SEEMOUS 2017

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Problem 1. Let $A \in \mathcal{M}_2(\mathbb{R})$. Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfies

$$a^2 + b^2 + c^2 + d^2 < \tfrac{1}{5}.$$

Show that $I + A$ is invertible.

Problem 2. Let $A, B \in \mathcal{M}_n(\mathbb{R})$.

a) Prove that there exists $a > 0$ such that for every $\varepsilon \in (-a, a)$, $\varepsilon \neq 0$, the matrix equation

$$AX + \varepsilon X = B, \quad X \in \mathcal{M}_n(\mathbb{R}),$$

has a unique solution $X(\varepsilon) \in \mathcal{M}_n(\mathbb{R})$.

b) Prove that if $B^2 = I_n$ and A *is diagonalisable*, then

$$\lim_{\varepsilon \to 0} \varepsilon \cdot \mathrm{Tr}(BX(\varepsilon)) = n - \operatorname{rank} A.$$

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^4 f(x(x-3)^2)\,dx = 2\int_1^3 f(x(x-3)^2)\,dx.$$

Problem 4. a) Let $n \geq 0$ be an integer. Calculate $\int_0^1 (1-t)^n e^t\,dt$.

b) Let $k \geq 0$ be a fixed integer and let $(x_n)_{n \geq k}$ be the sequence defined by

$$x_n = \sum_{i=k}^{n} \binom{i}{k} \left(e - 1 - \tfrac{1}{1!} - \tfrac{1}{2!} - \ldots - \tfrac{1}{i!}\right).$$

Prove that the sequence converges and find its limit.

Problem 1. Let $A \in \mathcal{M}_2(\mathbb{R})$. Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfies

$$a^2 + b^2 + c^2 + d^2 < \tfrac{1}{5}.$$

Show that $I + A$ is invertible.

Solution. We have

$$\det(I + A) = 1 + a + d + ad - bc.$$

Since

$\pm xy \geq -\frac{1}{2}(x^2 + y^2)$ for all $x, y \in \mathbb{R}$, we get

$$\det(I + A) \geq 1 + a + d - \tfrac{1}{2}(a^2 + b^2 + c^2 + d^2) > 1 + a + d - \tfrac{1}{10}.$$

Also, $a^2 < \frac{1}{5}$, so $|a| < \frac{1}{\sqrt{5}}$, and similarly for d. Therefore

$$\det(I + A) > 1 - \tfrac{2}{\sqrt{5}} - \tfrac{1}{10} > 0$$

so $I + A$ is invertible.

Remark. The problem is a particular case of a well known result in matrix theory: if $\|\cdot\|$ is a sub-multiplicative norm (that is, $\| XY \| \leq \| X \| \cdot \| Y \|$ for all matrices *X, Y)* and $\| A \| < 1$, then $I_n + A$ is invertible.

Problem 2. Let $A, B \in \mathcal{M}_n(\mathbb{R})$.

b) Prove that there exists $a > 0$ such that for every $\varepsilon \in (-a, a)$, $\varepsilon \neq 0$, the matrix equation

$$AX + \varepsilon X = B\,, \quad X \in \mathcal{M}_n(\mathbb{R})\,,$$

has a unique solution $X(\varepsilon) \in \mathcal{M}_n(\mathbb{R})$.

b) Prove that if $B^2 = I_n$ and A *is diagonalisable,* then

$$\lim_{\varepsilon \to 0} \varepsilon \cdot \mathrm{Tr}(BX(\varepsilon)) = n - \operatorname{rank} A.$$

Solution. a) Remark that the matrix $A + \varepsilon I_n$ has the eigenvalues $\lambda_1 + \varepsilon, ..., \lambda_n + \varepsilon,$ where by $\lambda_1, ..., \lambda_n$ we have denoted the eigenvalues of A. If all λ_i are nonzero, for $\varepsilon \neq 0$ sufficiently small in absolute value, $\lambda_1 + \varepsilon, ..., \lambda_n + \varepsilon$ are nonzero, hence the matrix $A + \varepsilon I_n$ is nonsingular. If 0 is eigenvalue for A, again, the matrix $A + \varepsilon I_n$ has as eigenvalues ε or $\lambda_i + \varepsilon$, with $\lambda_i \neq 0$, which are nonzero for $\varepsilon \neq 0$ sufficiently small in absolute value, therefore $A + \varepsilon I_n$ is nonsingular, again.

b) For every $\varepsilon \neq 0$ sufficiently small in absolute value, $A + \varepsilon I_n$ is invertible, and its inverse has eigenvalues $\dfrac{1}{\lambda_1 + \varepsilon}, \ldots, \dfrac{1}{\lambda_n + \varepsilon}$. We have

$$X(\varepsilon) = (A + \varepsilon I_n)^{-1} B$$

So,

$$\varepsilon \,\mathrm{Tr}(BX(\varepsilon)) = \varepsilon \,\mathrm{Tr}(B(A + \varepsilon I_n)^{-1} B) = \varepsilon \,\mathrm{Tr}(B(A + \varepsilon I_n)^{-1} B^{-1}) = \varepsilon \,\mathrm{Tr}((A + \varepsilon I_n)^{-1}) = \tfrac{\varepsilon}{\lambda_1 + \varepsilon} + ... + \tfrac{\varepsilon}{\lambda_n + \varepsilon}.$$

We used that $B^2 = I_n$, and the fact that traces of similar matrices are equal.

Therefore, $\lim_{\varepsilon \to 0} \varepsilon \cdot \mathrm{Tr}(BX(\varepsilon)) = k$, where k is the number of zero eigenvalues of the matrix A, i.e., $n - \operatorname{rank} A.$

Problem 3. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Prove that

$$\int_0^4 f(x(x-3)^2)dx = 2\int_1^3 f(x(x-3)^2)dx .$$

Solution. Let $g:[0,4]\to\mathbb{R}$ defined by $g(x)=x(x-3)^2$. Then $g'(x)=3(x-1)(x-3)$ and the behaviour of function g is given in the following table:

x	0		1		3		4
$g'(x)$	+	+	0	–	0	+	+
$g(x)$	0	↗	4	↘	0	↗	4

Let g_1, g_2, g_3 be the restrictions of g over $(0,1), (1,3)$ and $(3,4)$, respectively, and let h_1, h_2, h_3 be their inverses:

$$h_1:(0,4)\to(0,1),\ h_2:(0,4)\to(1,3),\ h_3:(0,4)\to(3,4)$$

where, for every $t\in(0,4)$,

$x_1 = h_1(t)$ is the solution of $x(x-3)^2 = t$ in $(0,1)$,

$x_2 = h_2(t)$ is the solution of $x(x-3)^2 = t$ in $(1,3)$,

$x_3 = h_3(t)$ is the solution of $x(x-3)^2 = t$ in $(3,4)$.

Using the changes of variable $x = h_i(t)$ $(i=1,2,3)$, we have that

$$\begin{aligned}\int_0^4 f(x(x-3)^2)dx - 2\int_1^3 f(x(x-3)^2)dx &= \int_0^1 f(g(x))dx - \int_1^3 f(g(x))dx + \int_3^4 f(g(x))dx \\ &= \int_0^4 f(t)\cdot h_1'(t)dt - \int_4^0 f(t)\cdot h_2'(t)dt + \int_0^4 f(t)\cdot h_3'(t)dt \\ &= \int_0^4 f(t)\cdot(h_1'(t)+h_2'(t)+h_3'(t))dt.\end{aligned}$$

Since the sum of the roots of the polynomial equation $x(x-3)^2 = t$ is 6, it follows that

$$h_1(t)+h_2(t)+h_3(t) = 6 \text{ for every } t\in(0,4),$$

hence

$$h_1'(t)+h_2'(t)+h_3'(t) = 0 \text{ for every } t\in(0,4),$$

which concludes the proof.

Remark. Since $g'(1) = g'(3) = 0$, it follows that $h_1'(4), h_2'(0), h_2'(4)$ and $h_3'(0)$ are infinite, hence the integrals $\int_0^4 f(t)\cdot h_1'(t)dt$, $\int_0^4 f(t)\cdot h_2'(t)dt$ and $\int_0^4 f(t)\cdot h_3'(t)dt$ are improper, yet convergent, because they where obtained from proper integrals by a change of variable.

Problem 4. a) Let $n \geq 0$ be an integer. Calculate $\int_0^1 (1-t)^n e^t dt$.

b) Let $k \geq 0$ be a fixed integer and let $(x_n)_{n \geq k}$ be the sequence defined by

$$x_n = \sum_{i=k}^{n} \binom{i}{k} \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{i!}\right).$$

Prove that the sequence converges and find its limit.

Solution. a) Let $I_n = \int_0^1 (1-t)^n e^t dt$, $n \geq 0$. We integrate by parts and we get that $I_n = -1 + nI_{n-1}$, $n \geq 1$ which implies that $\frac{I_n}{n!} = -\frac{1}{n!} + \frac{I_{n-1}}{(n-1)!}$. It follows that

$$\frac{I_n}{n!} = I_0 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{n!} = e - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{n!}.$$

Thus,

$$I_n = n!\left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{n!}\right), n \geq 0.$$

b) We have

$$x_{n+1} - x_n = \binom{n+1}{k}\left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{(n+1)!}\right) > 0$$

hence the sequence is strictly increasing.

On the other hand, based on Taylor's formula, we have that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{e^\theta}{(n+1)!}$$

for some $\theta \in (0,1)$. It follows that

$$0 < e - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{n!} < \frac{e}{(n+1)!}.$$

Therefore

$$x_n \leq \sum_{i=k}^{n} \binom{i}{k} \frac{e}{(i+1)!} \leq \frac{e}{k!} \sum_{i=k}^{n} \frac{1}{(i-k)!} = \frac{e}{k!}\left(\frac{1}{0!} + \frac{1}{1!} + \dots + \frac{1}{(n-k)!}\right) \leq \frac{e^2}{k!}$$

which implies the sequence is bounded. Since the sequence is bounded and increasing it converges.

To find $\lim_{n\to\infty} x_n$ we apply part a) of the problem and we have, since

$$e - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{i!} = \frac{1}{i!}\int_0^1 (1-t)^i e^t dt$$

that

$$x_n = \sum_{i=k}^{n} \binom{i}{k} \frac{1}{i!} \int_0^1 (1-t)^i e^t dt = \frac{1}{k!}\int_0^1 (1-t)^k e^t \left(\sum_{i=k}^{n} \frac{(1-t)^{i-k}}{(i-k)!}\right) dt.$$

Since $\lim_{n\to\infty} \sum_{i=k}^{n} \frac{(1-t)^{i-k}}{(i-k)!} = e^{1-t}$ and $\sum_{i=k}^{n} \frac{(1-t)^{i-k}}{(i-k)!} < e^{1-t}$, we get based on Lebesgue Dominated Convergence Theorem

$$\lim_{n\to\infty} x_n = \frac{1}{k!}\int_0^1 (1-t)^k e^t e^{1-t} dt = \frac{e}{(k+1)!}.$$

Remark. Part b) of the problem has an equivalent formulation

$$\sum_{i=k}^{\infty} \binom{i}{k}\left(e-1-\frac{1}{1!}-\frac{1}{2!}-\ldots-\frac{1}{i!}\right)=\frac{e}{(k+1)!}.$$