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Vektorsko rješenje jednog zadatka

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Sažetak. *U članku se analizira jedan planimetrijski zadatak objavljen u MFL-u, br. 174 (1993-94). Zadatak se rješava elementarno primjenom vektorskog računa.*

Ključne riječi: *planimetrija, vektori*

Vector solution of one task

Abstract. *The paper analyzes one planimetry task published in MFL, No. 174 (1993-94). The task is solved by using vector calculus.*

Key words: *planimetry, vectors*

Vjerni čitatelji naših matematičkih časopisa pronaći će u MFL-u br. 174 (1993-94) zgodan planimetrijski zadatak i dva njegova rješenja. Naime, radi se o zadatku br. 2265 i ovdje bih želio izložiti još jedno, vektorsko, rješenje toga zadatka. Rješenje je također elementarno i lako razumljivo svim srednjoškolcima.

Zadatak 1. *Iz vrha A kvadrata $ABCD$ unutar njega povučene su dvije zrake na koje su spuštene okomice BK, BL, DM i DN iz vrhova B i D. Dokažite da su dužine $\overline{KL}$ i $\overline{MN}$ okomite i da imaju jednake duljine.*

Rješenje. Da bi se uvjerili u okomitost vektora $\overrightarrow{MN}$ i $\overrightarrow{LK}$ dovoljno je pokazati da je

$$\overrightarrow{MN}\cdot\overrightarrow{LK}=0.$$

Sa slike je očito sljedeće: $\overrightarrow{AK}=\lambda\cdot\overrightarrow{AN}$, $\overrightarrow{AN}=\lambda_1\cdot\overrightarrow{AB}+\lambda_2\cdot\overrightarrow{AD}$, $\overrightarrow{AM}=\mu\cdot\overrightarrow{AL}$ te $\overrightarrow{AL}=\mu_1\cdot\overrightarrow{AB}+\mu_2\cdot\overrightarrow{AD}$.

Sada računamo:

$$\begin{aligned}
\overrightarrow{MN}\cdot\overrightarrow{LK}= & \\
&= (-\overrightarrow{AM}+\overrightarrow{AN})(-\overrightarrow{AL}+\overrightarrow{AK})\\
&= \Big(\lambda_1\overrightarrow{AB}+\lambda_2\overrightarrow{AD}-\mu(\mu_1\overrightarrow{AB}+\mu_2\overrightarrow{AD})\Big)\\
&\qquad\cdot\Big(\lambda(\lambda_1\overrightarrow{AB}+\lambda_2\overrightarrow{AD})-\mu_1\overrightarrow{AB}-\mu_2\overrightarrow{AD}\Big)\\
&= \Big((\lambda_1-\mu\mu_1)\overrightarrow{AB}+(\lambda_2-\mu\mu_2)\overrightarrow{AD}\Big)\\
&\qquad\cdot\Big((\lambda\lambda_1-\mu_1)\overrightarrow{AB}+(\lambda\lambda_2-\mu_2)\overrightarrow{AD}\Big)\\
&= ((\lambda_1-\mu\mu_1)(\lambda\lambda_1-\mu_1)+(\lambda_2-\mu\mu_2)(\lambda\lambda_2-\mu_2))\,|\overrightarrow{AB}|^2 \qquad (1)
\end{aligned}$$

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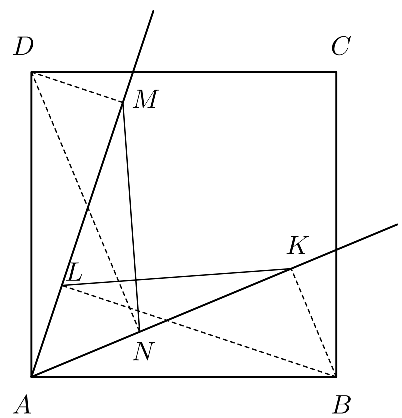

Slika 1.

Budući da (1) za sada više ne možemo pojednostavljivati, iskoristimo neke uvjete iz zadatka pa ćemo joj se kasnije vratiti.

i) $\overrightarrow{BK}\cdot\overrightarrow{AK}=0$

$$\Rightarrow\quad \Big(-\overrightarrow{AB}+\lambda(\lambda_1\overrightarrow{AB}+\lambda_2\overrightarrow{AD})\Big)\cdot\Big(\lambda\lambda_1\overrightarrow{AB}+\lambda\lambda_2\overrightarrow{AD}\Big)=0$$
$$\Rightarrow\quad \Big((\lambda\lambda_1-1)\overrightarrow{AB}+\lambda\lambda_2\overrightarrow{AD}\Big)\cdot\Big(\lambda\lambda_1\overrightarrow{AB}+\lambda\lambda_2\overrightarrow{AD}\Big)=0$$
$$\Rightarrow\quad \big(\lambda\lambda_1(\lambda\lambda_1-1)+\lambda^2\lambda_2^2\big)\cdot|\overrightarrow{AB}|^2=0$$
$$\Rightarrow\quad \lambda\lambda_1(\lambda\lambda_1-1)+\lambda^2\lambda_2^2=0$$
$$\Rightarrow\quad \lambda\left(\lambda_1(\lambda\lambda_1-1)+\lambda\lambda_2^2\right)=0$$

Za $\lambda=0$ je $K\equiv A$, $N\equiv D$ pa je $\overrightarrow{MD}\perp\overrightarrow{AL}$, što je trivijalno. Za $\lambda\neq 0$ je $\lambda_1(\lambda\lambda_1-1)+\lambda\lambda_2^2=0$ odakle je $\lambda\lambda_1^2+\lambda\lambda_2^2=\lambda_1$, tj.

$$\lambda(\lambda_1^2+\lambda_2^2)=\lambda_1. \tag{2}$$

ii) $\overrightarrow{BL}\cdot\overrightarrow{AL}=0$

$$\Rightarrow\quad \Big(-\overrightarrow{AB}+\mu_1\overrightarrow{AB}+\mu_2\overrightarrow{AD}\Big)\cdot\Big(\mu_1\overrightarrow{AB}+\mu_2\overrightarrow{AD}\Big)=0$$
$$\Rightarrow\quad \Big((\mu_1-1)\overrightarrow{AB}+\mu_2\overrightarrow{AD}\Big)\cdot\Big(\mu_1\overrightarrow{AB}+\mu_2\overrightarrow{AD}\Big)=0$$
$$\Rightarrow\quad \big(\mu_1(\mu_1-1)+\mu_2^2\big)\cdot|\overrightarrow{AB}|^2=0$$

odakle slijedi

$$\mu_1^2+\mu_2^2=\mu_1. \tag{3}$$

iii) $\overrightarrow{DM}\cdot\overrightarrow{AM}=0$

$$\Rightarrow\quad \Big(-\overrightarrow{AD}+\mu(\mu_1\overrightarrow{AB}+\mu_2\overrightarrow{AD})\Big)\cdot\Big(\mu\mu_1\overrightarrow{AB}+\mu\mu_2\overrightarrow{AD}\Big)=0$$
$$\Rightarrow\quad \Big(\mu\mu_1\overrightarrow{AB}+(\mu\mu_2-1)\overrightarrow{AD}\Big)\cdot\Big(\mu\mu_1\overrightarrow{AB}+\mu\mu_2\overrightarrow{AD}\Big)=0$$
$$\Rightarrow\quad \big(\mu^2\mu_1^2+\mu\mu_2(\mu\mu_2-1)\big)\cdot|\overrightarrow{AB}|^2=0$$
$$\Rightarrow\quad \mu\left(\mu\mu_1^2+\mu_2(\mu\mu_2-1)\right)=0$$

Za $\mu = 0$ je $M \equiv A$, $L \equiv B$ pa je $\overrightarrow{AN} \perp \overrightarrow{BK}$, što je trivijalno. Iz $\mu \neq 0$ slijedi

$$\mu(\mu_1^2 + \mu_2^2) = \mu_2. \tag{4}$$

iv) $\overrightarrow{DN} \cdot \overrightarrow{AN} = 0$

$$\Rightarrow \quad \left(-\overrightarrow{AD} + \lambda_1\overrightarrow{AB} + \lambda_2\overrightarrow{AD}\right) \cdot \left(\lambda_1\overrightarrow{AB} + \lambda_2\overrightarrow{AD}\right) = 0$$
$$\Rightarrow \quad \left(\lambda_1\overrightarrow{AB} + (\lambda_2 - 1)\overrightarrow{AD}\right) \cdot \left(\lambda_1\overrightarrow{AB} + \lambda_2\overrightarrow{AD}\right) = 0$$

odakle slijedi

$$\lambda_1^2 + \lambda_2^2 = \lambda_2 \tag{5}$$

Vratimo se sada jednakosti (1).

$$\begin{aligned}
\overrightarrow{MN} \cdot \overrightarrow{LK} &= \\
&= ((\lambda_1 - \mu\mu_1)(\lambda\lambda_1 - \mu_1) + (\lambda_2 - \mu\mu_2)(\lambda\lambda_2 - \mu_2))\,|\overrightarrow{AB}|^2 \\
&= \left(\lambda\lambda_1^2 - \lambda_1\mu_1 - \lambda\mu\lambda_1\mu_1 + \mu\mu_1^2 + \lambda\lambda_2^2 - \mu_2\lambda_2 - \lambda\mu\lambda_2\mu_2 + \mu\mu_2^2\right) \cdot |\overrightarrow{AB}|^2 \\
&= \left(\lambda(\lambda_1^2 + \lambda_2^2) + \mu(\mu_1^2 + \mu_2^2) - \lambda\mu(\lambda_1\mu_1 + \lambda_2\mu_2) - (\lambda_1\mu_1 + \mu_2\lambda_2)\right) \cdot |\overrightarrow{AB}|^2 \\
&= \left(\lambda(\lambda_1^2 + \lambda_2^2) + \mu(\mu_1^2 + \mu_2^2) - (1 + \lambda\mu)(\lambda_1\mu_1 + \lambda_2\mu_2)\right) \cdot |\overrightarrow{AB}|^2 \\
&= (\lambda_1 + \mu_2 - (\lambda_1\mu_1 + \lambda_2\mu_2) - \lambda\mu(\lambda_1\mu_1 + \lambda_2\mu_2)) \cdot |\overrightarrow{AB}|^2
\end{aligned} \tag{6}$$

Sada je

$$\begin{aligned}
\lambda\mu &= \left|\begin{matrix}\text{Koristimo} \\ \text{(2) i (4)}\end{matrix}\right| = \frac{\lambda_1}{\lambda_1^2 + \lambda_2^2} \cdot \frac{\mu_2}{\mu_1^2 + \mu_2^2} \\
&= \frac{\lambda_1\mu_2}{(\lambda_1^2 + \lambda_2^2)(\mu_1^2 + \mu_2^2)} = \left|\begin{matrix}\text{Koristimo} \\ \text{(3) i (5)}\end{matrix}\right| \\
&= \frac{\lambda_1\mu_2}{\lambda_2\mu_1}
\end{aligned} \tag{7}$$

Jednakost (6) dalje pišemo

$$\begin{aligned}
\overrightarrow{MN} \cdot \overrightarrow{LK} &= \\
&= (\lambda_1 + \mu_2 - (\lambda_1\mu_1 + \lambda_2\mu_2) - \lambda\mu(\lambda_1\mu_1 + \lambda_2\mu_2)) \cdot |\overrightarrow{AB}|^2 \\
&= \frac{1}{\lambda_2\mu_1}\left(\lambda_1\lambda_2\mu_1 + \lambda_2\mu_1\mu_2 - \lambda_1\lambda_2\mu_1^2 - \mu_1\mu_2\lambda_2^2 - \lambda_1^2\mu_1\mu_2 - \lambda_1\lambda_2\mu_2^2\right) \cdot |\overrightarrow{AB}|^2 \\
&= \frac{1}{\lambda_2\mu_1}\left(\lambda_1\lambda_2\mu_1 + \lambda_2\mu_1\mu_2 - \mu_1\mu_2(\lambda_1^2 + \lambda_2^2) - \lambda_1\lambda_2(\mu_1^2 + \mu_2^2)\right) \cdot |\overrightarrow{AB}|^2 \\
&= \frac{1}{\lambda_2\mu_1}(\lambda_1\lambda_2\mu_1 + \lambda_2\mu_1\mu_2 - \mu_1\mu_2\lambda_2 - \lambda_1\lambda_2\mu_1) \cdot |\overrightarrow{AB}|^2 \\
&= \frac{1}{\lambda_2\mu_1} \cdot 0 \cdot |\overrightarrow{AB}|^2 = 0
\end{aligned}$$

Dakle, vrijedi $\overrightarrow{MN} \cdot \overrightarrow{LK} = 0$ pa je i $\overrightarrow{MN} \perp \overrightarrow{LK}$.

Da bismo pokazali jednakost dužina $\overline{MN}$ i $\overline{LK}$ pretpostavit ćemo da je naš kvadrat upisan u koordinatni sustav, npr. tako da se vrh A nalazi u ishodištu, a stranice $\overline{AB}$ i $\overline{AD}$ leže na x odnosno y osi. Bez smanjenja općenitosti možemo pretpostaviti da je kvadrat jedinični, tj. $\overrightarrow{AB}$ možemo poistovjetiti s $\vec{\imath}$, a $\overrightarrow{AD}$ s $\vec{\jmath}$. Vratimo li se za tren na početak rješenja vidimo da je $\overrightarrow{MN} = (\lambda_1 - \mu\mu_1)\overrightarrow{AB} + (\lambda_2 - \mu\mu_2)\overrightarrow{AD}$ i $\overrightarrow{LK} = (\lambda\lambda_1 - \mu_1)\overrightarrow{AB} + (\lambda\lambda_2 - \mu_2)\overrightarrow{AD}$.

Iz $|\overrightarrow{MN}| = |\overrightarrow{LK}|$ slijedi $|\overrightarrow{MN}|^2 = |\overrightarrow{LK}|^2$, tj.

$$(\lambda_1 - \mu\mu_1)^2 + (\lambda_2 - \mu\mu_2)^2 = (\lambda\lambda_1 - \mu_1)^2 + (\lambda\lambda_2 - \mu_2)^2$$

odakle, sređivanjem, slijedi

$$\lambda_1^2 + \lambda_2^2 - \lambda^2(\lambda_1^2 + \lambda_2^2) + \mu^2(\mu_1^2 + \mu_2^2) - (\mu_1 + \mu_2) + 2(\lambda - \mu)(\lambda_1\mu_1 + \lambda_2\mu_2) = 0.$$

Zbog relacija (2) i (4) imamo

$$\lambda_1^2 + \lambda_2^2 - \frac{\lambda_1^2}{\lambda_1^2 + \lambda_2^2} + \frac{\mu_2^2}{\mu_1^2 + \mu_2^2} - (\mu_1 + \mu_2) + 2(\lambda - \mu)(\lambda_1\mu_1 + \lambda_2\mu_2) = 0.$$

Sada je zbog (3) i (4)

$$\lambda_2 - \frac{\lambda_1^2}{\lambda_2} + \frac{\mu_2^2}{\mu_1} - \mu_1 + 2(\lambda - \mu)(\lambda_1\mu_1 + \lambda_2\mu_2) = 0. \tag{8}$$

Iz (2) i (4) je

$$\lambda - \mu = \frac{\lambda_1}{\lambda_1^2 + \lambda_2^2} - \frac{\mu_2}{\mu_1^2 + \mu_2^2} = \frac{\lambda_1}{\lambda_2} - \frac{\mu_2}{\mu_1} = \frac{\lambda_1\mu_1 - \mu_2\lambda_2}{\lambda_2\mu_1}. \tag{9}$$

Koristeći (9) u (8) imamo

$$\lambda_2 - \frac{\lambda_1^2}{\lambda_2} + \frac{\mu_2^2}{\mu_1} - \mu_1 + 2\frac{\lambda_1\mu_1 - \mu_2\lambda_2}{\lambda_2\mu_1}(\lambda_1\mu_1 + \lambda_2\mu_2) = 0,$$

odnosno,

$$\frac{1}{\lambda_2\mu_1}(\lambda_2^2\mu_1 - \lambda_1^2\mu_1 + \mu_2^2\lambda_2 - \lambda_2\mu_1^2 + 2\lambda_1^2\mu_1^2 - 2\mu_2^2\lambda_2^2) = 0.$$

Konačno, koristeći (3) i (5) dobivamo

$$\frac{1}{\lambda_2\mu_1}\Big(\lambda_2^2(\mu_1^2 + \mu_2^2) - \lambda_1^2(\mu_1^2 + \mu_2^2) + \mu_2^2(\lambda_1^2 + \lambda_2^2) - \\ - \mu_1^2(\lambda_1^2 + \lambda_2^2) + 2\lambda_1^2\mu_1^2 - 2\mu_2^2\lambda_2^2 \Big) = 0.$$

Dakle, pretpostavka $|\overrightarrow{MN}| = |\overrightarrow{LK}|$ je točna i time smo u potpunosti riješili zadatak.

Teško je vjerovati da bi bilo kome, u prvu ruku, palo na um rješavati ovaj zadatak na ovaj način. Upravo ta netipičnost rješenja me ponukala da ga objavim u našem časopisu.