

The 14th Romanian Master of Mathematics Competition

Day 1: Wednesday, March 1st, 2023, Bucharest

Language: English

Problem 1. Determine all prime numbers p and all positive integers x and y satisfying

$$x^3 + y^3 = p(xy + p).$$

Problem 2. Fix an integer $n \geq 3$. Let \mathcal{S} be a set of n points in the plane, no three of which are collinear. Given different points A, B, C in \mathcal{S} , the triangle ABC is *nice for AB* if $\text{Area}(ABC) \leq \text{Area}(ABX)$ for all X in \mathcal{S} different from A and B . (Note that for a segment AB there could be several nice triangles.) A triangle is *beautiful* if its vertices are all in \mathcal{S} and it is nice for at least two of its sides.

Prove that there are at least $\frac{1}{2}(n - 1)$ beautiful triangles.

Problem 3. Let $n \geq 2$ be an integer, and let f be a $4n$ -variable polynomial with real coefficients. Assume that, for any $2n$ points $(x_1, y_1), \dots, (x_{2n}, y_{2n})$ in the Cartesian plane, $f(x_1, y_1, \dots, x_{2n}, y_{2n}) = 0$ if and only if the points form the vertices of a regular $2n$ -gon in some order, or are all equal.

Determine the smallest possible degree of f .

(Note, for example, that the degree of the polynomial

$$g(x, y) = 4x^3y^4 + yx + x - 2$$

is 7 because $7 = 3 + 4$.)

Each problem is worth 7 marks.

Time allowed: $4\frac{1}{2}$ hours.

Problem 1. Determine all prime numbers p and all positive integers x and y satisfying $x^3 + y^3 = p(xy + p)$.

SERBIA, DUSHAN DJUKITCH

Solution 1. Up to a swap of the first two entries, the only solutions are $(x, y, p) = (1, 8, 19)$, $(x, y, p) = (2, 7, 13)$ and $(x, y, p) = (4, 5, 7)$. The verification is routine.

Set $s = x + y$. Rewrite the equation in the form $s(s^2 - 3xy) = p(p + xy)$, and express xy :

$$xy = \frac{s^3 - p^2}{3s + p}. \quad (*)$$

In particular,

$$s^2 \geq 4xy = \frac{4(s^3 - p^2)}{3s + p},$$

or

$$(s - 2p)(s^2 + sp + 2p^2) \leq p^2 - p^3 < 0,$$

so $s < 2p$.

If $p \mid s$, then $s = p$ and $xy = p(p - 1)/4$ which is impossible for $x + y = p$ (the equation $t^2 - pt + p(p - 1)/4 = 0$ has no integer solutions).

If $p \nmid s$, rewrite $(*)$ in the form

$$27xy = (9s^2 - 3sp + p^2) - \frac{p^2(p + 27)}{3s + p}.$$

Since $p \nmid s$, this could be integer only if $3s + p \mid p + 27$, and hence $3s + p \mid 27 - s$.

If $s \neq 9$, then $|3s - 27| \geq 3s + p$, so $27 - 3s \geq 3s + p$, or $27 - p \geq 6s$, whence $s \leq 4$. These cases are ruled out by hand.

If $s = x + y = 9$, then $(*)$ yields $xy = 27 - p$. Up to a swap of x and y , all such triples (x, y, p) are $(1, 8, 19)$, $(2, 7, 13)$, and $(4, 5, 7)$.

Solution 2. Set again $s = x + y$. It is readily checked that $s \leq 8$ provides no solutions, so assume $s \geq 9$. Notice that $x^3 + y^3 = s(x^2 - xy + y^2) \geq \frac{1}{4}s^3$ and $xy \leq \frac{1}{4}s^2$. The condition in the statement then implies $s^2(s - p) \leq 4p^2$, so $s < p + 4$.

Notice that p divides one of s and $x^2 - xy + y^2$. The case $p \mid s$ is easily ruled out by the condition $s < p + 4$: The latter forces $s = p$,

so $x^2 - xy + y^2 = xy + p$, i. e., $(x - y)^2 = p$, which is impossible.

Hence $p \mid x^2 - xy + y^2$, so $x^2 - xy + y^2 = kp$ and $xy + p = ks$ for some positive integer k , implying

$$s^2 + 3p = k(3s + p). \quad (**)$$

Recall that $p \nmid s$ to infer that $3k \equiv s \pmod{p}$. We now present two approaches.

1st Approach. Write $3k = s + mp$ for some integer m and plug $k = \frac{1}{3}(s + mp)$ into $(**)$ to get $s = (9 - mp)/(3m + 1)$. The condition $s \geq 9$ then forces $m = 0$, so $s = 9$, in which case, up to a swap of the first two entries, the solutions turn out to be $(x, y, p) = (1, 8, 19)$, $(x, y, p) = (2, 7, 13)$ and $(x, y, p) = (4, 5, 7)$.

2nd Approach. Notice that $k = \frac{s^2 + 3p}{3s + p} = 3 + \frac{s(s-9)}{3s+p} \leq 3 + \frac{1}{3}(s-9) = \frac{1}{3}s \leq \frac{1}{3}(p+3)$, since $s < p+4$. Hence $3k \leq p + 3$, and the congruence $3k \equiv s \pmod{p}$ then forces either $3k = s - p$ or $3k = s$.

The case $3k = s - p$ is easily ruled out: Otherwise, $(**)$ boils down to $2s + p + 9 = 0$, which is clearly impossible.

Finally, if $3k = s$, then $(**)$ reduces to $s = 9$. In this case, up to a swap of the first two entries, the only solutions are $(x, y, p) = (1, 8, 19)$, $(x, y, p) = (2, 7, 13)$ and $(x, y, p) = (4, 5, 7)$.

Remark. The upper bound for k can equally well be established by considering the variation of $(s^2 + 3p)/(3s + p)$ for $1 \leq s \leq p + 3$. The maximum is achieved at $s = p + 3$:

$$\begin{aligned} \max_{1 \leq s \leq p+3} \frac{s^2 + 3p}{3s + p} &= \frac{(p+3)^2 + 3p}{3(p+3) + p} = \frac{p^2 + 9p + 9}{4p + 9} \\ &< \frac{p + 4}{3}, \end{aligned}$$

so the integer $3k \leq p + 3 < p + 9 \leq p + s$ and the remainder of the proof now goes along the few final lines above.

Problem 2. Fix an integer $n \geq 3$. Let \mathcal{S} be a set of n points in the plane, no three of which are collinear. Given different points A, B, C in \mathcal{S} , the triangle ABC is *nice for* AB if $\text{Area}(ABC) \leq \text{Area}(ABX)$ for all X in \mathcal{S} different from A and B . (Note that for a segment AB there could be several nice triangles.) A triangle is *beautiful* if its vertices are all in \mathcal{S} and it is nice for at least two of its sides.

Prove that there are at least $\frac{1}{2}(n - 1)$ beautiful triangles.

BULGARIA, ALEXANDER IVANOV

Solution. For convenience, a triangle whose vertices all lie in \mathcal{S} will be referred to as a triangle in \mathcal{S} . The argument hinges on the following observation:

Given any partition of \mathcal{S} , amongst all triangles in \mathcal{S} with at least one vertex in each part, those of minimal area are all adequate.

Indeed, amongst the triangles under consideration, one of minimal area is suitable for both sides with endpoints in different parts.

We now present two approaches for the lower bound.

1st Approach. By the above observation, the 3-uniform hypergraph of adequate triangles is connected. It is a well-known fact that such a hypergraph has at least $\frac{1}{2}(n - 1)$ hyperedges, whence the required lower bound

2nd Approach. For a partition $\mathcal{S} = \mathcal{A} \sqcup \mathcal{B}$, an area minimising triangle as above will be called $(\mathcal{A}, \mathcal{B})$ -minimal. Thus, $(\mathcal{A}, \mathcal{B})$ -minimal triangles are all adequate.

Consider now a partition of $\mathcal{S} = \mathcal{A} \sqcup \mathcal{B}$, where $|\mathcal{A}| = 1$. Choose an $(\mathcal{A}, \mathcal{B})$ -minimal triangle and add to \mathcal{A} its vertices from \mathcal{B} to obtain a new partition also written $\mathcal{S} = \mathcal{A} \sqcup \mathcal{B}$. Continuing, choose an $(\mathcal{A}, \mathcal{B})$ -minimal triangle and add to \mathcal{A} its vertices/vertex from \mathcal{B} and so on and so forth all the way down for at least another $\frac{1}{2}(n - 5)$ steps — this works at least as many times, since at each step, \mathcal{B} loses at most two points. Clearly, each step provides a new adequate triangle, so the overall number of adequate triangles is at least $\frac{1}{2}(n - 1)$, as required.

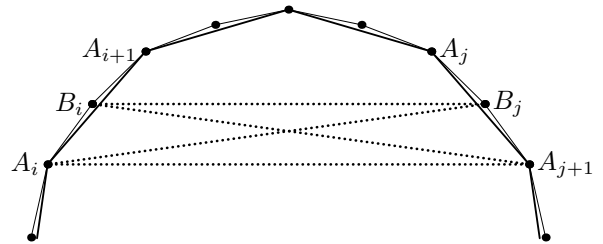
Remark. In fact, $\lfloor n/2 \rfloor$ is the smallest possible number of adequate triangles, as shown by the configurations described below.

Let first $n = 2k - 1$. Consider a regular n -gon

$\mathcal{P} = A_1 A_2 \dots A_n$. Choose a point B_i on the perpendicular bisector of $A_i A_{i+1}$ outside \mathcal{P} and sufficiently close to the segment $A_i A_{i+1}$. We claim that there are exactly $k - 1 = \lfloor n/2 \rfloor$ adequate triangles in the set

$$\mathcal{S} = \{A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_{k-1}\}.$$

Notice here that the arc $A_1 A_2 \dots A_k$ is less than half of the circumcircle of \mathcal{P} , so the angles $\angle A_u A_v A_w$, $1 \leq u < v < w \leq k$, are all obtuse.



To prove the claim, list the suitable triangles for each segment.

For segments $A_i A_{i+1}$, $A_i B_i$, and $B_i A_{i+1}$, it is $A_i B_i A_{i+1}$.

For segment $A_i A_{j+1}$, $j \geq i + 1$, those are $A_i B_i A_{j+1}$ and $A_i B_j A_{j+1}$.

For segment $A_i B_j$, $j \geq i + 1$, it is $A_i B_i B_j$.

For segment $B_i A_{j+1}$, $j \geq i + 1$, it is $B_i B_j A_{j+1}$.

For segment $B_i B_j$, $i < j$, those are $B_i A_{i+1} B_j$ and $B_i A_j B_j$.

It is easily seen that the only triangles occurring twice are $A_i B_i A_{i+1}$, hence they are the only adequate triangles.

For $n = 2k - 2$, just remove A_k from the above example. This removes the adequate triangle $A_{k-1} B_{k-1} A_k$ and provides only one new such instead, namely, $B_{k-2} A_{k-1} B_{k-1}$. Consequently, there are exactly $k - 1 = \lfloor n/2 \rfloor$ adequate triangles in the set

$$\mathcal{S} = \{A_1, A_2, \dots, A_{k-1}, B_1, B_2, \dots, B_{k-1}\}.$$

Problem 3. Let $n \geq 2$ be an integer, and let f be a $4n$ -variable polynomial with real coefficients. Assume that, for any $2n$ points $(x_1, y_1), \dots, (x_{2n}, y_{2n})$ in the plane, $f(x_1, y_1, \dots, x_{2n}, y_{2n}) = 0$ if and only if the points form the vertices of a regular $2n$ -gon in some order, or are all equal.

Determine the smallest possible degree of f .

USA

Solution. The smallest possible degree is $2n$. In what follows, we will frequently write $A_i = (x_i, y_i)$, and abbreviate $P(x_1, y_1, \dots, x_{2n}, y_{2n})$ to $P(A_1, \dots, A_{2n})$ or as a function of any $2n$ points.

Suppose that f is valid. First, we note a key property:

Claim (Sign of f). f attains wither only nonnegative values, or only nonpositive values.

Proof. This follows from the fact that the zero-set of f is very sparse: if f takes on a positive and a negative value, we can move A_1, \dots, A_{2n} from the negative value to the positive value without ever having them form a regular $2n$ -gon — a contradiction. \square

The strategy for showing $\deg f \geq 2n$ is the following. We will animate the points A_1, \dots, A_{2n} linearly in a variable t ; then $g(t) = f(A_1, \dots, A_{2n})$ will have degree at most $\deg f$ (assuming it is not zero). The claim above then establishes that any root of g must be a multiple root, so if we can show that there are at least n roots, we will have shown $\deg g \geq 2n$, and so $\deg f \geq 2n$.

Geometrically, our goal is to exhibit $2n$ linearly moving points so that they form a regular $2n$ -gon a total of n times, but not always form one.

We will do this as follows. Draw n mirrors through the origin, as lines making angles of $\frac{\pi}{n}$ with each other. Then, any point P has a total of $2n$ reflections in the mirrors, as shown below for $n = 5$. (Some of these reflections may overlap.)

Draw the n angle bisectors of adjacent mirrors. Observe that the reflections of P form a regular $2n$ -gon if and only if P lies on one of the bisectors.

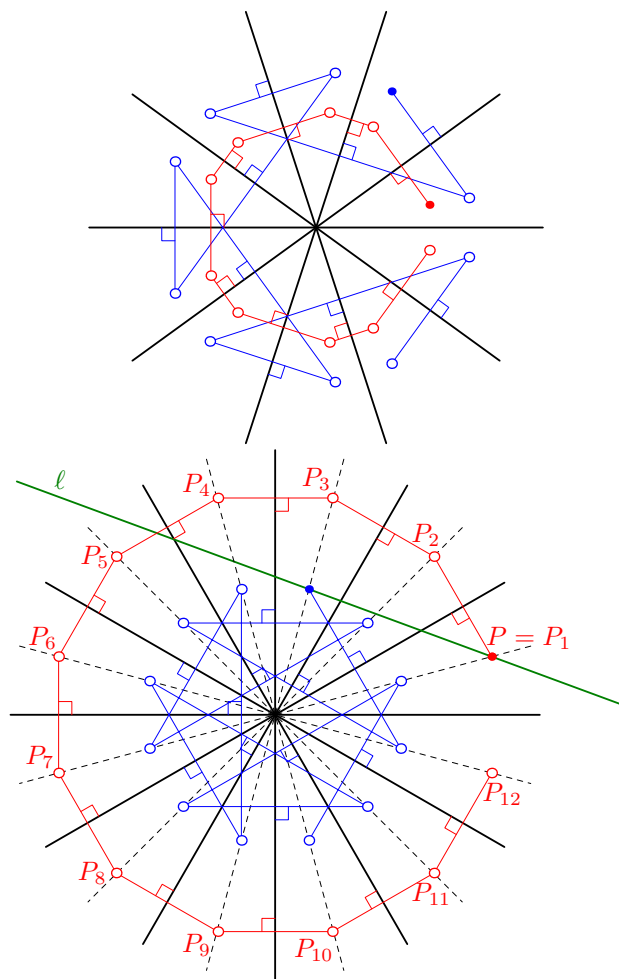
We will animate P on any line ℓ which intersects all n bisectors (but does not pass through the origin), and let P_1, \dots, P_{2n} be its reflections. Clearly, these are also all linearly animated, and because of the reasons above, they will form a regular $2n$ -gon exactly n times, when ℓ meets each bisector. So this establishes $\deg f \geq 2n$ for the reasons described previously.

Now we pass to constructing a polynomial f of degree $2n$ having the desired property. First of all, we will instead find a polynomial g which has this property, but only when points with sum zero are input. This still solves the problem, because then

we can choose

$$f(A_1, A_2, \dots, A_{2n}) = g(A_1 - \bar{A}, \dots, A_{2n} - \bar{A}),$$

where \bar{A} is the centroid of A_1, \dots, A_{2n} . This has the upshot that we can now always assume $A_1 + \dots + A_{2n} = 0$, which will simplify the ensuing discussion.



We will now construct a suitable g as a sum of squares. This means that, if we write $g = g_1^2 + g_2^2 + \dots + g_m^2$, then $g = 0$ if and only if $g_1 = \dots = g_m = 0$, and that if their degrees are d_1, \dots, d_m , then g has degree at most $2 \max(d_1, \dots, d_m)$.

Thus, it is sufficient to exhibit several polynomials, all of degree at most n , such that $2n$ points with zero sum are the vertices of a regular $2n$ -gon if and only if the polynomials are all zero at those points.

First, we will impose the constraints that all $|A_i|^2 = x_i^2 + y_i^2$ are equal. This uses multiple degree 2 constraints.

Now, we may assume that the points A_1, \dots, A_{2n} all lie on a circle with centre 0, and $A_1 + \dots + A_{2n} = 0$. If this circle has radius 0, then all A_i coincide, and we may ignore this case.

Otherwise, the circle has positive radius. We will use the following lemma.

Lemma. Suppose that a_1, \dots, a_{2n} are complex numbers of the same non-zero magnitude, and suppose that $a_1^k + \dots + a_{2n}^k = 0$, $k = 1, \dots, n$. Then a_1, \dots, a_{2n} form a regular $2n$ -gon centred at the origin. (Conversely, this is easily seen to be sufficient.)

Proof. Since all the hypotheses are homogeneous, we may assume (mostly for convenience) that a_1, \dots, a_{2n} lie on the unit circle. By Newton's sums, the k -th symmetric sums of a_1, \dots, a_{2n} are all zero for k in the range $1, \dots, n$.

Taking conjugates yields $a_1^{-k} + \dots + a_{2n}^{-k} = 0$, $k = 1, \dots, n$. Thus, we can repeat the above logic to obtain that the k -th symmetric sums of $a_1^{-1}, \dots, a_{2n}^{-1}$ are also all zero for $k = 1, \dots, n$. However, these are simply the $(2n - k)$ -th symmetric sums of a_1, \dots, a_{2n} (divided by $a_1 \dots a_{2n}$), so the first $2n - 1$ symmetric sums of a_1, \dots, a_{2n} are all zero. This implies that a_1, \dots, a_{2n} form a regular $2n$ -gon centred at the origin. \square

We will encode all of these constraints into our polynomial. More explicitly, write $a_r = x_r + y_r i$; then the constraint $a_1^k + \dots + a_{2n}^k = 0$ can be expressed as $p_k + q_k i = 0$, where p_k and q_k are real polynomials in the coordinates. To incorporate this, simply impose the constraints $p_k = 0$ and $q_k = 0$; these are conditions of degree $k \leq n$, so their squares are all of degree at most $2n$.

To recap, taking the sum of squares of all of these constraints gives a polynomial f of degree at most $2n$ which works whenever $A_1 + \dots + A_{2n} = 0$. Finally, the centroid-shifting trick gives a polynomial which works in general, as wanted.

Remark 1. Here is a more detailed approach of the mirror-reflection argument. Let $re^{i\theta}$ be the polar representation of the point P . The polar representations of its mirrored images are then

$$\begin{aligned} re^{i\theta}, \quad re^{-i\theta}, \quad re^{i\left(\frac{2\pi}{n} + \theta\right)}, \quad re^{i\left(\frac{2\pi}{n} - \theta\right)}, \\ \dots, \quad re^{i\left(\frac{2(n-1)\pi}{n} + \theta\right)}, \quad re^{i\left(\frac{2(n-1)\pi}{n} - \theta\right)}. \end{aligned}$$

Clearly, they are all linear with respect to P and lie on the circle of radius r centred at the origin. As listed above, the $2n$ images are not necessarily in

circular order around the circle. For convenience, assume $0 \leq \theta \leq \frac{\pi}{n}$, so the list now displays them in circular order. These images form the vertices of a regular $2n$ -gon if and only if the angle between every two consecutive terms in the list (read circularly) is $\frac{\pi}{n}$. This is clearly the case if and only if $\theta = \frac{\pi}{2n}$. Consequently, the images are the vertices of a regular $2n$ -gon if and only if P lies on the internal bisector of the angle formed by some pair of consecutive mirrors.

Remark 2. We sketch here some versions of the arguments in the solution above.

To show that $\deg f \geq 2n$, we use the same constancy of sign claim and the convention that the polynomial is a function of points (= pairs of coordinates) A_1, A_2, \dots, A_{2n} . Assume that the values of f are all non-negative.

Write $B(\phi) = (\cos \phi, \sin \phi)$. Choose a substitution $A_{2i-1} = B\left(\left(2i-1\right)\frac{\pi}{n} + \phi\right)$ and $A_{2i} = B\left(2i\frac{\pi}{n} - \phi\right)$, $i = 1, 2, \dots, n$. Notice that the coordinates of the points A_1, A_2, \dots, A_{2n} are all linear functions in $c = \cos \phi$ and $s = \sin \phi$, so, substituting these expressions into f , we get a polynomial $g(c, s)$ with $\deg g \leq \deg f$.

Now, the values of g are all non-negative (each being one of f), and, on the circle $c^2 + s^2 = 1$, it vanishes at exactly $2n$ points, namely, $(c, s) = \left(\cos \frac{\pi}{n} k, \sin \frac{\pi}{n} k\right)$, $k = 1, \dots, 2n$. We show that these properties already yield $\deg g \geq 2n$.

Obviously, if $g(c, s)$ possesses the properties listed above, then so does $g(c, -s)$, and hence so does $\bar{g}(c, s) = g(c, s) + g(c, -s)$.

The polynomial \bar{g} is even in s , so it in fact depends only on s^2 , and we may plug $s^2 = 1 - c^2$ into it, to obtain a polynomial $h(c)$ with $\deg h \leq \deg g$ which is non-negative on $[-1, 1]$ and vanishes on this segment exactly at $c = \cos \frac{\pi}{n} k$. These are $n + 1$ such points, and, except $c = \pm 1$, they should all be roots of h of even multiplicity, due to sign conservation. All in all, this provides $2n$ roots of h , counted with multiplicity, hence $\deg f \geq \deg g \geq \deg h \geq 2n$, as desired.

For a bit alternative construction of a suitable f , one may notice that the Lemma in the above solution can be changed to impose vanishing of the elementary symmetric polynomials $\sigma_i(a_1, a_2, \dots, a_{2n})$, $i = 1, 2, \dots, n$, instead of Newton sums. Indeed, if the σ_i all vanish, then so do the polynomials

$$\sigma_i(\bar{a}_1, a_2, \dots, \bar{a}_{2n}) = \frac{|a_1|^{2i} \sigma_{2n-i}(a_1, a_2, \dots, a_{2n})}{\bar{a}_1 \bar{a}_2 \dots \bar{a}_{2n}},$$

so $\sigma_i(a_1, \dots, a_{2n})$ also vanishes for $i = n + 1, \dots, 2n - 1$. Hence a_1, a_2, \dots, a_{2n} are the roots of $z^{2n} - |a_1|^n$, as desired.

The 14th Romanian Master of Mathematics Competition

Day 2: Thursday, March 2nd, 2023, Bucharest

Language: English

Problem 4. Given an acute triangle ABC , let H and O be its orthocentre and circumcentre, respectively. Let K be the midpoint of the line segment AH . Also let ℓ be a line through O , and let P and Q be the orthogonal projections of B and C onto ℓ , respectively.

Prove that $KP + KQ \geq BC$.

Problem 5. Let $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ be non-constant polynomials with real coefficients such that $P(Q(x)) = R(S(x))$. Suppose that the degree of $P(x)$ is divisible by the degree of $R(x)$.

Prove that there is a polynomial $T(x)$ with real coefficients such that

$$P(x) = R(T(x)).$$

Problem 6. Let r, g, b be non-negative integers. Let Γ be a connected graph on $r + g + b + 1$ vertices. The edges of Γ are each coloured red, green or blue. It turns out that Γ has

- a spanning tree in which exactly r of the edges are red,
- a spanning tree in which exactly g of the edges are green and
- a spanning tree in which exactly b of the edges are blue.

Prove that Γ has a spanning tree in which exactly r of the edges are red, exactly g of the edges are green and exactly b of the edges are blue.

(A *spanning tree* of Γ is a graph which has the same vertices as Γ , with edges which are also edges of Γ , for which there is exactly one path between each pair of different vertices.)

Each problem is worth 7 marks.

Time allowed: $4\frac{1}{2}$ hours.

Problem 4. Given a triangle ABC , let H and O be its orthocentre and circumcentre, respectively. Let K be the midpoint of the line segment AH . Let further ℓ be a line through O , and let P and Q be the orthogonal projections of B and C onto ℓ , respectively. Prove that $KP + KQ \geq BC$.

RUSSIA, VASILY MOKIN

Solution 1. Fix the origin at O and the real axis along ℓ . A lower case letter denotes the complex coordinate of the corresponding point in the configuration. For convenience, let $|a| = |b| = |c| = 1$.

Clearly, $k = a + \frac{1}{2}(b + c)$, $p = a + \frac{1}{2}(b + \frac{1}{b})$ and $q = a + \frac{1}{2}(c + \frac{1}{c})$.

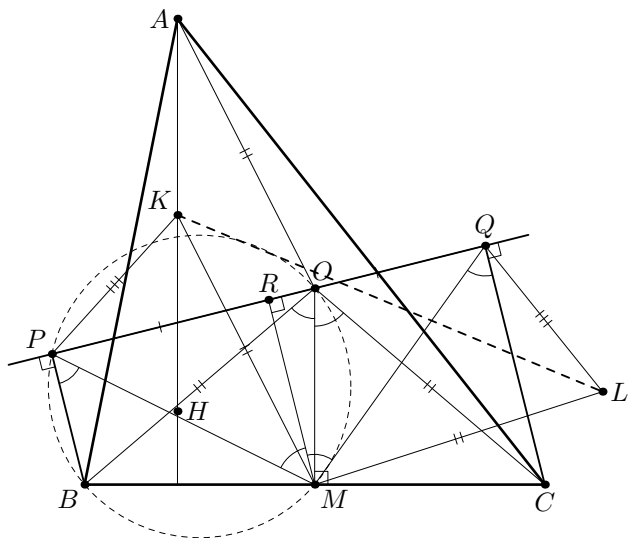
Then $|k - p| = |a + \frac{1}{2}(c - \frac{1}{b})| = \frac{1}{2}|2ab + bc - 1|$, since $|b| = 1$.

Similarly, $|k - q| = \frac{1}{2}|2ac + bc - 1|$, so, since $|a| = 1$,

$$\begin{aligned} |k - p| + |k - q| &= \frac{1}{2}|2ab + bc - 1| + \frac{1}{2}|2ac + bc - 1| \\ &\geq \frac{1}{2}|2a(b - c)| = |b - c|, \end{aligned}$$

as required.

Solution 2. Let M be the midpoint of BC , and let R be the projection of M onto ℓ . In other words, R is the midpoint of PQ . Since $\angle BPO = \angle BMO = 90^\circ$, the points B, P, O , and M are concyclic, so $\angle(OM, OB) = \angle(PM, PB) = \angle(PM, MR)$, so the right triangles MRP and OMB are similar and have different orientation. Similarly, the triangles MRQ and OMC are similar and have different orientation, hence so are the triangles OBC and MPQ .



Recall that $\overrightarrow{AH} = 2\overrightarrow{OM}$, so $\overrightarrow{OM} = \overrightarrow{AK}$. Hence $AOMK$ is a parallelogram, so $MK = OA = OB = OC$.

Consider the rotation through $\angle(\overrightarrow{OC}, \overrightarrow{OB})$ about M . It maps P to Q ; let it map K to some point L . Then $MK = ML = OB = OC$ and $\angle LMK = \angle BOC$, so the triangles OBC and MKL are congruent. Hence $BC = KL \leq KQ + LQ = KQ + KP$, as required.

Solution 3. Let $\alpha = \angle(PB, BC) = \angle(QC, BC)$. Since P lies on the circle of diameter OB , $\angle(OP, OM) = \alpha$. Since also Q lies on the circle of diameter OC , it immediately follows that

$MP = MQ = R \sin \alpha$ by sine theorem in triangles $\triangle OPM$ and $\triangle OQM$.

Because PQ is the projection of BC on line ℓ , it follows that $PQ = BC \sin \alpha$. Just like in the first solution, $KM = AO = R$ (the circumradius of triangle $\triangle ABC$).

Now apply Ptolemy's inequality for the quadrilateral $KPMQ$: $KP \cdot MQ + KQ \cdot MP \geq PQ \cdot KM$, and now substitute the relations from above, leading to

$$R \sin \alpha (KP + KQ) \geq R \sin \alpha \cdot BC,$$

which is precisely the conclusion whenever $\sin \alpha \neq 0$. The case when $\sin \alpha = 0$ can be treated either directly, or via a limit argument.

Solution 4. Denote by R and O the circumradius and the circumcentre of triangle ABC , respectively. As in Solution 1, we see that $MK = R$.

Assume now that ℓ is fixed, while A moves along the fixed circle (ABC) . Then K will move along a circle centred at M with radius R . We must show that for each point K on this circle we have $BC \leq KP + KQ$. In doing so, we prove that the afore-mentioned circle contains an ellipse with foci at Q and P with distance BC .

Let S be the foot of the perpendicular from M to PQ , it is easy to verify that S is the center of the ellipse. We shall then consider it as the origin. Let $u = \frac{BC}{2}$ and $t = \frac{PQ}{2}$; notice that u is the major semi-axis of the ellipse and $\sqrt{u^2 - t^2}$ is the minor one. Assume $X(x, y)$ is a point on this ellipse. We now need to prove $MX \leq R$.

Since X is on the ellipse, we can write $(x, y) = (u \cos \theta, \sqrt{u^2 - t^2} \sin \theta)$, for some $\theta \in (0, 2\pi)$. Since $MX^2 = x^2 + (y + MS)^2$, we can expand and obtain

$$MX^2 = u^2 + MS^2 - t^2 \cdot \sin^2 \theta + 2MS \cdot \sqrt{u^2 - t^2} \cdot \sin \theta.$$

Add and subtract $MS^2(u^2 - t^2)/t^2$ in order to obtain a square on the right hand side: $MX^2 = u^2 + MS^2 + \frac{MS^2(u^2 - t^2)}{t^2} - \left(t \sin \theta - \frac{MS \sqrt{u^2 - t^2}}{t}\right)^2$. It now suffices to show that $u^2 + MS^2 + \frac{MS^2(u^2 - t^2)}{t^2} = R^2$, since then it would immediately follow that $MX^2 \leq R^2$.

Applying *Pythagorean* theorem in triangles OBM and OSM , we obtain $R^2 = u^2 + OM^2$ and $OM^2 = MS^2 + OS^2$, so it remains to prove that $OS^2 = \frac{MS^2(u^2 - t^2)}{t^2}$. Let $\alpha = \angle(OP, BM)$, then $OS/MS = \tan \alpha$ and $t/u = \cos \alpha$, so $OS^2 = MS^2 \tan^2 \alpha = MS^2 \left(\frac{1 - \cos^2 \alpha}{\cos^2 \alpha}\right) = MS^2 \cdot \frac{u^2 - t^2}{t^2}$, which is the desired result.

Problem 5. Let $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ be non-constant polynomials with real coefficients such that $P(Q(x)) = R(S(x))$. Suppose that the degree of $P(x)$ is divisible by the degree of $R(x)$.

Prove that there is a polynomial $T(x)$ with real coefficients such that $P(x) = R(T(x))$.

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Solution 1. Degree comparison of $P(Q(x))$ and $R(S(x))$ implies that $q = \deg Q \mid \deg S = s$. We will show that $S(x) = T(Q(x))$ for some polynomial T . Then $P(Q(x)) = R(S(x)) = R(T(Q(x)))$, so the polynomial $P(t) - R(T(t))$ vanishes upon substitution $t = S(x)$; it therefore vanishes identically, as desired.

Choose the polynomials $T(x)$ and $M(x)$ such that

$$S(x) = T(Q(x)) + M(x), \quad (*)$$

where $\deg M$ is minimised; if $M = 0$, then we get the desired result. For the sake of contradiction, suppose $M \neq 0$. Then $q \nmid m = \deg M$; otherwise, $M(x) = \beta Q(x)^{m/q} + M_1(x)$, where β is some number and $\deg M_1 < \deg M$, contradicting the choice of M . In particular, $0 < m < s$ and hence $\deg T(Q(x)) = s$.

Substitute now $(*)$ into $R(S(x)) - P(Q(x)) = 0$; let α be the leading coefficient of $R(x)$ and let $r = \deg R(x)$. Expand the brackets to get a sum of powers of $Q(x)$ and other terms including powers of $M(x)$ as well. Amongst the latter, the unique term of highest degree is $\alpha r M(x) T(Q(x))^{r-1}$. So, for some polynomial $N(x)$, $N(Q(x)) = \alpha r M(x) T(Q(x))^{r-1} +$ a polynomial of lower degree.

This is impossible, since q divides the degree of the left-hand member, but not that of the right-hand member.

Solution 2. All polynomials in the solution have real coefficients. As usual, the degree of a polynomial $f(x)$ is denoted $\deg f(x)$.

Of all pairs of polynomials $P(x)$, $R(x)$, satisfying the conditions in the statement, choose one, say, $P_0(x)$, $R_0(x)$, so that $P_0(Q(x)) = R_0(S(x))$ has a minimal (positive) degree. We will show that $\deg R_0(x) = 1$, say, $R_0(x) = \alpha x + \beta$ for some real numbers $\alpha \neq 0$ and β , so $P_0(Q(x)) = \alpha S(x) + \beta$. Hence $S(x) = T(Q(x))$ for some polynomial $T(x)$.

Now, if $P(x)$ and $R(x)$ are polynomials satisfying $P(Q(x)) = R(S(x))$, then $P(Q(x)) = R(T(Q(x)))$. Since $Q(x)$ is not constant, it takes infinitely many values, so $P(x)$ and $R(T(x))$ agree at infinitely many points, implying that $P(x) = R(T(x))$, as required.

It is therefore sufficient to solve the problem in the particular case where $F(x) = P(Q(x)) = R(S(x))$ has a minimal degree. Let $d = \gcd(\deg Q(x), \deg S(x))$ to write $\deg Q(x) = ad$ and $\deg S(x) = bd$, where $\gcd(a, b) = 1$. Then

$\deg P(x) = bc$, $\deg R(x) = ac$ and $\deg F(x) = abcd$ for some positive integer c . We will show that minimality of $\deg F(x)$ forces $c = 1$, so $\deg P(x) = b$, $\deg R(x) = a$ and $\deg F(x) = abd$. The conditions $a = \deg R(x) \mid \deg P(x) = b$ and $\gcd(a, b) = 1$ then force $a = 1$, as stated above.

Consequently, the only thing we are left with is the proof of the fact that $c = 1$. For convenience, we may and will assume that $P(x)$, $Q(x)$, $R(x)$, $S(x)$ are all monic; hence so is $F(x)$. The argument hinges on the lemma below.

Lemma. If $f(x)$ is a monic polynomial of degree mn , then there exists a degree n monic polynomial $g(x)$ such that $\deg(f(x) - g(x)^m) < (m-1)n$. (If $m = 0$ or 1 , or $n = 0$, the conclusion is still consistent with the usual convention that the identically zero polynomial has degree $-\infty$.)

Proof. Write $f(x) = \sum_{k=0}^{mn} \alpha_k x^k$, $\alpha_{mn} = 1$, and seek $g(x) = \sum_{k=0}^n \beta_k x^k$, $\beta_n = 1$, so as to fit the bill. To this end, notice that, for each positive integer $k \leq n$, the coefficient of x^{mn-k} in the expansion of $g(x)^m$ is of the form $m\beta_{n-k} + \varphi_k(\beta_n, \dots, \beta_{n-k+1})$, where $\varphi_k(\beta_n, \dots, \beta_{n-k+1})$ is an algebraic expression in $\beta_n, \dots, \beta_{n-k+1}$. Recall that $\beta_n = 1$ to determine the β_{n-k} recursively by requiring $\beta_{n-k} = \frac{1}{m}(a_{mn-k} - \varphi_k(\beta_n, \dots, \beta_{n-k+1}))$, $k = 1, \dots, n$.

The outcome is then the desired polynomial $g(x)$.

We are now in a position to prove that $c = 1$. Suppose, if possible, that $c > 1$. By the lemma, there exist monic polynomials $U(x)$ and $V(x)$ of degree b and a , respectively, such that $\deg(P(x) - U(x)^c) < (c-1)b$ and $\deg(R(x) - V(x)^c) < (c-1)a$. Then $\deg(F(x) - U(Q(x))^c) = \deg(P(Q(x)) - U(Q(x))^c) < (c-1)abd$, $\deg(F(x) - V(S(x))^c) = \deg(R(S(x)) - V(S(x))^c) < (c-1)abd$, so $\deg(U(Q(x))^c - V(S(x))^c) = \deg((F(x) - V(S(x))^c) - (F(x) - U(Q(x))^c)) < (c-1)abd$.

On the other hand, $U(Q(x))^c - V(S(x))^c = (U(Q(x)) - V(S(x)))(U(Q(x))^{c-1} + \dots + V(S(x))^{c-1})$.

By the preceding, the degree of the left-hand member is (strictly) less than $(c-1)abd$ which is precisely the degree of the second factor in the right-hand member. This forces $U(Q(x)) = V(S(x))$, so $U(Q(x)) = V(S(x))$ has degree $abd < abcd = \deg F(x)$ — a contradiction. Consequently, $c = 1$. This completes the argument and concludes the proof.

Problem 6. Let r, g, b be non-negative integers. Let Γ be a connected graph on $r + g + b + 1$ vertices. The edges of Γ are each coloured red, green or blue. It turns out that Γ has

- a spanning tree in which exactly r of the edges are red,
- a spanning tree in which exactly g of the edges are green and
- a spanning tree in which exactly b of the edges are blue.

Prove that Γ has a spanning tree in which exactly r of the edges are red, exactly g of the edges are green and exactly b of the edges are blue.

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Solution 1. Induct on $n = r + g + b$. The base case, $n = 1$, is clear.

Let now $n > 1$. Let V denote the vertex set of Γ , and let T_r , T_g , and T_b be the trees with exactly r red edges, g green edges, and b blue edges, respectively. Consider two cases.

Case 1: There exists a partition $V = A \sqcup B$ of the vertex set into two non-empty parts such that the edges joining the parts all bear the same colour, say, blue.

Since Γ is connected, it has a (necessarily blue) edge connecting A and B . Let e be one such.

Assume that T , one of the three trees, does not contain e . Then the graph $T \cup \{e\}$ has a cycle C through e . The cycle C should contain another edge e' connecting A and B ; the edge e' is also blue. Replace e' by e in T to get another tree T' with the same number of edges of each colour as in T , but containing e .

Performing such an operation to all three trees, we arrive at the situation where the three trees T'_r , T'_g , and T'_b all contain e . Now shrink e by identifying its endpoints to obtain a graph Γ^* , and set $r^* = r$, $g^* = g$, and $b^* = b - 1$. The new graph satisfies the conditions in the statement for those new values — indeed, under the shrinking, each of the trees T'_r , T'_g , and T'_b loses a blue edge. So Γ^* has a spanning tree with exactly r red, exactly g

green, and exactly $b - 1$ blue edges. Finally, pass back to Γ by restoring e , to obtain the a desired spanning tree in Γ .

Case 2: There is no such a partition.

Consider all possible collections (R, G, B) , where R , G and B are acyclic sets consisting of r red edges, g green edges, and b blue edges, respectively. By the problem assumptions, there is at least one such collection. Amongst all such collections, consider one such that the graph on V with edge set $R \cup G \cup B$ has the smallest number k of components. If $k = 1$, then the collection provides the edges of a desired tree (the number of edges is one less than the number of vertices).

Assume now that $k \geq 2$; then in the resulting graph some component K contains a cycle C . Since R , G , and B are acyclic, C contains edges of at least two colours, say, red and green. By assumption, the edges joining $V(K)$ to $V \setminus V(K)$ bear at least two colours; so one of these edges is either red or green. Without loss of generality, consider a red such edge e .

Let e' be a red edge in C and set $R' = R \setminus \{e'\} \cup \{e\}$. Then (R', G, B) is a valid collection providing a smaller number of components. This contradicts minimality of the choice above and concludes the proof.

Problem 6. Let r, g, b be non-negative integers. Let Γ be a connected graph on $r + g + b + 1$ vertices. The edges of Γ are each coloured red, green or blue. It turns out that Γ has

- a spanning tree in which exactly r of the edges are red,
- a spanning tree in which exactly g of the edges are green and
- a spanning tree in which exactly b of the edges are blue.

Prove that Γ has a spanning tree in which exactly r of the edges are red, exactly g of the edges are green and exactly b of the edges are blue.

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Solution 2. For a spanning tree T in Γ , denote by $r(T)$, $g(T)$, and $b(T)$ the number of red, green, and blue edges in T , respectively.

Assume that \mathcal{C} is some collection of spanning trees in Γ . Write

$$\begin{aligned} r(\mathcal{C}) &= \min_{T \in \mathcal{C}} r(T), & g(\mathcal{C}) &= \min_{T \in \mathcal{C}} g(T), \\ b(\mathcal{C}) &= \min_{T \in \mathcal{C}} b(T), & R(\mathcal{C}) &= \max_{T \in \mathcal{C}} r(T), \\ G(\mathcal{C}) &= \max_{T \in \mathcal{C}} g(T), & B(\mathcal{C}) &= \max_{T \in \mathcal{C}} b(T). \end{aligned}$$

Say that a collection \mathcal{C} is *good* if $r \in [r(\mathcal{C}), R(\mathcal{C})]$, $g \in [g(\mathcal{C}), G(\mathcal{C})]$, and $b \in [b(\mathcal{C}), B(\mathcal{C})]$. By the problem conditions, the collection of all spanning trees in Γ is good.

For a good collection \mathcal{C} , say that an edge e of Γ is *suspicious* if e belongs to some tree in \mathcal{C} but not to all trees in \mathcal{C} . Choose now a good collection \mathcal{C} minimizing the number of suspicious edges. If \mathcal{C} contains a desired tree, we are done. Otherwise, without loss of generality, $r(\mathcal{C}) < r$ and $G(\mathcal{C}) > g$.

We now distinguish two cases.

Case 1: $B(\mathcal{C}) = b$.

Let T^0 be a tree in \mathcal{C} with $g(T^0) = g(\mathcal{C}) \leq g$. Since $G(\mathcal{C}) > g$, there exists a green edge e contained in some tree in \mathcal{C} but not in T^0 ; clearly, e is suspicious. Fix one such green edge e .

Now, for every T in \mathcal{C} , define a spanning tree T_1 of Γ as follows. If T does not contain e , then $T_1 = T$; in particular, $(T^0)_1 = T^0$. Otherwise, the graph $T \setminus \{e\}$ falls into two components. The tree T_0 contains some edge e' joining those components; this edge is necessarily suspicious. Choose one such edge and define $T_1 = T \setminus \{e\} \cup \{e'\}$.

Let $\mathcal{C}_1 = \{T_1 : T \in \mathcal{C}\}$. All edges suspicious for \mathcal{C}_1 are also suspicious for \mathcal{C} , but no tree in \mathcal{C}_1 con-

tains e . So the number of suspicious edges for \mathcal{C}_1 is strictly smaller than that for \mathcal{C} .

We now show that \mathcal{C}_1 is good, reaching thereby a contradiction with the choice of \mathcal{C} . For every T in \mathcal{C} , the tree T_1 either coincides with T or is obtained from it by removing a green edge and adding an edge of some colour. This already shows that $g(\mathcal{C}_1) \leq g(\mathcal{C}) \leq g$, $G(\mathcal{C}_1) \geq G(\mathcal{C}) - 1 \geq g$, $R(\mathcal{C}_1) \geq R(\mathcal{C}) \geq r$, $r(\mathcal{C}_1) \leq r(\mathcal{C}) + 1 \leq r$, and $B(\mathcal{C}_1) \geq B(\mathcal{C}) \geq b$. Finally, we get $b(T^0) \leq B(\mathcal{C}) = b$; since \mathcal{C}_1 contains T^0 , it follows that $b(\mathcal{C}_1) \leq b(T^0) \leq b$, which concludes the proof.

Case 2: $B(\mathcal{C}) > b$.

Consider a tree T^0 in \mathcal{C} satisfying $r(T^0) = R(\mathcal{C}) \geq r$. Since $r(\mathcal{C}) < r$, the tree T^0 contains a suspicious red edge. Fix one such edge e .

Now, for every T in \mathcal{C} , define a spanning tree T_2 of Γ as follows. If T contains e , then $T_2 = T$; in particular, $(T^0)_2 = T^0$. Otherwise, the graph $T \cup \{e\}$ contains a cycle C through e . This cycle contains an edge e' absent from T^0 (otherwise T^0 would contain the cycle C), so e' is suspicious. Choose one such edge and define $T_2 = T \setminus \{e'\} \cup \{e\}$.

Let $\mathcal{C}_2 = \{T_2 : T \in \mathcal{C}\}$. All edges suspicious for \mathcal{C}_2 are also suspicious for \mathcal{C} , but all trees in \mathcal{C}_2 contain e . So the number of suspicious edges for \mathcal{C}_2 is strictly smaller than that for \mathcal{C} .

We now show that \mathcal{C}_2 is good, reaching again a contradiction. For every T in \mathcal{C} , the tree T_2 either coincides with T or is obtained from it by removing some edge and adding a red edge. This shows that $r(\mathcal{C}_2) \leq r(\mathcal{C}) + 1 \leq r$, $R(\mathcal{C}_2) \geq R(\mathcal{C}) \geq r$, $G(\mathcal{C}_2) \geq G(\mathcal{C}) - 1 \geq g$, $g(\mathcal{C}_2) \leq g(\mathcal{C}) \leq g$, $b(\mathcal{C}_2) \leq b(\mathcal{C}) \leq b$ and $B(\mathcal{C}_2) \geq B(\mathcal{C}) - 1 \geq b$. This concludes the proof.