THE 1991 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let G be the centroid of triangle ABC and M be the midpoint of BC. Let X be on AB and Y on AC such that the points X, Y, and G are collinear and XY and BC are parallel. Suppose that XC and GB intersect at Q and YB and GC intersect at P. Show that triangle MPQ is similar to triangle ABC.

Question 2

Suppose there are 997 points given in a plane. If every two points are joined by a line segment with its midpoint coloured in red, show that there are at least 1991 red points in the plane. Can you find a special case with exactly 1991 red points?

Question 3

Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \cdots + \frac{a_n^2}{a_n+b_n} \geq \frac{a_1+a_2+\cdots+a_n}{2} \, .$$

Question 4

During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of n for which eventually, perhaps after many rounds, all children will have at least one candy each.

Question 5

Given are two tangent circles and a point P on their common tangent perpendicular to the lines joining their centres. Construct with ruler and compass all the circles that are tangent to these two circles and pass through the point P.

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Lee Yiu Sing

1 Solution

Problem 1. As $XY||BC$, by Ceva's theorem, AM, BY and CX are concurrent. By sine law, $\dfrac{BP}{sin\angle BCP} = \dfrac{BC}{sin\angle BPC}$ and

$\dfrac{PY}{sin\angle ACP} = \dfrac{CY}{sin\angle CPY} = \dfrac{\frac{1}{3}AC}{sin\angle BPC}$. Hence, $\dfrac{BP}{PY} = \dfrac{BCsin\angle BCP}{\frac{1}{3}sin\angle ACP}$. Let M be the midpoint of AB. Using the similar arguments, we have
$\dfrac{BM}{AM} = 1 = \dfrac{BCsin\angle BCP}{ACsin\angle ACP}$. Hence, $\dfrac{BP}{PY} = 3$. From Ceva's theorem, it follows that $QP||BC||XY$. Hence, $\dfrac{BQ}{QC} = 3$. Let N be the midpoint of AC. It follows that Q is the midpoint of BN. Hence, $QM||AC$. Using similar arguments $PM||AB$, it follows that ΔABC and ΔMPQ have parallel sides. Therefore, they are similar. ■

Problem 2. Consider a rectangular coordinate system on the set of points such that all the points have distinct y-coordinates. Denote these points $y_1, y_2, \cdots, y_{997}$ in increasing order. Let A_i be the point with y-coordinate y_i.
Now, consider the midpoints of $\overline{A_iA_{i+1}}$ for $1 \leq i \leq 996$. There are 996 such midpoints. By order all of them have distinct y-coordinate. Hence, they do not coincide. Consider the midpoints of $\overline{A_iA_{i+2}}$ for $1 \leq i \leq 995$. These points cannot coincide with any of the former midpoints as all y_i are distinct. This yeids $996 + 995 = 1991$ distinct midpoints.

Problem 3. **Solution 1.** By Cauchy-Schwarz Inequality, we have

$$(\sum_{i=1}^{n} a_i)^2 \leq \sum_{i=1}^{n}(\frac{a_i}{\sqrt{a_i+b_i}})^2 \sum_{i=1}^{n}(\sqrt{a_i+b_i})^2$$

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \frac{a_i^2}{a_i+b_i} \sum_{i=1}^{n}(a_i+b_i)$$

By dividing each side by $\sum_{i=1}^{n}(a_i+b_i)$ each side and
$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$, it follows that

$$\frac{a_1^2}{a_1+b_1}+\frac{a_2^2}{a_2+b_2}+\cdots+\frac{a_n^2}{a_n+b_n}\geq\frac{a_1+a_2+\cdots+a_n}{2}$$

which completes the proof. ■

Solution 2. By Titu's Lemma, we have

$$\frac{a_1^2}{a_1+b_1}+\frac{a_2^2}{a_2+b_2}+\cdots+\frac{a_n^2}{a_n+b_n}\geq\frac{(a_1+a_2+\cdots+a_n)^2}{a_1+a_2+\cdots+a_n+b_1+b_2+\cdots+b_n}$$

Since $a_1+a_2+\cdots+a_n=b_1+b_2+\cdots+b_n$, we get

$$\frac{a_1^2}{a_1+b_1}+\frac{a_2^2}{a_2+b_2}+\cdots+\frac{a_n^2}{a_n+b_n}\geq\frac{a_1+a_2+\cdots+a_n}{2}$$

which completes the proof. ■

Problem 4. Denote each child the number of the set $0,1,2,\cdots,n-1$ in closewise direction. The first child which receive a candy is 1. The k-th is the remainder when $\dfrac{k(k+1)}{2}$ is divided by n.

If n is an odd number, then $\dfrac{(n+1)(n+2)}{2}\equiv 1\ (mod\ n)$. This implies that $(n+1)$-th child is the first one so that the teacher will take the same steps. It is obvious that the $(n-1)$-th and n-th children are the same child. Therefore, there is one child who didn't take a candy as there is one who took two on the first round.

Consider the case that n is an even number. Let $C_{(1,i)}=\{i,i+\dfrac{n}{2}\}$, for each $i\in\{1,2,\ldots,\dfrac{n}{2}\}$. Consider a circle with $C_{(1,i)}$ written in clockwise direction. Note that the steps taken on the circle with n children will be seen as if the teacher had $\dfrac{n}{2}$ children on the latter circle.. Hence, $\dfrac{n}{2}$ must be an even number. Otherwise, there exist a set $C_{(1,i)}$ which wasn't awarded with a candy.

If $\dfrac{n}{2}$ is an even number, call $C_{(2,i)}=\{C_{(1,i)},C_{(1,i+\frac{n}{4})}\}$. The same argument can be taken. Following this, the only possibility for each child to get a candy is $n=2^k$ where $k\in\mathbb{Z}$ and $k\geq 0$.

Problem 5. Let C be the intersection of two circles. Let l be one of the common external tangents. The circle we are searching for is the inverse of the line l with respect to circle (P,CP). We can obtain one more circle if we use the other external common tangent.