  $H$  is the orthocenter and point  $M$  is the midpoint of the segment  $BC$ . The median  $AM$  intersects the circumcircle of triangle  $ABC$  at  $X$ . The line  $CH$  intersects the perpendicular bisector of  $BC$  at  $E$  and the circumcircle of the triangle  $ABC$  again at  $F$ . Point  $J$  lies on circle  $\omega$ , passing through  $X$ ,  $E$ , and  $F$ , such that  $BCHJ$  is a trapezoid ( $CB \parallel HJ$ ). Prove that  $JB$  and  $EM$  meet on  $\omega$ .

(! p.19)

**Problem 4.** Triangle  $ABC$  is given. An arbitrary circle with center  $J$ , passing through  $B$  and  $C$ , intersects the sides  $AC$  and  $AB$  at  $E$  and  $F$ , respectively. Let  $X$  be a point such that triangle  $FXB$  is similar to triangle  $EJC$  (with the same order) and the points  $X$  and  $C$  lie on the same side of the line  $AB$ . Similarly, let  $Y$  be a point such that triangle  $EYC$  is similar to triangle  $FJB$  (with the same order) and the points  $Y$  and  $B$  lie on the same side of the line  $AC$ . Prove that the line  $XY$  passes through the orthocenter of the triangle  $ABC$ .

(/ p.21)

**Problem 5.** Find all numbers  $n \geq 4$  such that there exists a convex polyhedron with exactly  $n$  faces, whose all faces are right-angled triangles. (Note that the angle between any pair of adjacent faces in a convex polyhedron is less than  $180^\circ$ .)

(/ p.23)

# Solutions

**Problem 1.** A trapezoid  $ABCD$  is given where  $AB$  and  $CD$  are parallel. Let  $M$  be the midpoint of the segment  $AB$ . Point  $N$  is located on the segment  $CD$  such that  $\angle ADN = \frac{1}{2}\angle MNC$  and  $\angle BCN = \frac{1}{2}\angle MND$ . Prove that  $N$  is the midpoint of the segment  $CD$ .

*Proposed by Alireza Dadgarnia*

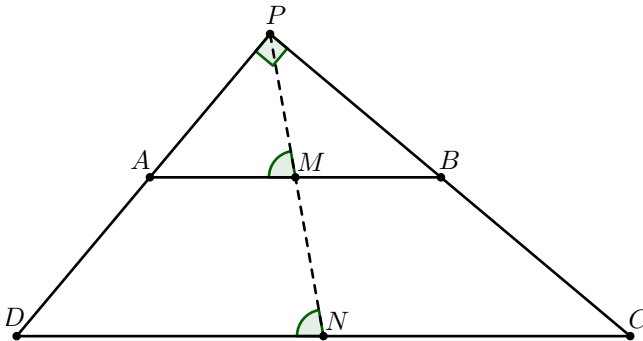
**Solution.** We have

$$\angle BCN + \angle ADN = \frac{1}{2}(\angle MND + \angle BDN) = 90^\circ .$$

Hence,  $AD$  and  $BC$  intersect in point  $P$  such that  $\angle DPC = 90^\circ$  . Since  $M$  is the midpoint of  $AB$ ,

$$\angle PMA = 2\angle PBA = 2\angle PCD = \angle MND.$$

Note that  $AB$  and  $CD$  are parallel, therefore,  $PM$  and  $MN$  are parallel to and  $M, N$  and  $P$  lie on a straight line, hence  $N$  is the midpoint of segment  $CD$ .



**Problem 2.** Let  $ABC$  be an isosceles triangle ( $AB = AC$ ) with its circum-center  $O$ . Point  $N$  is the midpoint of the segment  $BC$  and point  $M$  is the reflection of the point  $N$  with respect to the side  $AC$ . Suppose that  $T$  is a point so that  $ANBT$  is a rectangle. Prove that  $\angle OMT = \frac{1}{2}\angle BAC$ .

*Proposed by Ali Zamani*

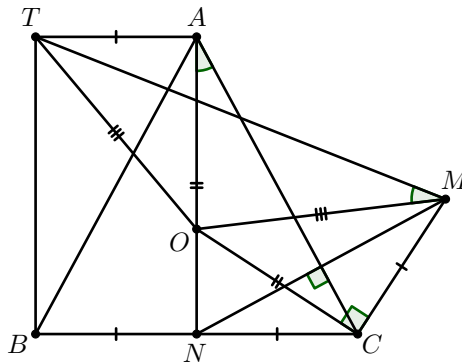
Since  $\triangle ABC$  is an isosceles triangle, we have  $\angle ANC = 90^\circ$ . Therefore,

$$\angle OCM = \angle OCA + \angle MCA = \angle OAC + \angle NCA = 90^\circ = \angle TAO.$$

Also we have  $CM = CN = BN = AT$  and  $OC = OA$ ; So triangles  $\triangle OCM$  and  $\triangle OAT$  are congruent. Which leads to  $OT = OM$  and

$$\angle AOT = \angle MOC \Rightarrow \angle TOM = \angle AOC.$$

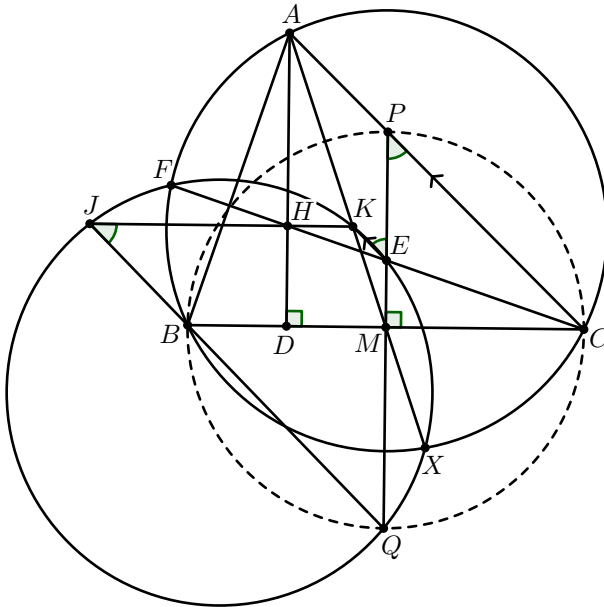
Thus,  $\triangle AOC \cong \triangle MOT$  and  $\angle OMT = \angle OAC = \frac{1}{2}\angle A$ .



**Problem 3.** In acute-angled triangle  $ABC$  ( $AC > AB$ ), point  $H$  is the orthocenter and point  $M$  is the midpoint of the segment  $BC$ . The median  $AM$  intersects the circumcircle of triangle  $ABC$  at  $X$ . The line  $CH$  intersects the perpendicular bisector of  $BC$  at  $E$  and the circumcircle of the triangle  $ABC$  again at  $F$ . Point  $J$  lies on circle  $\omega$ , passing through  $X, E$ , and  $F$ , such that  $BCHJ$  is a trapezoid ( $CB \parallel HJ$ ). Prove that  $JB$  and  $EM$  meet on  $\omega$ .

*Proposed by Alireza Dadgarnia*

**Solution.** Let  $D$  be the foot of altitude passing through  $A$  and  $P, K$  be the intersection of lines  $EM, AC$  and  $JH, AM$ , respectively.



From parallel lines, we have

$$\frac{ME}{EP} = \frac{DH}{HA} = \frac{MK}{KA} \Rightarrow EK \parallel AC. \tag{1}$$

Note that  $\angle XKE = \angle XAC = \angle XFE$ . So  $K$  lies on  $\omega$ . Let  $Q$  be the second intersection point of line  $EM$  and circle  $\omega$ . We have

$$\angle KJQ = \angle KEP \stackrel{(1)}{=} \angle EPC = \angle QPC.$$

Note that it suffices to prove that  $\angle KJQ = \angle CBQ$  or prove that  $CPBQ$  is a cyclic quadrilateral. Which is equivalent to  $MP \cdot MQ = MB \cdot MC$ . Also,

noting the parallel lines we can write  $MA = \frac{MK \cdot MP}{ME}$ . Using this equation and power of the point  $M$  with respect to the circumcircle of triangle  $\triangle ABC$ , we have

$$MB \cdot MC = MA \cdot MX = \frac{MK \cdot MX}{ME} \cdot MP = MQ \cdot MP.$$

Where the last equation comes from power of the point  $M$  with respect to circle  $\omega$ . Hence proved.

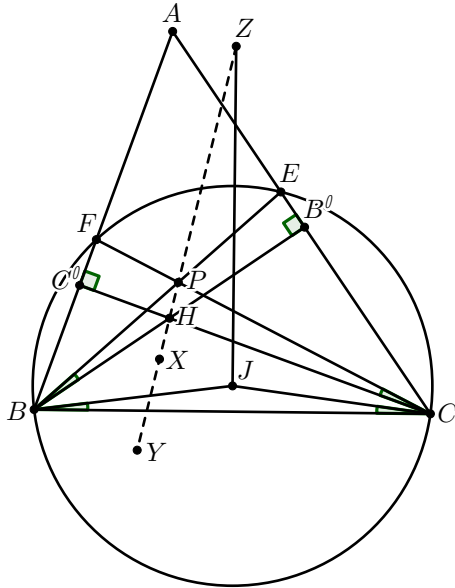
**Comment.** The same proof can be used to obtain the following generalised result:

*In triangle  $ABC$  point  $P$  is an arbitrary point and point  $D$  lies on the line  $BC$ . The line  $AD$  intersects the circumcircle of triangle  $ABC$  at  $X$ . The line  $CP$  intersects the line parallel to  $AP$  through  $D$  at  $E$  and the circumcircle of triangle  $ABC$  again at  $F$ . Suppose that  $P$  lies inside of circle  $\omega$ , passing through  $X$ ,  $E$ , and  $F$ . Point  $J$  lies on  $\omega$  such that  $BCPJ$  is a trapezoid ( $CB \parallel PJ$ ). Then  $JB$  and  $ED$  meet on  $\omega$ .*

**Problem 4.** Triangle  $ABC$  is given. An arbitrary circle with center  $J$ , passing through  $B$  and  $C$ , intersects the sides  $AC$  and  $AB$  at  $E$  and  $F$ , respectively. Let  $X$  be a point such that triangle  $FXB$  is similar to triangle  $EJC$  (with the same order) and the points  $X$  and  $C$  lie on the same side of the line  $AB$ . Similarly, let  $Y$  be a point such that triangle  $EYC$  is similar to triangle  $FJB$  (with the same order) and the points  $Y$  and  $B$  lie on the same side of the line  $AC$ . Prove that the line  $XY$  passes through the orthocenter of the triangle  $ABC$ .

*Proposed by Nguyen Van Linh - Vietnam*

**Solution.** Let  $H$  be the orthocenter of triangle  $ABC$ ,  $P$  be the intersection of  $BE$  and  $CF$ .  $PH$  cuts the perpendicular bisector of  $BC$  at  $Z$ .



We have

$$\angle HBP = \angle ABH \quad \angle ABP = 90^\circ - \angle BAC \quad \angle ABP = 90^\circ - \angle BEC = \angle JBC.$$

Then  $BH$  and  $BJ$  are isogonal lines with respect to angle  $\angle PBC$ . Similarly,  $CH$  and  $CJ$  are isogonal lines with respect to angle  $\angle PCB$ . From this, we deduce that  $H$  and  $J$  are isogonal conjugate with respect to triangle  $ABC$ . Then  $\angle HPB = \angle JPC$ . But  $ZB = ZC$ ,  $JF = JE$  and  $\angle PFE = \angle PBC$ . Therefore  $\angle PFE = \angle JZg = \angle PBC = \angle ZPg$ . Which follows that  $\angle JEF$



$\triangle ZCB$ .

Let  $B^0, C^0$  be the intersections of  $BH$  and  $AC$ ,  $CH$  and  $AB$ , respectively. We have

$$P_{(BE)}^H = HB \quad HB^0 = HC \quad HC^0 = P_{(CF)}^H,$$

$$P_{(BE)}^P = PB \quad PE = PC \quad PF = P_{(CF)}^P.$$

We get  $Z$  lies on  $HP$ , which is the radical axis of circles with diameters  $BE$  and  $CF$ . Analogously,  $X, Y$  also lie on  $HP$ . Therefore  $XY$  passes through the orthocenter of triangle  $\triangle ABC$ .

**Problem 5.** Find all numbers  $n \geq 4$  such that there exists a convex polyhedron with exactly  $n$  faces, whose all faces are right-angled triangles. (Note that the angle between any pair of adjacent faces in a convex polyhedron is less than  $180^\circ$ .)

*Proposed by Hesam Rajabzadeh*

**Solution.** If such a polyhedron exists for some  $n$ , the total number of sides of faces is from one hand equal to  $3n$ , and on the other is twice the number of edges. So  $3n$  is divisible by 2 and  $n$  must be even. We will give an example of such a polyhedron for any even number  $n \geq 4$ .

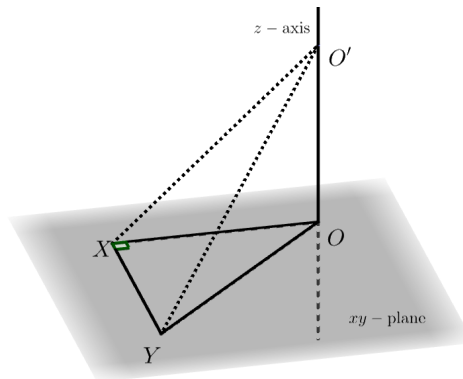
To this purpose, we need the following lemma.

**Lemma 1.** Let  $O$  be the origin in the 3-dimensional space and suppose  $X, Y$  are two distinct points (different from  $O$ ) in the  $xy$ -plane so that  $\angle OXY = 90^\circ$ . Then for any point  $O'$  on the  $z$ -axis, the triangle  $O'OXY$  is right-angled (with  $\angle O'OXY = 90^\circ$ ).

*Proof.* The proof is based on the Pythagorean Theorem. If  $O' = O$ , there is nothing to prove. If  $O' \neq O$ , the line  $OO'$  (the  $z$ -axis) is perpendicular to the  $xy$ -plane and so is perpendicular to every line in this plane passing through  $O$ . In particular two triangles,  $O'OX$  and  $O'OY$  are right-angled. According to the Pythagorean Theorem in these two triangles together with triangle  $OXY$ , we have

$$O'OY^2 = O'O^2 + OY^2 = O'O^2 + OX^2 + XY^2 = O'O^2 + XY^2.$$

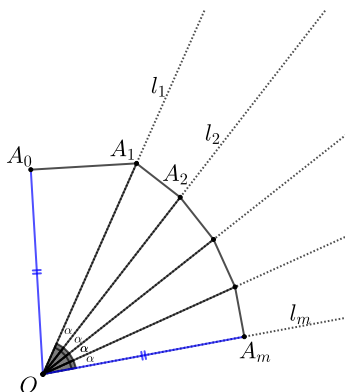
which implies  $\angle O'OXY = 90^\circ$ .



□

Now we return to the main problem. If  $n = 4$ , the tetrahedron with vertices  $O^\theta, O, X, Y$  as in the lemma works (above figure). So we can assume  $n \geq 6$ . Take  $m = \frac{n-2}{2} - 2$ . First, we construct a convex  $(m + 2)$ -gon  $OA_0A_1 \dots A_m$  in the  $xy$ -plane (take  $O$  to be the origin) so that

- $OA_0 = OA_m$ .
- All the triangles of the form  $OA_iA_{i+1}$  (for  $0 \leq i \leq m - 1$ ) are right-angled.



Consider  $m$  different rays with initial point  $O$  (denote them by  $l_1, \dots, l_m$  respectively in clockwise order) so that for a sufficiently small value of  $\alpha$ ,

$$\angle l_1Ol_2 = \angle l_2Ol_3 = \dots = \angle l_{m-1}Ol_m = \alpha. \tag{1}$$

Take an arbitrary point on the ray  $l_1$  and call it  $A_1$ . Start from  $A_1$  and inductively by drawing perpendiculars from  $A_i$  to  $l_{i+1}$  define the points  $A_2, A_3, \dots, A_m$  so that

$$\angle OA_2A_1 = \angle OA_3A_2 = \dots = \angle OA_mA_{m-1} = 90^\circ. \tag{2}$$

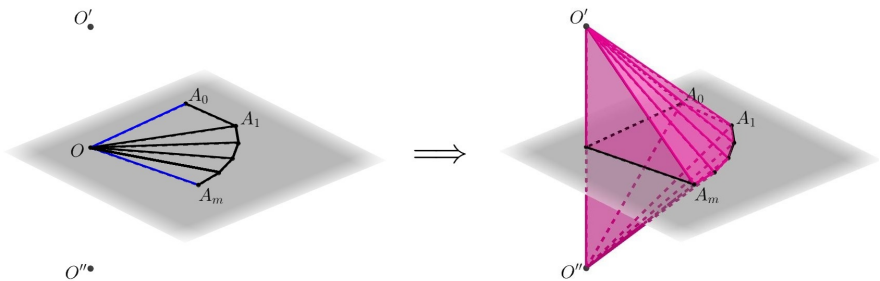
By (1) and (2) all the triangles  $OA_1A_2, OA_2A_3, \dots, OA_{m-1}A_m$  are similar. Therefore  $\frac{OA_m}{OA_{m-1}} = \dots = \frac{OA_3}{OA_2} = \frac{OA_2}{OA_1}$ . We denote this common value by  $r < 1$ . Note that  $r$  can be arbitrarily close to 1 by taking  $\alpha$  small. Now we have

$$OA_m = \frac{OA_m}{OA_{m-1}} \cdot \frac{OA_3}{OA_2} \cdot \frac{OA_2}{OA_1} \cdot OA_1 = r^m \cdot OA_1.$$

Note that since  $\alpha$  is small all the points  $A_2, A_3, \dots, A_m$  are on the same side of the line  $OA_1$ . Take the point  $A_0$  on the other side of this line so that  $\angle OA_0A_1 = 90^\circ$  and  $OA_0 = r^m \cdot OA_1$  ( $A_0$  is one of the intersection points

of the circle with diameter  $OA_1$  and the circle with center  $O$  and radius  $r^m \cdot OA_1$ . If  $r$  is sufficiently close to 1 (equivalently  $\alpha$  sufficiently close to zero),  $r^m$  will be close to one and we can ensure that  $\angle A_0OA_1$  is small and so the polygon satisfies all desired properties.

After construction of the polygon, consider two points  $O^0, O^{00}$  on the  $z$ -axis (on different sides of the  $xy$ -plane) with  $OO^0 = OO^{00} = OA_0 = OA_m$ . Then the polyhedron with vertices  $O^0, O^{00}, A_0, A_1, \dots, A_m$  (convex hull of these points) has exactly  $n = 2m + 2$  faces, and all are right-angled triangles. Indeed, it has  $2m$  faces of the form  $O^0A_iA_{i+1}$  and  $O^{00}A_iA_{i+1}$  which are all right-angled according to the lemma and two faces  $O^0A_0O^{00}$  and  $O^0A_mO^{00}$  that are isosceles right triangles.





# Advanced Level



# Problems

**Problem 1.** Let  $M$ ,  $N$ , and  $P$  be the midpoints of sides  $BC$ ,  $AC$ , and  $AB$  of triangle  $ABC$ , respectively.  $E$  and  $F$  are two points on the segment  $BC$  so that  $\angle NEC = \frac{1}{2} \angle AMB$  and  $\angle PFB = \frac{1}{2} \angle AMC$ . Prove that  $AE = AF$ .

(! p.31)

**Problem 2.** Let  $ABC$  be an acute-angled triangle with its incenter  $I$ . Suppose that  $N$  is the midpoint of the arc  $BAC$  of the circumcircle of triangle  $ABC$ , and  $P$  is a point such that  $ABPC$  is a parallelogram. Let  $Q$  be the reflection of  $A$  over  $N$ , and  $R$  the projection of  $A$  on  $QI$ . Show that the line  $AI$  is tangent to the circumcircle of triangle  $PQR$ .

(! p.33)

**Problem 3.** Assume three circles mutually outside each other with the property that every line separating two of them have intersection with the interior of the third one. Prove that the sum of pairwise distances between their centers is at most  $2\sqrt{2}$  times the sum of their radii.

(A line separates two circles, whenever the circles do not have intersection with the line and are on different sides of it.)

*Note.* Weaker results with  $2\sqrt{2}$  replaced by some other  $c$  may be awarded points depending on the value of  $c > 2\sqrt{2}$ .

(! p.35)

**Problem 4.** Convex circumscribed quadrilateral  $ABCD$  with incenter  $I$  is given such that its incircle is tangent to  $AD$ ,  $DC$ ,  $CB$ , and  $BA$  at  $K$ ,  $L$ ,  $M$ , and  $N$ . Lines  $AD$  and  $BC$  meet at  $E$  and lines  $AB$  and  $CD$  meet at  $F$ . Let  $KM$  intersect  $AB$  and  $CD$  at  $X$  and  $Y$ , respectively. Let  $LN$  intersect  $AD$  and  $BC$  at  $Z$  and  $T$ , respectively. Prove that the circumcircle of triangle  $XFY$  and the circle with diameter  $EI$  are tangent if and only if



the circumcircle of triangle  $TEZ$  and the circle with diameter  $FI$  are tangent.

(! p.37)

**Problem 5.** Consider an acute-angled triangle  $ABC$  ( $AC > AB$ ) with its orthocenter  $H$  and circumcircle  $\omega$ . Points  $M$  and  $P$  are the midpoints of the segments  $BC$  and  $AH$ , respectively. The line  $AM$  meets  $\omega$  again at  $X$  and point  $N$  lies on the line  $BC$  so that  $NX$  is tangent to  $\omega$ . Points  $J$  and  $K$  lie on the circle with diameter  $MP$  such that  $\angle AJP = \angle HNM$  ( $B$  and  $J$  lie on the same side of  $AH$ ) and circle  $\omega_1$ , passing through  $K, H,$  and  $J$ , and circle  $\omega_2$ , passing through  $K, M,$  and  $N$ , are externally tangent to each other. Prove that the common external tangents of  $\omega_1$  and  $\omega_2$  meet on the line  $NH$ .

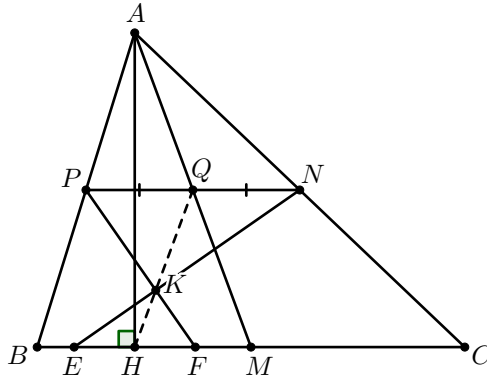
(! p.43)

# Solutions

**Problem 1.** Let  $M, N,$  and  $P$  be the midpoints of sides  $BC, AC,$  and  $AB$  of triangle  $ABC,$  respectively.  $E$  and  $F$  are two points on the segment  $BC$  so that  $\angle NEC = \frac{1}{2}\angle AMB$  and  $\angle PFB = \frac{1}{2}\angle AMC.$  Prove that  $AE = AF.$

*Proposed by Alireza Dadgarnia*

**Solution.** Let  $H$  be the foot of the altitude passing through  $Q,$   $A$  be the midpoint of  $NP$  and  $K$  be the intersection point of  $NE$  and  $PF.$



If we prove that points  $K, H$  and  $Q$  are collinear, using parallel lines ,we get that  $H$  is the midpoint of  $EF$  which is equivalent to the problem. Clearly,  $AM$  passes through  $Q$  and  $H$  is the reflection of  $A$  with respect to  $NP.$  Therefore,  $\angle PQH = \angle AQP = \angle AMB.$  So it suffices to show that  $\angle P Q K = \angle AMB.$  Note that

$$\angle NEC + \angle PFB = \frac{1}{2}(\angle AMB + \angle AMC) = 90^\circ \Rightarrow \angle EKF = 90^\circ .$$

So  $KQ$  is a median on the hypotenuse in triangle  $\triangle PKN$  and we'll get

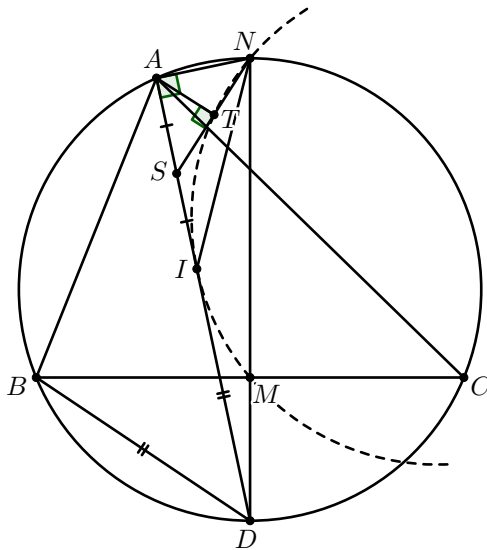
$$\angle PQQ = 2\angle PNK = 2\angle NEC = \angle AMB$$

which completes the proof.

**Problem 2.** Let  $ABC$  be an acute-angled triangle with its incenter  $I$ . Suppose that  $N$  is the midpoint of the arc  $BAC$  of the circumcircle of triangle  $ABC$ , and  $P$  is a point such that  $ABPC$  is a parallelogram. Let  $Q$  be the reflection of  $A$  over  $N$ , and  $R$  the projection of  $A$  on  $QI$ . Show that the line  $AI$  is tangent to the circumcircle of triangle  $PQR$ .

*Proposed by Patrik Bak - Slovakia*

**Solution.** Let  $M, S$  be the midpoint of segments  $BC, AI$ , respectively. By a homothety with center  $A$  and ratio  $\frac{1}{2}$ ,  $P$  goes to  $M$ ,  $Q$  to  $N$  and  $R$  to  $T$ ; Where  $T$  is the projection of  $A$  on  $SN$ . So it suffices to show that the circumcircle of triangle  $AMNT$  is tangent to  $AI$ .



We claim that this circle is tangent to  $AI$  at point  $I$ . We know that  $\angle NAS = 90^\circ$ , So by the similarity of two triangles  $\triangle ASN, \triangle TSA$ , we'll get

$$ST \cdot SN = SA^2 = SI^2.$$

Therefore,  $SI$  is tangent to the circumcircle of triangle  $\triangle ITN$ . Now if we show that  $SI$  is tangent to the circumcircle of triangle  $\triangle NIM$  as well, our proof is completed; Because the circle passing through  $I$  and  $N$  and tangent to  $SI$  is unique. Let  $D$  be the second intersection point of  $AI$  and circumcircle of triangle  $\triangle ABC$ . Note that  $\angle DBM = \angle DCB = \angle DNB$ . Therefore,

$$DM \cdot DN = DB^2 = DI^2.$$

Thus,  $DI$  is tangent to the circumcircle of triangle  $ANIM$  and we're done.

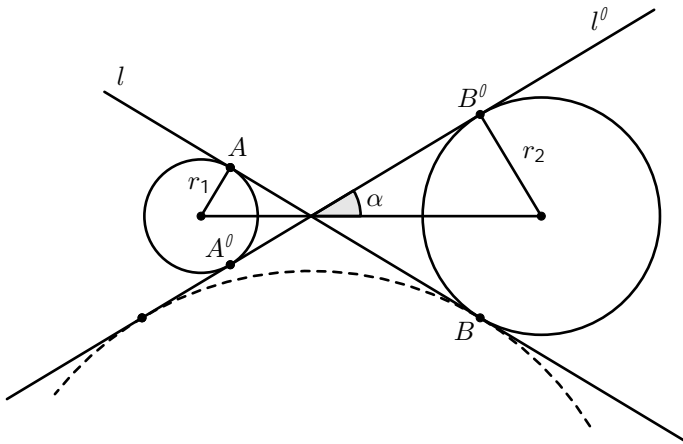
**Problem 3.** Assume three circles mutually outside each other with the property that every line separating two of them have intersection with the interior of the third one. Prove that the sum of pairwise distances between their centers is at most  $2\sqrt{2}$  times the sum of their radii.

(A line separates two circles, whenever the circles do not have intersection with the line and are on different sides of it.)

*Note.* Weaker results with  $2\sqrt{2}$  replaced by some other  $c$  may be awarded points depending on the value of  $c > 2\sqrt{2}$ .

*Proposed by Morteza Saghanian*

**Solution.** According to the figure, we denote the radii of the circles by  $r_1, r_2, r_3$  and the distance  $O_i O_j$  by  $d_{ij}$ . Moreover, let  $l, l^0$  be two interior common tangents of circles  $\omega_1$  and  $\omega_2$ . We denote the tangency points of  $l$  and  $l^0$  as in the figure. Obviously  $d_{12} = \frac{r_1+r_2}{\sin \alpha}$  ( $\alpha$  is defined in the figure). Without loss of generality we assume that  $r_1 \leq r_2$ .



By assumption we can deduce that both lines  $l$  and  $l^0$  must intersect the third circle ( $\omega_3$ ). If the intersection point of  $l$  and  $\omega_3$  lies outside between  $A$  and  $B$ , we can find a line separating  $\omega_1$  and  $\omega_2$  so which does not intersect  $\omega_3$  and this is a contradiction with the assumptions. We have similar arguments for  $l^0$ . So we can assume that the intersection of  $\omega_3$  with  $l$  and  $l^0$  is below  $B$  and  $A^0$  respectively. Therefore,  $r_3$  is at least the radius of the circle tangent to  $l$  at  $B$  and also is tangent to  $l^0$  (why?). The radius of this circle is  $r_2 \cot^2 \alpha$ . Hence

$$r_3 \geq r_2 \cot^2 \alpha = r_2 \left( \frac{1 - \sin^2 \alpha}{\sin^2 \alpha} \right) = \frac{r_1 + r_2}{2} \left( \frac{d_{12}^2}{(r_1 + r_2)^2} - 1 \right).$$

Consequently,

$$d_{12}^2 = (r_1 + r_2)^2 + 2r_3(r_1 + r_2), \quad ( )$$

We have similar equations for  $d_{13}$  and  $d_{23}$ . Summing these three together with Cauchy-Shwarz Inequality gives the assertion. Indeed,

$$\left(\sum d_{ij}\right)^2 \leq 3\sum d_{ij}^2 = 6\sum r_i^2 + 18\sum r_i r_j = 8\left(\sum r_i\right)^2$$

Here the first and third inequality are coming from Cauchy-Shwarz Inequality and the second inequality is the consequence of summing ( ) and two other similar inequalities.

**Remark.** Using upper bound  $(r_1 + r_2 + r_3)^2$  for the right-hand side of ( ) gives  $d_{12} \leq r_1 + r_2 + r_3$ . Summing these, gives a weaker result with 3 replaced by  $2\sqrt{2}$ .

**Problem 4.** Convex circumscribed quadrilateral  $ABCD$  with incenter  $I$  is given such that its incircle is tangent to  $AD$ ,  $DC$ ,  $CB$ , and  $BA$  at  $K$ ,  $L$ ,  $M$ , and  $N$ . Lines  $AD$  and  $BC$  meet at  $E$  and lines  $AB$  and  $CD$  meet at  $F$ . Let  $KM$  intersects  $AB$  and  $CD$  at  $X$  and  $Y$ , respectively. Let  $LN$  intersects  $AD$  and  $BC$  at  $Z$  and  $T$ , respectively. Prove that the circumcircle of triangle  $XFY$  and the circle with diameter  $EI$  are tangent if and only if the circumcircle of triangle  $TEZ$  and the circle with diameter  $FI$  are tangent.

*Proposed by Mahdi Etesamifard*

**Solution.** First, let us prove these lemmas:

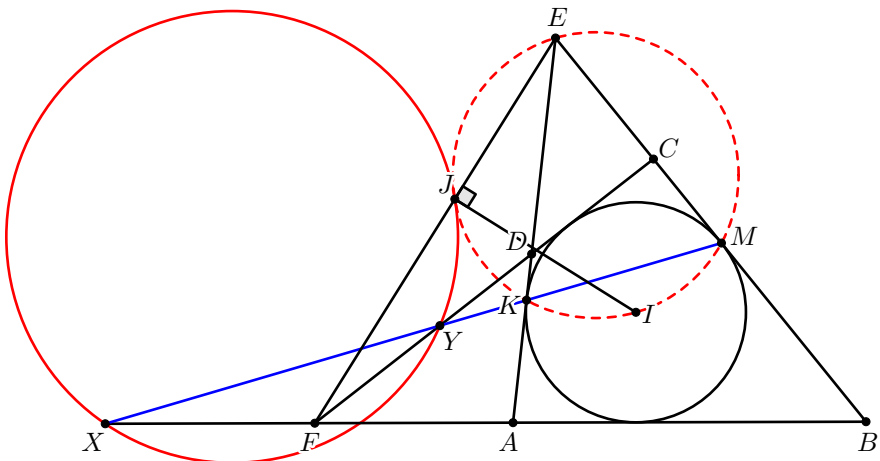
**Lemma 1.** *Lines  $AC$ ,  $BD$ ,  $KM$  and  $LN$  are concurrent.*

*Proof.* Using Brianchon's Theorem in quadrilateral  $ABCD$ , one can simply conclude the fact that  $AC$ ,  $BD$ ,  $KM$  and  $LN$  are concurrent.  $\square$

**Lemma 2.** *Let  $P$  be the point of concurrency of lines in Lemma 1. Therefore,  $P$  is also the intersection point of quadrilateral  $ABCD$ 's diagonals and we have  $IP \perp EF$ .*

*Proof.* We know that polar of point  $P$  is in fact line  $EF$ . Therefore, we'll get  $IP \perp EF$ .  $\square$

**Lemma 3.** *A circle with diameter  $EI$  and the circumcircle of triangle  $XFY$  are tangent.*





*Proof.* For the proof of tangency of circumcircle of triangle  $\triangle XYJ$  to the circle with diameter  $EI$  (circle  $\omega_2$ ), it suffices that the equation of Casey's Theorem hold for points  $X, Y, J$  and circle  $\omega_2$ .

$$XY \cdot P_{\omega_2}^J + XJ \cdot P_{\omega_2}^Y + YJ \cdot P_{\omega_2}^X = 0.$$

Since  $P_{\omega_2}^J = 0$ , Therefore,

$$XJ \cdot \overline{YK \cdot YM} = YJ \cdot \overline{XK \cdot XM} \tag{1}$$

Since  $X, Y$  lie on the radical axis of two circles  $\omega$  and  $\omega_2$ , We have:

$$YK \cdot YM = YL^2, \quad XK \cdot XM = XN^2 \stackrel{(1)}{=} XJ \cdot YL = YJ \cdot XN \tag{2}$$

So, we have to prove equation (2). Using Menelaus's Theorem for triangle  $\triangle XFY$  and line  $LNP$ , We have:

$$\frac{XN}{FN} \cdot \frac{FL}{YL} \cdot \frac{YP}{XP} \stackrel{FN=FL}{=} \frac{XN}{YL} = \frac{XP}{YP}.$$

From equation (2), we get:

$$\frac{XJ}{YJ} = \frac{XN}{YL} = \frac{XP}{YP}.$$

Therefore we need to prove that  $JP$  is the exterior angle bisector of angle  $\angle XJY$ . Since  $JQ \perp JP$ , we need to prove that  $(XY, PQ) = 1$ .

$$(XY, PQ) = F(XY, PQ) \stackrel{NL}{=} (NL, PU) = 1.$$

And since point  $U$  lies on  $EF$  (polar of  $P$ ), the last equation holds and we're done. □

**Lemma 4.**  *$AK$  is tangent to the circumcircle of triangle  $\triangle ABC$  if and only if*

$$\frac{BK}{KC} = \left( \frac{AB}{AC} \right)^2.$$

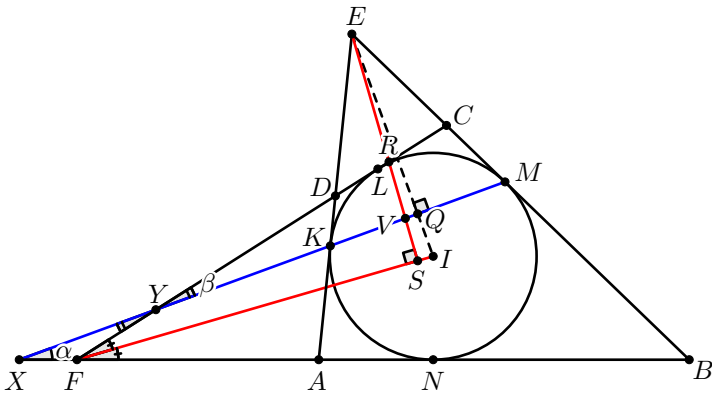
*Proof.* Using The Law of Sines and Ratio Lemma, one can simply get the desired results. □

**Lemma 5.** *If angle bisectors of angles  $\angle E$  and  $\angle F$  are perpendicular, then  $ABCD$  is a cyclic quadrilateral.*



Now, by putting them in equation (4), We'll get:

$$\begin{aligned}
 & XF \left( FI \cos\left(\frac{\alpha + \beta}{2}\right) - FY \right) \\
 & \quad YF \left( FI \cos\left(\frac{\alpha + \beta}{2}\right) - XF \right) - XY \sqrt{FS \cdot FI} = 0 \\
 \Rightarrow & FI \left( XF + YF \right) \cos\left(\frac{\alpha + \beta}{2}\right) = XY \sqrt{FS \cdot FI} \\
 \Rightarrow & FI \left( \frac{XF + YF}{XY} \right) \cos\left(\frac{\alpha + \beta}{2}\right) = \sqrt{FS \cdot FI} \\
 \Rightarrow & \cos\left(\frac{\alpha + \beta}{2}\right) \left( \frac{\sin \alpha + \sin \beta}{\sin \alpha + \beta} \right) = \sqrt{\frac{FS}{FI}} \\
 \Rightarrow & \cos^2\left(\frac{\alpha + \beta}{2}\right) = \frac{FS}{FI}. \tag{5}
 \end{aligned}$$



Also, we have:

$$\begin{aligned}
 \sphericalangle FRS = 90^\circ & \quad \left(\frac{\alpha + \beta}{2}\right) & \sphericalangle QVR = 90^\circ & \quad \left(\frac{\alpha + \beta}{2}\right) \\
 & & & \sphericalangle EIF = 90^\circ & \quad \left(\frac{\alpha + \beta}{2}\right).
 \end{aligned}$$

So, by equation (5), we have:

$$\sin^2(\angle EIF) = \frac{FS}{FI}. \tag{6}$$

We consider three cases for point  $S$  on line  $FI$ :

**Case 1)**  $\angle EIF = 90^\circ$ . Which gives us that  $S$  and  $I$  coincide.

$$\sin^2(\angle EIF) = \frac{FS}{FI} = 1.$$

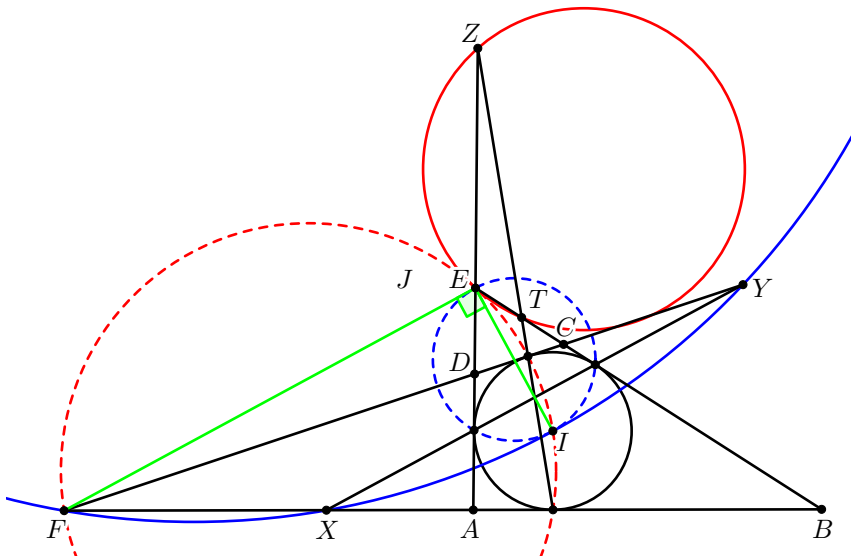
Now, by Lemma 5,  $ABCD$  is a cyclic quadrilateral. On the other hand,  $ABCD$  is circumscribed and every equation resulted from Casey's Theorem for the circumcircle of triangle  $AXFY$  and the circle with diameter  $EI$ , can be written for the circumcircle of triangle  $ATEZ$  and the circle with diameter  $FI$  as well. So by Casey's Theorem, these two circles are tangent to each other.

**Case 2)**  $\angle EIF < 90^\circ$ .

$$\sin^2(\angle EIF) = \left(\frac{ES}{EI}\right) = \frac{FS}{FI}.$$

Now by Lemma 4, we get that  $EF$  is tangent to the circumcircle of triangle  $ATESI$  and

$$\angle FES = \angle FIF \Rightarrow \angle IEF = 90^\circ.$$



Now since  $\angle IEF = 90^\circ$ , the foot of perpendicular line to  $EF$  passing through  $I$ , (Point  $J$ ) coincides with point  $E$ . By Lemma 3, the circumcircle of triangle  $ATJZ$  (which is also the circumcircle of triangle

$\angle TEZ$ ), will be tangent to the circle with diameter  $FI$ . In this case, tangency point of the circumcircle of triangle  $\triangle TEZ$  and the circle with diameter  $EI$ , will be point  $I$  and tangency point of the circumcircle of triangle  $\triangle TEZ$  and the circle with diameter  $FI$ , will be point  $E$ .

**Case 3)**  $\angle EIF > 90^\circ$ . Since

$$\sin^2(\angle EIF) = \frac{FS}{FI} > 1,$$

this case will never happen.

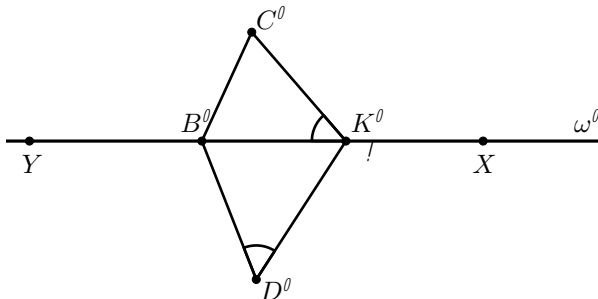
**Problem 5.** Consider an acute-angled triangle  $ABC$  ( $AC > AB$ ) with its orthocenter  $H$  and circumcircle  $\omega$ . Points  $M$  and  $P$  are the midpoints of the segments  $BC$  and  $AH$ , respectively. The line  $AM$  meets  $\omega$  again at  $X$  and point  $N$  lies on the line  $BC$  so that  $NX$  is tangent to  $\omega$ . Points  $J$  and  $K$  lie on the circle with diameter  $MP$  such that  $\angle AJP = \angle HNM$  ( $B$  and  $J$  lie on the same side of  $AH$ ) and circle  $\omega_1$ , passing through  $K, H$ , and  $J$ , and circle  $\omega_2$ , passing through  $K, M$ , and  $N$ , are externally tangent to each other. Prove that the common external tangents of  $\omega_1$  and  $\omega_2$  meet on the line  $NH$ .

*Proposed by Alireza Dadgarnia*

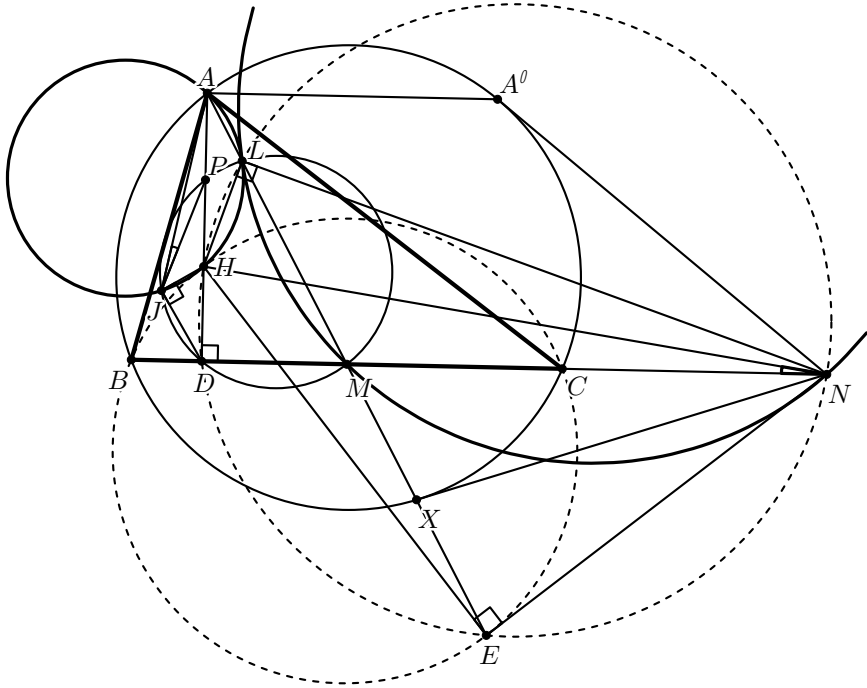
**Solution 1.** Let  $D$  be the intersection of  $AH$  and  $BC$ . Denote  $\omega$  by the circle with diameter  $PM$ . It's obvious that  $D$  lies on  $\omega$ . Also since  $ABC$  is acute,  $H$  lies on the segment  $PD$  and so inside of  $\omega$ .  $N$  lies on the extension of  $DM$  and so outside of  $\omega$ . We claim that there are at most two possible cases for  $K$ . The following lemma proves our claim.

**Lemma.** Given a circle  $\omega$  and four points  $A, B, C$ , and  $D$ , such that  $A$  and  $B$  lie on the circle,  $C$  inside and  $D$  outside of the circle. There are exactly two points like  $K$  on  $\omega$  so that the circumcircles of triangles  $ACK$  and  $BDK$  are tangent to each other.

*Proof.* Invert the whole diagram at center  $A$  with arbitrary radius, the images of points and circle are denoted by primes. Since  $A$  lies on  $\omega$ ,  $\omega^0$  is a line, passes through  $B^0$  and  $K^0$ . Notice that  $C^0$  and  $D^0$  lie on the different sides of  $\omega^0$ . Since the circumcircles of triangles  $ACK$  and  $BDK$  are tangent to each other, we have  $C^0K^0$  is tangent to the circumcircle of triangle  $B^0D^0K^0$ . It means  $\angle C^0K^0B^0 = \angle B^0D^0K^0$ . Let  $X$  and  $Y$  be two arbitrary points, lie on  $\omega^0$  and the different sides of  $B^0$ .



First assume that  $K^0 \in B^0$  so  $\angle C^0 B^0 Y = \angle C^0 K^0 B^0 > 0 = \angle K^0 D^0 B^0$  and when  $K^0$  moves along the ray  $B^0 X$ ,  $\angle C^0 K^0 B^0$  decreases and  $\angle K^0 D^0 B^0$  increases. It yields there is exactly one point  $K^0$  on the ray  $B^0 X$  so that  $\angle C^0 K^0 B^0 = \angle B^0 D^0 K^0$ . In the same way we get there is only one possible case for  $K^0$  on the ray  $B^0 Y$  and the result follows.  $\square$



Denote  $\omega_1$  and  $\omega_2$  by the circumcircles of triangles  $AJP$  and  $HND$ . Let  $H$  be the indirect homothety that sends  $\omega_1$  to  $\omega_2$ . Notice that  $J$  and  $N$  lie on the different sides of  $AH$ . Now since the arc  $AP$  of  $\omega_1$  is equal to the arc  $HD$  of  $\omega_2$  and  $AP \parallel HD$ ,  $H$  sends  $A$  to  $D$  and  $P$  to  $H$  therefore  $(A, H)$  and  $(P, D)$  are anti-homologous pairs. Let  $L$  be the anti-homologous point of  $J$  under  $H$ . It's well-known that the pairs of anti-homologous points lie on a circle so  $ALHJ$  and  $LPJD$  are cyclic quadrilaterals.

Let  $E$  be the reflection of  $A$  over the point  $M$ . We claim that  $HDEN$  is cyclic.  $A^0$  lies on  $\omega_1$  so that  $AA^0 \parallel BC$ . We know that  $(A^0 X, BC) = 1$  hence  $NA^0$  is tangent to  $\omega_1$ . Also by symmetry  $NE$  is tangent to the circumcircle of triangle  $CEB$ . Now since  $HE$  is the diameter of this circle, we have  $\angle NEH = 90^\circ = \angle NDH$  and our claim is proved. The line  $AM$  meets the

circumcircle of triangle  $PDM$  again at  $L^0$ . We have

$$AL^0 \cdot AM = AP \cdot AD \Rightarrow AL^0 \cdot AE = AH \cdot AD$$

it follows that  $L^0HEDEN$  is cyclic so  $L^0 \in L$ . We have

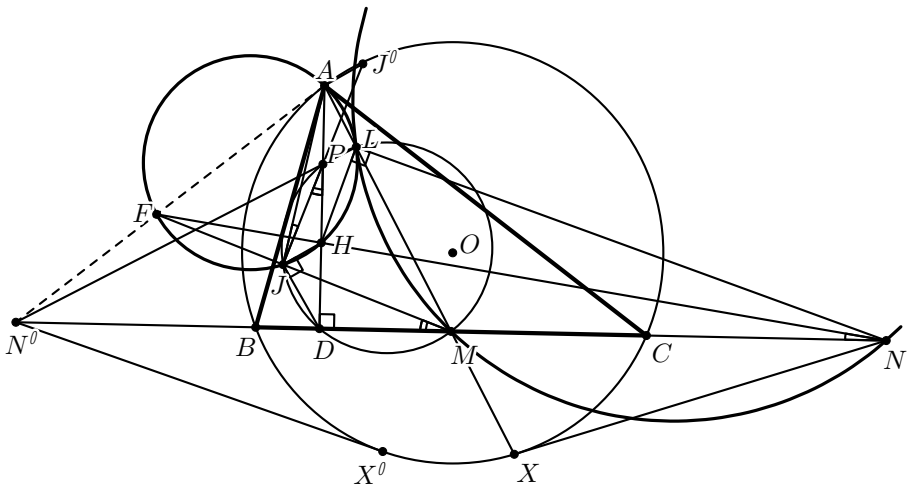
$$\begin{aligned} \angle PJH &= \angle AJH & \angle AJP &= \angle HLM & \angle HND \\ & & &= \angle HLM & \angle HLD = \angle DLM = \angle DJM \end{aligned}$$

therefore  $\angle HJD = 90^\circ$ . From this we can conclude that the circumcircles of triangles  $DHJ$  and  $DMN$  are tangent to each other and the common external tangents of them are concurrent at  $H$  since the tangent line to the circumcircle of triangle  $DHJ$  through  $H$  is parallel to  $DMN$ . So the problem is proved for  $K = D$ , now suppose that  $K \neq D$ . Since  $\angle AHL = \angle LNM$  the circumcircles of triangles  $LHJ$  and  $LMN$  are tangent to each other. So  $L \in K$ . Denote  $O_1$  and  $O_2$  by the circumcenters of triangles  $LHJ$  and  $LMN$ . It's obvious that  $O_1, L$ , and  $O_2$  are collinear so  $\angle O_1LH + \angle O_2LN = 90^\circ$ . It yields

$$\angle HO_1L = 180^\circ - 2\angle O_1LH = 2\angle O_2LN = 180^\circ - \angle LO_2N \Rightarrow O_1H \perp O_2N$$

therefore the direct homothety that sends  $(O_1)$  to  $(O_2)$ , sends  $H$  to  $N$  and the conclusion follows.

**Solution 2.** Let  $D$  be the intersection of  $AH$  and  $BC$ . Denote  $\omega$  by the circle with diameter  $PM$ . It's obvious that  $D$  lies on  $\omega$ .





Let  $F$  be the intersection of  $NH$  and  $MJ$ . Since  $J$  and  $B$  lie on the same side of  $PD$ ,  $J$  lies on the arc  $PD$  (the one that does not contain  $M$ ) so  $J$  and  $H$  lie on the same side of  $BC$ . Also

$$\angle HNM = \angle AJP < \angle JPD = \angle JMD$$

therefore  $F$  and  $J$  lie on the same side of  $NM$  and we have  $\angle FMN = \angle APJ$  since  $\angle JPD = \angle JMD$ . It follows that  $A, F, H$ , and  $J$  are concyclic. Let  $J^0$  and  $N^0$  be the reflections of  $J$  and  $N$  over the points  $P$  and  $M$ , respectively. Since  $P$  is the midpoint of  $AH$ ,  $AJ^0HJ$  is a parallelogram. The  $A$  symmedian meets  $JH$  again at  $X^0$ . Since  $XX^0 \perp BC$ , by symmetry  $N^0X^0$  is tangent to  $\omega$ , too. Also we know that  $(AX^0, BC) = 1$  so  $N^0A$  is tangent to  $\omega$ . Now  $\angle FMN = \angle APJ$  yields  $\angle FMN^0 = \angle APJ^0$ . It follows that

$$\angle N^0FM = \angle J^0AP = \angle AHJ = 180^\circ - \angle AFJ$$

hence  $A, F$ , and  $N^0$  are collinear. Again from  $\angle FMN^0 = \angle APJ^0$  we get

$$\angle PJH = \angle AJ^0P = \angle FN^0M = 90^\circ \quad \angle PMN^0 = \angle DPM = \angle DJM$$

In the third equality we used that  $MP \perp AN^0$  (It's a well-known property, if we let  $O$  be the center of  $\omega$  then  $APMO$  is a parallelogram). It yields  $\angle HJD = \angle PJM = 90^\circ$ . Like the first solution we know that there are at most two possible cases for  $K$  and we can conclude that  $D$  is one of them. Now we suppose that  $K \neq D$ . Let  $AM$  meet  $JH$  again at  $L$ . We have

$$\angle LAH = 90^\circ \quad \angle LMD = \angle LJD = 90^\circ = \angle LJH$$

therefore  $ALHJ$  is cyclic. Since  $MP \perp AN^0$  and  $AP \perp MN^0$ ,  $P$  is the orthocenter of triangle  $AN^0M$  and  $N^0P \perp AM$ . It follows that  $N^0, P$  and  $L$  lie on a same line. Now since  $\angle ALP = \angle N^0LM = 90^\circ$  and  $\angle APL = \angle N^0ML$ , we have  $\angle APL = \angle N^0ML$ . It yields  $\angle LMN = \angle LPH$ . Hence

$$\angle MLN = \angle PLH \Rightarrow \angle HLN = \angle PLM = 90^\circ$$

so  $LNDH$  is cyclic and  $\angle AHL = \angle LNM$ . It follows that the circumcircles of triangles  $LHJ$  and  $LMN$  are tangent to each other. So  $L \in K$ . Denote  $O_1$  and  $O_2$  by the circumcenters of triangles  $LHJ$  and  $LMN$ . It's obvious that  $O_1, L$ , and  $O_2$  are collinear so  $\angle O_1LH + \angle O_2LN = 90^\circ$ . It yields

$$\angle HO_1L = 180^\circ - 2\angle O_1LH = 2\angle O_2LN = 180^\circ - \angle LO_2N \Rightarrow O_1H \perp O_2N$$

therefore the direct homothety that sends  $(O_1)$  to  $(O_2)$ , sends  $H$  to  $N$  and the conclusion follows.

**Comment.** We can also prove  $LHDN$  is cyclic by angle-chasing. We have

$$\angle DLM = \angle DPM = 90^\circ \quad \angle PMD = \angle PJD = 90^\circ = \angle PJH$$

also  $\angle HLM = \angle AJH$  so  $\angle HLD = \angle AJP = \angle HND$  and it follows that  $LHDN$  is cyclic.