The 16^{th} Romanian Master of Mathematics Competition

Day 1: 12 February, 2025, Bucharest

Language: English

Problem 1. Let $n > 10$ be an integer, and let $A_1, A_2, \ldots, A_n$ be distinct points in the plane such that the distances between the points are pairwise different. Define $f_{10}(j,k)$ to be the 10^{th} smallest of the distances from A_j to $A_1, A_2, \ldots, A_k$, excluding A_j if $k \geq j$. Suppose that for all j and k satisfying $11 \leq j \leq k \leq n$, we have $f_{10}(j, j-1) \geq f_{10}(k, j-1)$. Prove that $f_{10}(j,n) \geq \frac{1}{2} f_{10}(n,n)$ for all j in the range $1 \leq j \leq n-1$.

Problem 2. Consider an infinite sequence of positive integers $a_1, a_2, a_3, \ldots$ such that $a_1 > 1$ and $(2^{a_n} - 1)a_{n+1}$ is a square for all positive integers n. Is it possible for two terms of such a sequence to be equal?

Problem 3. Fix an integer $n \geq 3$. Determine the smallest positive integer k satisfying the following condition:

> For any tree T with vertices $v_1, v_2, \ldots, v_n$ and any pairwise distinct complex numbers $z_1, z_2, \ldots, z_n$, there is a polynomial $P(X,Y)$ with complex coefficients of total degree at most k such that for all $i \neq j$ satisfying $1 \leq i, j \leq n$, we have $P(z_i, z_j) = 0$ if and only if there is an edge in T joining v_i to v_j.

Note, for example, that the total degree of the polynomial

$$9X^3Y^4 + XY^5 + X^6 - 2$$

is 7 *because* $7 = 3 + 4$.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

The 16^{th} Romanian Master of Mathematics Competition

Day 2: 13 February, 2025, Bucharest

Language: English

Problem 4. Let $\mathbb{Z}$ denote the set of integers, and let $S \subset \mathbb{Z}$ be the set of integers that are at least 10^{100}. Fix a positive integer c. Determine all functions $f\colon S \to \mathbb{Z}$ satisfying $f(xy + c) = f(x) + f(y)$ for all $x, y \in S$.

Problem 5. Let ABC be an acute triangle with $AB < AC$, and let H and O be its orthocentre and circumcentre, respectively. Let Γ be the circumcircle of triangle BOC. Circle Γ intersects line AO at points O and A', and Γ intersects the circle of radius AO with centre A at points O and F. Prove that the circle which has diameter AA', the circumcircle of triangle AFH, and Γ pass through a common point.

Problem 6. Let k and m be integers greater than 1. Consider k pairwise disjoint sets $S_1, S_2, \dots, S_k$, each of which has exactly $m + 1$ elements: one red and m blue. Let $\mathcal{F}$ be the family of all subsets T of $S_1 \cup S_2 \cup \dots \cup S_k$ such that, for every i, the intersection $T \cap S_i$ is monochromatic. Determine the largest possible number of sets in a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that no two sets in $\mathcal{G}$ are disjoint.

A set is monochromatic if all of its elements have the same colour. In particular, the empty set is monochromatic.

Each problem is worth 7 marks.
Time allowed: $4\frac{1}{2}$ hours.

The 16th Romanian Master of Mathematics Competition

Day 1 — Solutions

Problem 1. Let $n > 10$ be an integer, and let $A_1, A_2, \dots A_n$ be distinct points in the plane such that the distances between the points are pairwise different. Define $f_{10}(j,k)$ to be the 10^{th} smallest of the distances from A_j to $A_1, A_2, \dots, A_k$, excluding A_j if $k \geq j$. Suppose that for all j and k satisfying $11 \leq j \leq k \leq n$, we have $f_{10}(j, j-1) \geq f_{10}(k, j-1)$. Prove that $f_{10}(j,n) \geq \frac{1}{2} f_{10}(n,n)$ for all j in the range $1 \leq j \leq n-1$.

Iran, Morteza Saghafian

Solution 1. For every i, denote $a_i = f_{10}(i, i-1)$ and $b_i = f_{10}(i,n)$. So, we need to show that $b_n \leq 2b_i$ for all i. Notice that $a_i \geq b_i$ for all i.

To prove this, choose an arbitrary $i < n$, and let $A_iA_{j_1}, A_iA_{j_2}, \dots, A_iA_{j_{10}}$ be the ten smallest numbers among the A_iA_j with $j \neq i$, ordered so that $j_1 < j_2 < \cdots < j_{10}$.

If $j_{10} < i$, then $i > 10$, and the problem condition yields

$$b_i = \max_{1 \leq k \leq 10} A_iA_{j_k} = a_i \geq a_n \geq b_n,$$

which is even stronger than we need.

Otherwise, set $j = j_{10} > i$ (in this case we also have $j_{10} > 10$), and denote $m = b_i = \max\limits_{1 \leq k \leq 10} A_iA_{j_k}$. By the problem condition, we have $a_j \geq a_n = b_n$. On the other hand, we have

$$a_j \leq \max\left(A_jA_i, \max_{1 \leq k \leq 9} A_jA_{j_k}\right) \leq \max\left(A_jA_i, \max_{1 \leq k \leq 9} (A_jA_i + A_iA_{j_k})\right) \leq 2m,$$

as $A_jA_i, A_iA_{j_k} \leq m$. So $b_n \leq a_j \leq 2m = 2b_i$, as desired.

Solution 2. Let $d_j = f_{10}(j,n)$, $j = 1, \dots, n$. To prove that $2d_j \geq d_n$ for every $j = 1, \dots, n-1$, induct on n.

Consider the base case, $n = 11$. Note that each $d_j = \max_{i \neq j} A_iA_j$, as f_{10} is 10-variate. Let $d_{11} = A_{11}A_k$ for some index $k \leq 10$. Clearly, $2d_k \geq 2A_kA_{11} = 2d_{11} \geq d_{11}$ and if $j \neq k$ then $2d_j \geq A_jA_k + A_jA_{11} \geq A_kA_{11} = d_{11}$, by the triangle inequality.

For the induction step, let $n > 11$ and note that

$$\max_{n \geq \ell \geq k} f_{10}(\ell, k-1) = \max_{n-1 \geq \ell \geq k} f_{10}(\ell, k-1),$$

as both maxima are achieved at $\ell = k$, by the condition in the statement. Hence $A_1, A_2, \dots, A_{n-1}$ also satisfy this condition.

Let $d'_j = f_{10}(j, n-1)$, $j = 1, \dots, \leq n-1$. By the induction hypothesis, $2d'_j \geq d'_{n-1}$ for all $j \leq n-2$. Note that $d'_{n-1} = f_{10}(n-1, n-2) \geq f_{10}(n, n-2) \geq f_{10}(n, n-1) = d_n$; the first inequality holds by the condition in the statement for $k = n-1$ and the second because adding more variables to f_{10} does not increase its value.

Let now Δ_i be the closed disc (interior and boundary) of radius d_i, centred at A_i. By the definition of d_i, each Δ_i contains at least 11 points, of which at most 10 (A_i, inclusive) lie strictly inside.

Finally, suppose, if possible, that $2d_j < d_n$ for some index $j < n$. If A_jA_n is not among the first 10 distances from A_j to the other points, then $d_j = d'_j$ and this leads to a contradiction with the induction hypothesis. So A_jA_n has to be among the first 10 distances from A_j to the other points. This means that $d_j \geq A_jA_n$, so $d_n > 2d_j \geq 2A_jA_n$. Hence Δ_j lies strictly inside Δ_n. This is a contradiction, as Δ_j contains at least 11 points, whereas Δ_n contains at most 10 strictly inside. The conclusion follows.

Problem 2. Consider a sequence of integers $a_1, a_2, a_3, \dots$ such that $a_1 > 1$ and $(2^{a_n} - 1)a_{n+1}$ is a square for all positive integers n. Is it possible that two terms of such a sequence be equal?

Russia, Pavel Kozlov

Solution. The answer is in the negative. Notice first that, if $a_n > 1$, then $2^{a_n} - 1 \equiv 3 \pmod 4$; since $(2^{a_n} - 1)a_{n+1}$ is a perfect square, we should have $a_{n+1} \equiv 0 \pmod 4$ or $a_{n+1} \equiv 3 \pmod 4$, so in particular $a_{n+1} > 1$. As $a_1 > 1$, we conclude that all terms of the sequence are greater than 1.

Denote the largest prime divisor of an integer $k > 1$ by $g(k)$. We will show that $g(a_{n+1}) > g(a_n)$ for all n, which yields the desired result. To this end, usage is made of the lemma below.

Lemma: For any prime p, each prime divisor of $2^p - 1$ is greater than p.

Proof. Let q be a prime factor of $2^p - 1$; then q is odd. The multiplicative order d of 2 modulo q divides p and is larger than 1, so $d = p$. On the other hand, by Fermat's little theorem, $2^{q-1} \equiv 1 \pmod q$, so $p = d \mid q - 1$ and the lemma follow.

Choose now any positive integer n, and denote, for convenience, $k = a_m$ and $\ell = a_{n+1}$. Let $p = g(k)$; then $2^p - 1 \mid 2^k - 1$. Since $2^p - 1 \equiv 3 \pmod 4$, this number is not a square, so there exists a prime q such that $v_q(2^p - 1)$ is odd. By the Lemma, $q > p$, so in particular $q \nmid k$. Therefore, by the Lifting Exponent Lemma,

$$v_q(2^k - 1) = v_q(2^p - 1) + v_q(k/p) = v_q(2^p - 1) + 0,$$

so $v_q(2^k - 1)$ is odd as well. Since $(2^k - 1)\ell$ is a perfect square, we should then have $q \mid \ell$, so $g(\ell) \geq q > p = g(k)$, as desired.

Problem 3. Fix an integer $n \geq 3$. Determine the smallest positive integer k satisfying the following condition:

> For any tree T with vertices $v_1, v_2, \ldots, v_n$ and any pairwise distinct complex numbers $z_1, z_2, \ldots, z_n$, there is a polynomial $P(X, Y)$ with complex coefficients of total degree at most k such that for all $i \neq j$ satisfying $1 \leq i, j \leq n$, we have $P(z_i, z_j) = 0$ if and only if there is an edge in T joining v_i to v_j.

Note, for example, that the total degree of the polynomial

$$9X^3Y^4 + XY^5 + X^6 - 2$$

is 7 *because* $7 = 3 + 4$.

Romania, Andrei Chiriţă

Solution 1. First we provide a proof that $k \geq n-1$. Let T be the path where v_i and v_{i+1} are adjacent for all $1 \leq i \leq n-1$. Let ω be a primitive root of unity of order n and let $a_i = \omega^i$ for all $1 \leq i \leq n$.

If $f(X) = P(X, \omega X)$, then for all $1 \leq i \leq n-1$ we have $f(\omega^i) = P(a_i, a_{i+1}) = 0$. Since $f(1) = P(a_n, a_1) \neq 0$, f is non-zero and has at least $n-1$ roots. This means that $\deg P \geq \deg f \geq n-1$, proving $k \geq n-1$.

It remains to prove that $k = n-1$ is sufficient i.e. for any tree T and any $a_1, a_2, \ldots, a_n$ we can find a polynomial P of degree at most $n-1$. For brevity, we call a two-variable polynomial $A(X, Y)$ *symmetric* if $A(X, Y) = A(Y, X)$.

We begin with the following observation. Suppose that A and B are two variable polynomials of degree at most d. Then we can find $\alpha \in \mathbb{C}$ such that for any $1 \leq i, j \leq n$, $A(a_i, a_j) + \alpha B(a_i, a_j) = 0$ if and only if $A(a_i, a_j) = B(a_i, a_j) = 0$. This means that we can "merge" two conditions of degree at most d into a condition of degree at most d (note that this produces a symmetric polynomial if the initial polynomials are symmetric).

For any integer $t \geq 2$, let a *star* of size t be a collection of t edges for which there is a vertex which belongs to all edges. We will prove the following claims.

Claim 1. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$ and E edges. Suppose that we can partition the edges of G into a number of stars. Then for any distinct complex numbers $a_1, a_2, \ldots, a_n$ we can find a symmetric polynomial P of degree at most E such that for all $1 \leq i, j \leq n, i \neq j$, $P(a_i, a_j) = 0$ if and only if there is an edge between v_i and v_j in G.

Proof. We will first prove the claim when G consists of a star of size $E \leq n-1$ and some isolated vertices. Without loss of generality, let $v_1v_2, v_1v_3, \ldots, v_1v_{E+1}$ be the edges of G. Also let $s_1 = a_1 + a_2, s_2 = a_1 + a_3, \ldots, s_E = a_1 + a_{E+1}$.

Consider merging the polynomials $(X - a_1)(Y - a_1)$ and $(X + Y - s_1)(X + Y - s_2) \ldots (X + Y - s_E)$ into a polynomial of degree at most E (which is of course symmetric). They both vanish at a pair a_i, a_j if and only if $1 \in \{i, j\}$ and $a_i + a_j \in \{s_1, s_2, \ldots, s_E\}$. These two happen if and only if v_i and v_j are adjacent, so this produces a valid polynomial.

For the general case, let $S_1 \cup S_2 \cup \cdots \cup S_k$ be the partition of the edges of G. For each $1 \leq i \leq k$, we can find a two variable polynomial P_i of degree at most $|S_i|$ which vanishes only at the edges of S_i. Then we can let $P = P_1P_2 \ldots P_k$, which satisfies the claim as $\deg P \leq |S_1| + |S_2| + \cdots + |S_k| = E$, as desired.

Claim 2. Any tree Γ with odd number of vertices can be partitioned into stars of size 2.

Proof. We prove this by induction on the number of the vertices of Γ. The base case is clear, since Γ is a star of size 2 when Γ has three vertices.

For the inductive step, let Γ be a tree with $2m + 1$ vertices, where $m \geq 2$. Let $u_1u_2 \ldots u_t$ be a path of maximal length in Γ (of course, $t \geq 3$). Then any neighbour of u_2 except for maybe u_3 must have degree 1, otherwise we can delete u_1 and insert two edges, contradicting the maximality of t. If $\deg u_2 = 2$, we can form the star u_1u_2, u_2u_3 and apply the inductive hypothesis on $\Gamma \setminus \{u_1, u_2\}$. If $\deg u_2 \geq 3$, let $u \neq u_1, u_3$ be a neighbour of u_2. Then create the star u_1u_2, uu_2 and apply the inductive hypothesis on $\Gamma \setminus \{u, u_1\}$. This proves the claim.

The case where n is odd becomes trivial, since T has $n-1$ edges. Assume that n is even. Without loss of generality, let $\deg v_n = 1$ and let v_n be adjacent to v_{n-1}.

Consider the graph T' formed by deleting the edge $v_{n-1}v_n$ from T (but not perturbing the n vertices). Clearly, T' consists of v_n and a tree on $v_1, v_2, \ldots, v_{n-1}$. As $n-1$ is odd, the edges of T' can be partitioned into stars of size 2 from Claim 2. From Claim 1 it follows that there is a symmetric polynomial $Q(X, Y)$ of degree at most $n-2$ such that $Q(a_i, a_j) = 0$ if and only if $i, j \leq n-1$ and v_i, v_j are adjacent in T.

Let $g(X)$ be the polynomial of degree $n-1$ such that $g(a_1) = g(a_2) = \cdots = g(a_{n-1}) = 0$ and $g(a_n) = -Q(a_{n-1}, a_n) = -Q(a_n, a_{n-1})$. Let P be the polynomial of degree at most $n-1$ obtained by merging $F_1(X, Y) = Q(X, Y) + g(X) + g(Y)$ and $F_2(X, Y) = Q(X, Y)(X + Y - s)$, where $s = a_n + a_{n-1}$. It is easy to see that P vanishes at each (a_i, a_j) for which v_i, v_j are adjacent in T.

Suppose that v_i and v_j are not adjacent in T. If $i, j \leq n-1$, then $F_1(a_i, a_j) = Q(a_i, a_j) \neq 0$. If $n \in \{i, j\}$, then $a_i + a_j \neq s$ and $Q(a_i, a_j) \neq 0$, so $F_2(a_i, a_j) \neq 0$. Hence $P(a_i, a_j) \neq 0$. This proves that P satisfies the required conditions, as desired.

Solution 2. Establish the lower bound as in Solution 1. We now address the upper bound differently. Let G be a graph on vertices $v_1, \ldots, v_n$. Say that a polynomial $P(X, Y)$ is G-good if it satisfies the conditions in the statement of the problem. We prove the more general fact below:

Claim. Let d_i be the degree of v_i. Assume that these degrees satisfy $d_i \leq n-i$ for all $i \leq n-1$, and $d_n = 1$. Then there is a G-admissible polynomial of degree at most $n-1$.

Notice here that, if G is a tree with vertices ordered so that $d_1 \geq d_2 \geq \cdots \geq d_n$, then it satisfies the conditions in the Claim. Indeed, we have $d_n = 1$, and if $d_i > n-i$ for some $i \leq n-1$, then we have

$$2n - 2\sum_{j=1}^{n} d_j \geq \sum_{j=1}^{i}(n-i+1) + \sum_{j=i+1}^{n} 1 = i(n-i+1) + (n-i) = (i+1)(n-i+1) - 1 \geq 2n-1,$$

which is a contradiction. So, it suffices to prove the Claim.

Proof of the Claim. Let

$$P(X, Y) = \sum_{j=0}^{n-1} R_j(Y) X^j$$

be the sought polynomial; set $R_{n-1}(X) = 1$. Denote

$$Q_i(X) = P(X, a_i) = X^{n-1} + \sum_{j=0}^{n-2} q_{ij} X^j, \qquad \text{where } q_{ij} = R_j(a_i).$$

So, we will seek for the sequences $C_j = (q_{1j}, q_{2j}, \ldots, q_{nj})$ such that there exists a polynomial R_j with $\deg R_j \leq n-j-1$ such that $q_{ij} = R_j(a_i)$. Notice that the first $n-j$ terms of such a sequence determine it uniquely; in particular, there are no restrictions on the sequence C_0.

We know that the polynomial Q_i has d_i prescribed roots. For every $i \leq n-1$, augment this list by some numbers not from the set $A = \{a_1, a_2, \ldots, a_n\}$ to the list $b_{i1}, b_{i2}, \ldots, b_{i,n-i}$. Also, denote by b_{n1} the unique prescribed root of Q_n. Thus, we should have

$$Q_i(X) = S_i(X) \prod_{j=1}^{\max(n-i,1)} (X - b_{ij}), \tag{1}$$

where S_i is a monic polynomial with $\deg S_i = i-1$ for $i \le n-1$ and $\deg S_n = n-2$. The only extra conditions we have are that each S_i should achieve non-zero values at the prescribed finite subset A_i of A (containing no prescribed roots of Q_i).

The polynomial Q_1 is uniquely determined by (1).

Assume that the polynomials $Q_1, \dots, Q_{i-1}$ have already been determined, for some $i \le n-2$. This means that the sequences $S_1, \dots, S_{i-1}$ are also determined. This determines the polynomial S_i up to the constant term. So we may choose this constant term so that S_i has no prohibited roots, thus defining Q_i.

It remains to deal with the indices $i = n-1, n$. At this moment, both polynomials S_{n-1} and S_n are determined up to constant terms. The conditions they must obey are: (i) they should not have roots inside A_{n-1} and A_n, respectively; and (ii) the sequence C_1 should be a sequence of values of a polynomial R_1 with $\deg R_1 \le n-2$. Notice that all such polynomials R_1 are determined up to an additive polynomial of the form $\alpha \prod_{j \le n-2}(X - a_j)$. The coefficients $q_{n-1,1}$ and $q_{n,1}$ depend linearly on α with a non-zero linear term. Hence the constant terms of S_{n-1} and S_n depend linearly on α. Now, again, there are only finitely many restrictions which remove finitely many values of α; any other value fits the bill.

Solution 3. The answer is $n-1$. Use the same argument as the official solution for the lower bound.

We can construct such a polynomial via a method that uses no graph theory other than the fact that T has $n-1$ edges.

Lemma 1. Let k be a positive integer and let $A, B \subseteq \mathbb{C}^2$ be two disjoint finite sets. Suppose $|A| = 2k$ and no line intersects $A \cup B$ in more than $k+1$ points. Then there exists a polynomial of degree k that vanishes on A and is non-zero on B.

Proof. We first argue that we may reduce to the case when $|B| = 1$. This is since if P and P' are polynomials that are zero on A and nonzero on B and B', then a generic linear combination of P and P' is nonzero on A and nonzero on $B \cup B'$. Now suppose $B = \{b\}$.

For $a, a' \in A$, write $a \sim a'$ if a, a', b are collinear. This is an equivalence relation, and each equivalence class has at most k elements. Thus we may pair up the elements of A such that no two paired elements are collinear with b. Now let P be the polynomial vanishing on the union of the k lines determined by the pairs, which is nonzero at b by construction.

Lemma 2. Let $a_1, a_2, \dots, a_n$ be distinct complex numbers. Then a line can intersect at most n points of the form (a_i, a_j).

Proof. If not, then by the pigeonhole principle such a line must contain two points with the same x-coordinate. But then it is vertical and thus can only contain n points.

We are done by applying Lemma 1 with

$$A = \{(a_i, a_j) \colon v_i v_j \in E(T)\} \quad \text{and} \quad B = \{(a_i, a_j) \colon i \ne j, v_i v_j \notin E(T)\}.$$

The 16$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 2 — Solutions

Problem 4. Let $\mathbb{Z}$ denote the set of integers and let $S \subset \mathbb{Z}$ be the set of integers that are at least 10^{100}. Fix a positive integer c. Determine all functions $f\colon S \to \mathbb{Z}$ satisfying

$$f(xy+c) = f(x) + f(y) \quad \text{for all } x, y \in S.$$

United Kingdom

Solution. Observe that if $x_1, y_1, x_2, y_2 \in S$ with $x_1y_1 = x_2y_2$ then

$$f(x_1) + f(y_1) = f(x_2) + f(y_2). \tag{1}$$

This tells us that for $u, v, w \in S$,

$$f(uv) + f(w) = f(u) + f(vw), \quad \text{so} \quad f(uv) - f(u) - f(v) = f(vw) - f(w) - f(v).$$

Notice the RHS is independent of u so the same must be true of the LHS. By replicating the argument with u and v switched, we also see the LHS is independent of v so in fact

$$f(uv) - f(u) - f(v) = k \quad \text{for some constant} \quad k \in \mathbb{Z} \tag{2}$$

Using (1) again we have, for $y, z \in S$

$$\begin{aligned} f(cz) + f(y) &= f(z) + f(cy), \\ \text{so} \quad f(cy) - f(y) = l &\quad \text{for some constant} \quad \ell \in \mathbb{Z}. \end{aligned} \tag{3}$$

Setting $x = cz$ in the original functional equation for $z \in S$ shows

$$\begin{gathered} f(c(yz+1)) \overset{(3)}{=} f(yz+1) + \ell = f(cz) + f(y) \overset{(3)}{=} f(z) + f(y) + \ell, \\ \text{so} \quad f(yz+1) = f(y) + f(z). \end{gathered}$$

Let $x \in S$ and set $y = x$, $z = x+2$ in the above to get

$$\begin{gathered} f\big((x+1)^2\big) \overset{(2)}{=} 2f(x+1) + k = f(x) + f(x+2) \\ \Rightarrow f(x) + f(x+2) - 2f(x+1) = -k = \text{constant} \end{gathered}$$

which forces f to be a quadratic. By setting $x = y$ in the original functional equation and considering the degree of both sides, we see f must be in fact be constant. The only constant function that satisfies the condition is $f \equiv 0$.

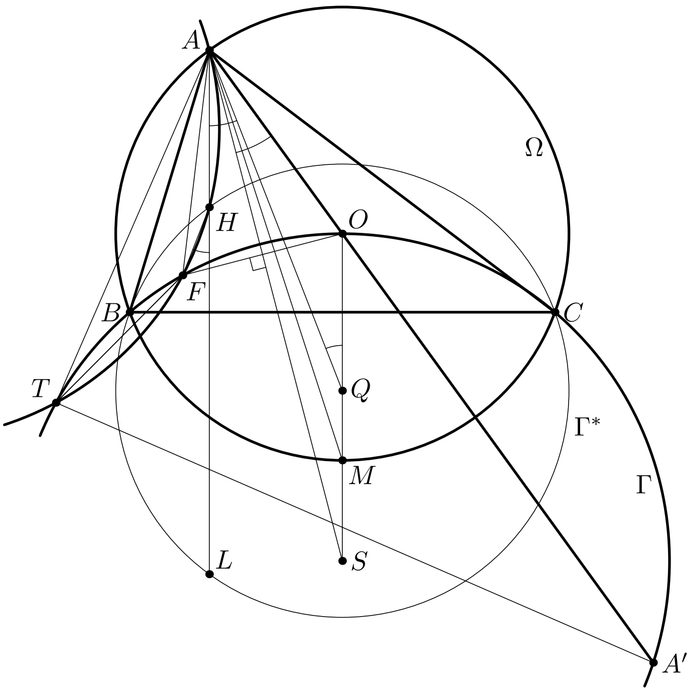
Problem 5. Let ABC be an acute triangle with $AB < AC$ and let H and O be its orthocentre and circumcentre, respectively. Let Γ be the circle BOC. The line AO and the circle of radius AO centred at A cross Γ again at A' and F, respectively. Prove that Γ, the circle on diameter AA' and the circle AFH are concurrent.

Romania, Radu-Andrei Lecoiu

Solution 1. Let Ω denote the circle (ABC) (centred at O), and let M be the midpoint of the minor arc BC of that circle.

Consider the composition ι of an inversion centered at A and the reflection in the bisector AM that swaps B and C. Then ι swaps the circle Ω with the line BC, hence it swaps O with the reflection L of A in BC. Hence $\iota(\Gamma)$ is the circle $\Gamma^* = (BCL)$, i.e., the reflection of Ω in BC which passes through H. Let S and Q be the centers of Γ and Γ^*, respectively; then AM is the angle bisector of $\angle QAS$.

Since ι swaps Γ and Γ^*, they are seen from A at the same angle, so there exists a rotational homothety h centred at A mapping Γ to Γ^*; the angle of h is $\angle SAQ$. Notice that the rays AH and AF are obtained from AO by reflections in AM and AS, respectively, so $\angle HAF = 2\angle MAS = \angle QAS$. This easily yields that $H = h(F)$. Hence the triangles AHF and AQS are similar, and $\angle AHF = \angle AQS$.



Now, let the circle (AHF) meet Γ again at T. Using directed angles, we get $\angle ATF = \angle AHF = \angle AQS$; next, since $FO \perp AS$, we have

$$\angle FTA' = \angle FOA' = \pi/2 - \angle OAS = \pi/2 - \angle QAL = \pi/2 - \angle AQS,$$

(here the equality $\angle OAS = \angle QAL$ holds because these angles are symmetric to each other with respect to AM), so $\angle ATA' = \angle ATF + \angle FTA' = \pi/2$, as desired.

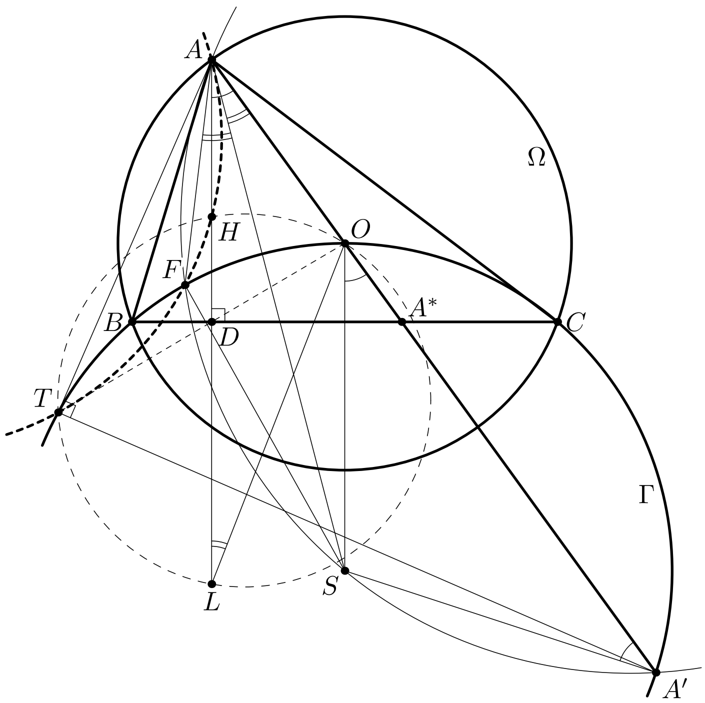
Remark. Existence of the rotational homothety h may be shown in various ways. E.g., one may notice that Ω is an Apollonius circle of the segment QS, so the ratio of the radii of Γ and Γ^* is $BS/BQ = AS/AQ$, which also yields that h exists.

Solution 2. Let S be the centre of Γ and let the circle on diameter AA' cross Γ again at T. It is sufficient to prove that T lies on circle AFH.

The points O and F are reflections of one another in AS; hence S lies on the internal angle bisectrix of $\angle FAA'$. On the other hand, since $SF = SA'$, it lies on the perpendicular bisectrix of FA'; so S is the midpoint of the arc $A'F$ on circle $AA'F$ not containing A. In particular, $AFSA'$ is cyclic.

Let D be the orthogonal projection of A on BC. We will prove that O, D, T are collinear. Invert from O with radius OB. This fixes B and C, so Γ maps to line BC. It follows that A maps to $A^* = AA' \cap BC$. Note that A is fixed under this inversion, as $OA = OB$, so the image of the circle on diameter AA' is a circle δ through A and A^* — and, in fact, δ is the circle of diameter AA^*, as AA' passes through O. Hence T maps to one of the points where line BC crosses δ. As $T \neq A'$, its image is D, so O, D, T are indeed collinear.

Letting L be the reflection of A in BC, we now prove that $HTLO$ is cyclic. As circle HBC is the reflection of Γ in BC, the quadrangle $HLBC$ is cyclic, so $HD \cdot DL = BD \cdot DC = OD \cdot DT$, whence $HTLO$ is indeed cyclic.



Next, we show that triangles ALO and $A'AS$ are similar. Let $\angle BAC = \alpha$ and $\angle CBA = \beta$. As $OS \parallel AL$, it follows that $\angle OAL = \angle A'OS = \angle AA'S$, so by the sine law:

$$\frac{AL}{AA'} = 2 \cdot \frac{AD}{AB} \cdot \frac{AB}{AA'} = 2 \cdot \sin\beta \cdot \frac{\cos\alpha}{\sin\beta} = 2\cos\alpha \quad \text{and} \quad \frac{AO}{A'S} = \frac{BO}{BS} = \frac{\sin 2\alpha}{\sin\alpha} = 2\cos\alpha.$$

Consequently, $AL/AA' = AO/A'S$, so $AL/AO = AA'/A'S$, implying the desired similarity. In particular, $\angle ALO = \angle A'AS$.

Finally, combine the properties established above to chase angles and write successively

$$\begin{aligned}\angle FTH = \angle FTO - \angle HTO = \frac{1}{2}\angle FSO - \angle HLO = \angle ASO - \angle A'AS \\ = \angle ASO + \angle A'AS - 2\angle A'AS = \angle SOA' - \angle FAO = \angle HAO - \angle FAO \\ = \angle HAF\end{aligned}$$

and conclude that T lies on circle AFH, as stated in the first paragraph. This completes the solution.

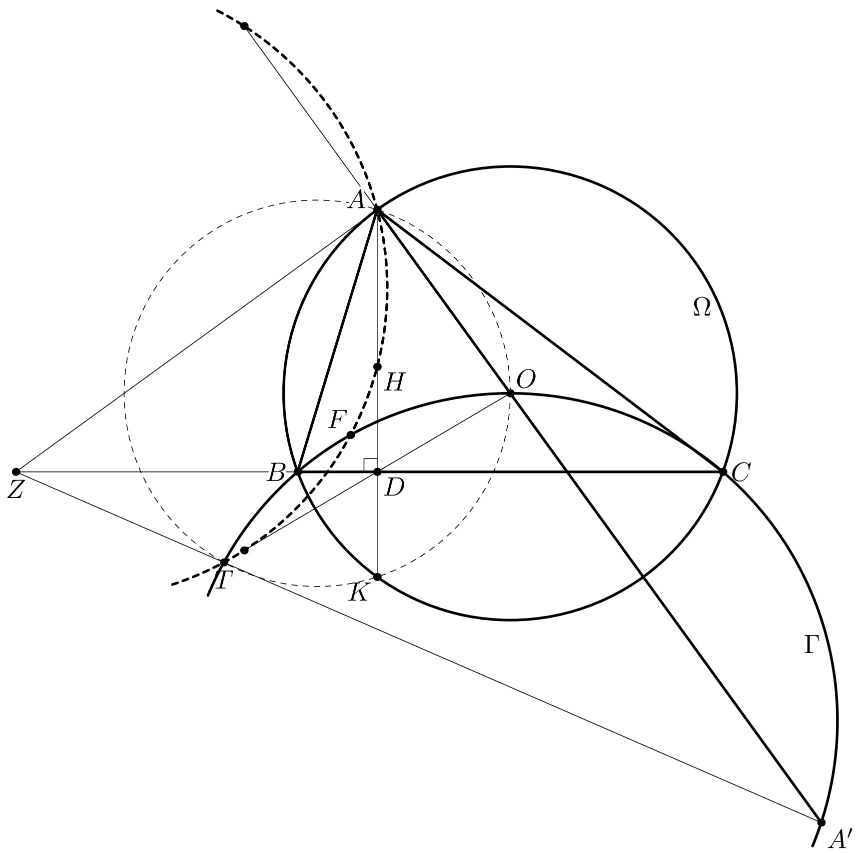
Remark. Let T be the desired intersection point. The property that the points O, D, and T is also useful in other approaches to the problem; in fact, it may be proved for both definitions of point T (as in Solutions 1 and 2), thus providing a different solution.

Here we list several other properties of the figure which appear to be useful in other approaches to the problem.

Let the tangent to the circle (ABC) at A meet BC at Z. Then the desired common intersection point T lies on ZA'.

Let AH meet the circle (ABC) again at K. Then the points A, O, K, and T are concyclic.

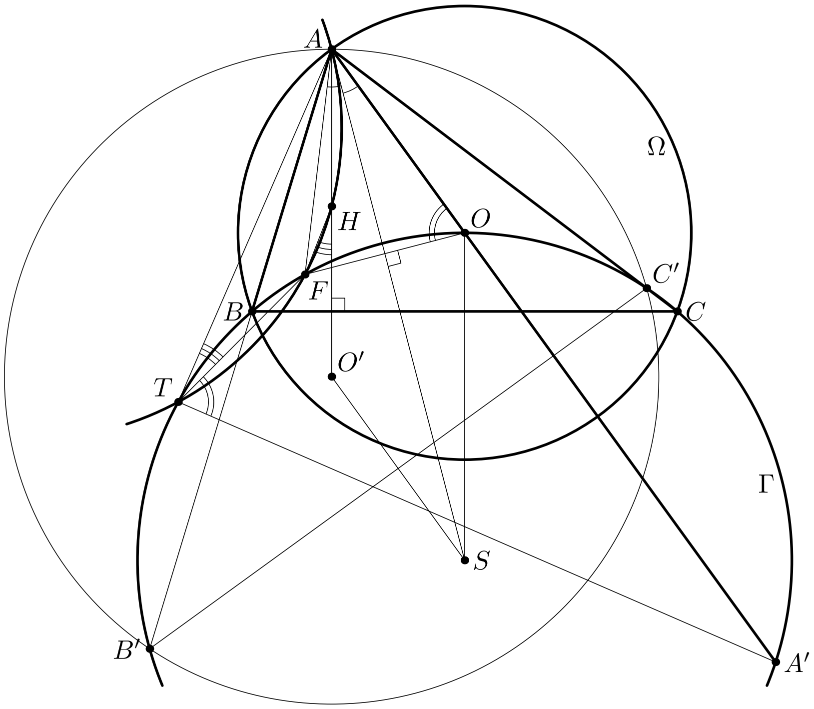
Finally, the circle $(AHFT)$ also passes through K, as well as through the reflection of O in D.



Solution 3. Let S be the centre of Γ and let circle AFH meet Γ again at $T \neq F$. In the sequel, angles are all orientated.

We are to prove that $\angle ATF + \angle FTA' = 90^\circ$. To this end, note the equivalences below:

$$\begin{aligned}
\angle ATF + \angle FTA' = 90^\circ &\Leftrightarrow \angle AHF + \angle FOA = 90^\circ \Leftrightarrow \angle AHF = \angle OAS \quad (\text{as } AS \perp OF) \\
&\Leftrightarrow \angle AHF = \angle SAF \Leftrightarrow AS \text{ is tangent to circle } AFH \\
&\Leftrightarrow \frac{AH}{\sin \angle HAS} = \frac{AF}{\sin \angle FAS} \\
&\Leftrightarrow \frac{AH}{AO} = \frac{\sin \angle HAS}{\sin \angle SAO}. \qquad (*)
\end{aligned}$$



To prove (∗), let AB and AC meet Γ again at B' and C', respectively. An easy angle chase shows that O is the orthocentre of triangle $AB'C'$.

As triangles ABC and $AC'B'$ are similar, AH passes through the centre O' of circle $AB'C'$; and as circles $AB'C'$ and $BOCC'B'$ are reflections of one another in $B'C'$ and $AO'SO$ is a parallelogram, it follows that

$$\frac{\sin\angle HAS}{\sin\angle SAO} = \frac{\sin\angle O'AS}{\sin\angle ASO'} = \frac{O'S}{AO'} = \frac{AO}{AO'}. \qquad (**)$$

Further on, as triangles ABC and $AC'B'$ are similar, (∗∗) implies equal corresponding length ratios, so $AO/AO' = AH/AO$. This establishes (∗) and concludes the solution.

Remark. Relation (∗) is equivalent to AS being the A-symmedian of triangle AOH. This might very well be known and can actually be proved in several different ways.

Problem 6. Let k and m be integers greater than 1. Consider k pairwise disjoint sets $S_1, S_2, \ldots, S_k$; each of these sets has exactly $m+1$ elements, one of which is red and the other m are all blue. Let $\mathcal{F}$ be the family of all subsets F of $S_1 \cup S_2 \cup \ldots \cup S_k$ such that, for every i, the intersection $F \cap S_i$ is monochromatic; the empty set is monochromatic. Determine the largest possible cardinality of a subfamily $\mathcal{G} \subseteq \mathcal{F}$, no two sets of which are disjoint.

Russia, Andrey Kupavskii and Maksim Turevskii

Solution. The required maximum is $2^{m-1}(2^m+1)^{k-1}$ and is achieved if, for instance, $\mathcal{G}$ consists of all sets in $\mathcal{F}$ containing a fixed blue element.

We now prove that $|\mathcal{G}| \leq 2^{m-1}(2^m+1)^{k-1}$ for any $\mathcal{G}$ satisfying the conditions in the statement. For convenience, write $M = 2^m + 1$. Let r_i denote the red element of S_i, and let B_i be the set of blue elements in S_i.

For every subset $X_i \subset B_i$ and every $j \in \mathbb{Z}_M$, define the sets

$$T_{X_i,j} = \begin{cases} \{r_i\}, & \text{if } j = 0; \\ X_i, & \text{if } j \neq 0 \text{ and } j \text{ is even (considered as a number in } [1, M-1]); \\ B_i \setminus X_i, & \text{if } j \neq 0 \text{ and } j \text{ is odd (considered as a number in } [1, M-1]). \end{cases}$$

Note that, for every i and every j, the sets $T_{X_i,j}$ and $T_{X_i,j+1}$ are disjoint. Now, for every sets $X_i \subset B_i$ and every elements $j_i \in \mathbb{Z}_M$, $i = 1, 2, \ldots, k$, denote

$$F(X_1, X_2, \ldots, X_k, j_1, j_2, \ldots, j_k) = \bigcup_{i=1}^{k} T_{X_i,j_i}. \qquad (*)$$

Claim. Every set $F \in \mathcal{F}$ has exactly 2^{mk} representations of the form $(*)$.

Proof. Set $F_i = F \cap S_i$. If $F_i = \{r_i\}$, then there are 2^m possible choices for X_i, and one should necessarily have $j_i = 0$. Otherwise, there are only two possible choices for X_i, namely $X_i = F_i$ and $X_i = B_i \setminus F_i$, and for each of them there are 2^{m-1} possible choices for j_i. So, whatever F, there are 2^m possible choices for each pair (X_i, j_i) all of which can be made independently, whence a total of 2^{mk} possible tuples $(X_1, X_2, \ldots, X_k, j_1, j_2, \ldots, j_k)$. This proves the Claim.

The Claim implies that each $F \in \mathcal{F}$ has the same number of representations of the form $(*)$. Thus, it suffices to show that, among all $N = 2^{km}(2^m+1)^k$ tuples $(X_1, X_2, \ldots, X_k, j_1, j_2, \ldots, j_k)$, at most $\dfrac{2^{m-1}}{2^m+1}N$ satisfy $F(X_1, X_2, \ldots, X_k, j_1, j_2, \ldots, j_k) \in \mathcal{G}$.

To this end, split all these tuples into length M cycles

$$\big(F(X_1, X_2, \ldots, X_k, j_1, j_2, \ldots, j_k), F(X_1, X_2, \ldots, X_k, j_1+1, j_2+1, \ldots, j_k+1), \ldots, \\ F(X_1, X_2, \ldots, X_k, j_1+M-1, j_2+M-1, \ldots, j_k+M-1)\big),$$

and note that any two adjacent sets of a cycle are disjoint. Hence each cycle contains at most $\lfloor M/2 \rfloor = 2^{m-1}$ sets from $\mathcal{G}$. This provides the desired upper bound and completes the solution.