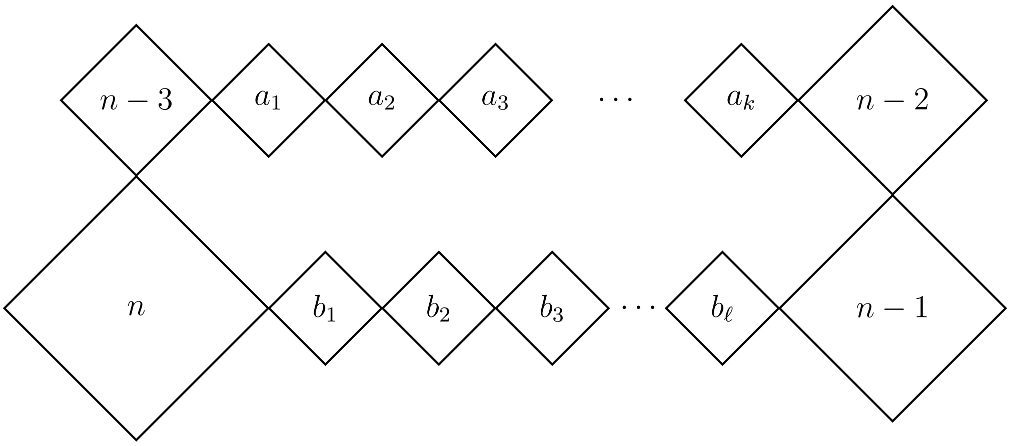
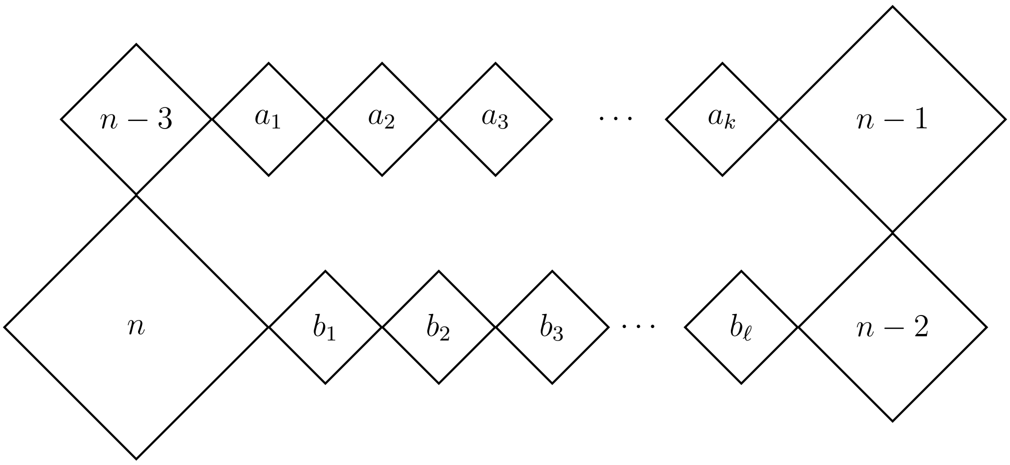
APMO 2023 – Problems and Solutions

Problem 1
Let $n \geq 5$ be an integer. Consider n squares with side lengths $1, 2, \ldots, n$, respectively. The squares are arranged in the plane with their sides parallel to the x and y axes. Suppose that no two squares touch, except possibly at their vertices.
Show that it is possible to arrange these squares in a way such that every square touches exactly two other squares.

Solution 1
Set aside the squares with sidelengths $n-3$, $n-2$, $n-1$, and n and suppose we can split the remaining squares into two sets A and B such that the sum of the sidelengths of the squares in A is 1 or 2 units larger than the sum of the sidelengths of the squares in B.
String the squares of each set A, B along two parallel diagonals, one for each diagonal. Now use the four largest squares along two perpendicular diagonals to finish the construction: one will have sidelengths n and $n-3$, and the other, sidelengths $n-1$ and $n-2$. If the sum of the sidelengths of the squares in A is 1 unit larger than the sum of the sidelengths of the squares in B, attach the squares with sidelengths $n-3$ and $n-1$ to the A-diagonal, and the other two squares to the B-diagonal. The resulting configuration, in which the A and B-diagonals are represented by unit squares, and the sidelengths a_i of squares from A and b_j of squares from B are indicated within each square, follows:



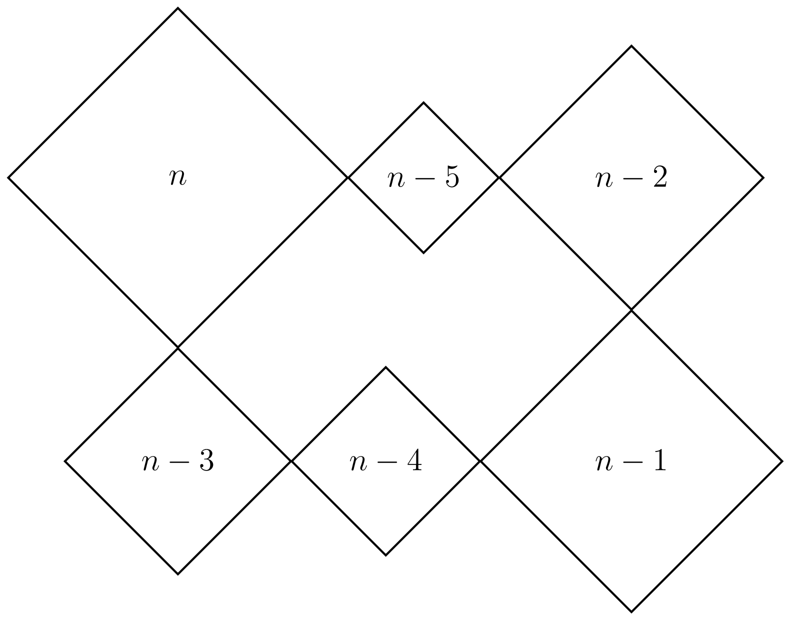
Since $(a_1 + a_2 + \cdots + a_k)\sqrt{2} + \frac{((n-3)+(n-2))\sqrt{2}}{2} = (b_1 + b_2 + \cdots + b_\ell + 2)\sqrt{2} + \frac{(n+(n-1))\sqrt{2}}{2}$, this case is done.
If the sum of the sidelengths of the squares in A is 1 unit larger than the sum of the sidelengths of the squares in B, attach the squares with sidelengths $n-3$ and $n-2$ to the A-diagonal, and the other two squares to the B-diagonal. The resulting configuration follows:



Since $(a_1 + a_2 + \cdots + a_k)\sqrt{2} + \frac{((n-3)+(n-1))\sqrt{2}}{2} = (b_1 + b_2 + \cdots + b_\ell + 1)\sqrt{2} + \frac{(n+(n-2))\sqrt{2}}{2}$, this case is also done.
In both cases, the distance between the A-diagonal and the B-diagonal is $\frac{((n-3)+n)\sqrt{2}}{2} = \frac{(2n-3)\sqrt{2}}{2}$. Since $a_i, b_j \le n-4$, $\frac{(a_i+b_j)\sqrt{2}}{2} < \frac{(2n-4)\sqrt{2}}{2} < \frac{(2n-3)\sqrt{2}}{2}$, and therefore the A- and B-diagonals do not overlap.
Finally, we prove that it is possible to split the squares of sidelengths 1 to $n-4$ into two sets A and B such that the sum of the sidelengths of the squares in A is 1 or 2 units larger than the sum of the sidelengths of the squares in B. One can do that in several ways; we present two possibilities:

- *Direct construction:* Split the numbers from 1 to $n-4$ into several sets of four consecutive numbers $\{t, t+1, t+2, t+3\}$, beginning with the largest numbers; put squares of sidelengths t and $t+3$ in A and squares of sidelengths $t+1$ and $t+2$ in B. Notice that $t + (t+3) = (t+1) + (t+2)$. In the end, at most four numbers remain.
  - If only 1 remains, put the corresponding square in A, so the sum of the sidelengths of the squares in A is one unit larger that those in B;
  - If 1 and 2 remains, put the square of sidelength 2 in A and the square of sidelength 1 in B (the difference is 1);
  - If 1, 2, and 3 remains, put the squares of sidelengths 1 and 3 in A, and the square of sidelength 2 in B (the difference is 2);
  - If 1, 2, 3, and 4 remains, put the squares of sidelengths 2 and 4 in A, and the squares of sidelengths 1 and 3 in B (the difference is 2).
- *Indirect construction:* Starting with A and B as empty sets, add the squares of sidelengths $n-4, n-3, \ldots, 2$ to either A or B in that order such that at each stage the difference between the sum of the sidelengths in A and the sum of the sidelengths of B is minimized. By induction it is clear that after adding an integer j to one of the sets, this difference is at most j. In particular, the difference is 0, 1 or 2 at the end. Finally adding the final 1 to one of the sets can ensure that the final difference is 1 or 2. If necessary, flip A and B.

Solution 2
Solve the problem by induction in n. Construct examples for $n = 5, 6, 7, 8, 9, 10$ (one can use the constructions from the previous solution, for instance). For $n > 10$, set aside the six larger squares and arrange them in the following fashion:



By the induction hypothesis, one can arrange the remaining $n - 6$ squares away from the six larger squares, so we are done.

Problem 2
Find all integers n satisfying $n \geq 2$ and $\frac{\sigma(n)}{p(n)-1} = n$, in which $\sigma(n)$ denotes the sum of all positive divisors of n, and $p(n)$ denotes the largest prime divisor of n.

Answer: $n = 6$.

Solution
Let $n = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ be the prime factorization of n with $p_1 < \ldots < p_k$, so that $p(n) = p_k$ and $\sigma(n) = (1 + p_1 + \cdots + p_1^{\alpha_1}) \ldots (1 + p_k + \cdots + p_k^{\alpha_k})$. Hence

$$p_k - 1 = \frac{\sigma(n)}{n} = \prod_{i=1}^{k} \left(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{\alpha_i}}\right) < \prod_{i=1}^{k} \frac{1}{1 - \frac{1}{p_i}} = \prod_{i=1}^{k} \left(1 + \frac{1}{p_i - 1}\right) \leq \prod_{i=1}^{k} \left(1 + \frac{1}{i}\right) = k+1,$$

that is, $p_k - 1 < k+1$, which is impossible for $k \geq 3$, because in this case $p_k - 1 \geq 2k - 2 \geq k+1$. Then $k \leq 2$ and $p_k < k + 2 \leq 4$, which implies $p_k \leq 3$.
If $k = 1$ then $n = p^\alpha$ and $\sigma(n) = 1 + p + \cdots + p^\alpha$, and in this case $n \nmid \sigma(n)$, which is not possible. Thus $k = 2$, and $n = 2^\alpha 3^\beta$ with $\alpha, \beta > 0$. If $\alpha > 1$ or $\beta > 1$,

$$\frac{\sigma(n)}{n} > \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) = 2.$$

Therefore $\alpha = \beta = 1$ and the only answer is $n = 6$.

Comment: There are other ways to deal with the case $n = 2^\alpha 3^\beta$. For instance, we have $2^{\alpha+2}3^\beta = (2^{\alpha+1} - 1)(3^{\beta+1} - 1)$. Since $2^{\alpha+1} - 1$ is not divisible by 2, and $3^{\beta+1} - 1$ is not divisible by 3, we have

$$\begin{cases} 2^{\alpha+1} - 1 = 3^\beta \\ 3^{\beta+1} - 1 = 2^{\alpha+2} \end{cases} \iff \begin{cases} 2^{\alpha+1} - 1 = 3^\beta \\ 3 \cdot (2^{\alpha+1} - 1) - 1 = 2 \cdot 2^{\alpha+1} \end{cases} \iff \begin{cases} 2^{\alpha+1} = 4 \\ 3^\beta = 3 \end{cases},$$

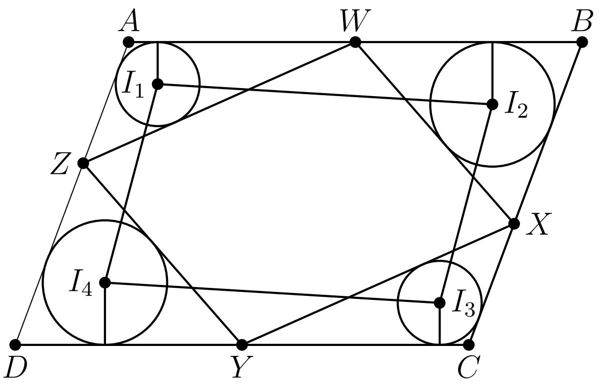
and $n = 2^\alpha 3^\beta = 6$.

Problem 3
Let $ABCD$ be a parallelogram. Let W, X, Y, and Z be points on sides AB, BC, CD, and DA, respectively, such that the incenters of triangles AWZ, BXW, CYX and DZY form a parallelogram. Prove that $WXYZ$ is a parallelogram.

Solution
Let the four incenters be I_1, I_2, I_3, and I_4 with inradii r_1, r_2, r_3, and r_4 respectively (in the order given in the question). Without loss of generality, let I_1 be closer to AB than I_2. Let the acute angle between I_1I_2 and AB (and hence also the angle between I_3I_4 and CD) be θ. Then

$$r_2 - r_1 = I_1I_2 \sin\theta = I_3I_4 \sin\theta = r_4 - r_3,$$

which implies $r_1 + r_4 = r_2 + r_3$. Similar arguments show that $r_1 + r_2 = r_3 + r_4$. Thus we obtain $r_1 = r_3$ and $r_2 = r_4$.



Now let's consider the possible positions of W, X, Y, Z. Suppose $AZ \neq CX$. Without loss of generality assume $AZ > CX$. Since the incircles of AWZ and CYX are symmetric about the centre of the parallelogram $ABCD$, this implies $CY > AW$. Using similar arguments, we have

$$CY > AW \implies BW > DY \implies DZ > BX \implies CX > AZ,$$

which is a contradiction. Therefore $AZ = CX \implies AW = CY$ and $WXYZ$ is a parallelogram.

Comment: There are several ways to prove that $r_1 = r_3$ and $r_2 = r_4$. The proposer shows the following three alternative approaches:

Using parallel lines: Let O be the centre of parallelogram $ABCD$ and P be the centre of parallelogram $I_1I_2I_3I_4$. Since AI_1 and CI_3 are angle bisectors, we must have $AI_1 \parallel CI_3$. Let ℓ_1 be the line through O parallel to AI_1. Since $AO = OC$, ℓ_1 is halfway between AI_1 and CI_3. Hence P must lie on ℓ_1.

Similarly, P must also lie on ℓ_2, the line through O parallel to BI_2. Thus P is the intersection of ℓ_1 and ℓ_2, which must be O. So the four incentres and hence the four incircles must be symmetric about O, which implies $r_1 = r_3$ and $r_2 = r_4$.

Using a rotation: Let the bisectors of $\angle DAB$ and $\angle ABC$ meet at X and the bisectors of $\angle BCD$ and $\angle CDA$ meet at Y. Then I_1 is on AX, I_2 is on BX, I_3 is on CY, and I_4 is on DY. Let O be the centre of $ABCD$. Then a 180 degree rotation about O takes $\triangle AXB$ to $\triangle CYD$. Under the same transformation I_1I_2 is mapped to a parallel segment $I_1'I_2'$ with I_1' on CY and I_2' on DY. Since $I_1I_2I_3I_4$ is a parallelogram, $I_3I_4 = I_1I_2$ and $I_3I_4 \parallel I_1I_2$. Hence $I_1'I_2'$ and I_3I_4 are parallel, equal length segments on sides CY, DY and we conclude that $I_1' = I_3$, $I_2' = I_4$. Hence the centre of $I_1I_2I_3I_4$ is also O and we establish that by rotational symmetry that $r_1 = r_3$ and $r_2 = r_4$.

Using congruent triangles: Let AI_1 and BI_2 intersect at E and let CI_3 and DI_4 intersect at F. Note that $\triangle ABE$ and $\triangle CDF$ are congruent, since $AB = CD$ and corresponding pairs of angles are equal (equal opposite angles parallelogram $ABCD$ are each bisected).

Since $AI_1 \parallel CI_3$ and $I_1I_2 \parallel I_4I_3$, $\angle I_2I_1E = \angle I_4I_3F$. Similarly $\angle I_1I_2E = \angle I_3I_4F$. Furthermore $I_1I_2 = I_3I_4$. Hence triangles I_2I_1E and $I4I_3F$ are also congruent.
Hence $ABEI_1I_2$ and $DCFI_3I_4$ are congruent. Therefore, the perpendicular distance from I_1 to AB equals the perpendicular distance from I_3 to CD, that is, $r_1 = r_3$. Similarly $r_2 = r_4$.

Problem 4

Let $c > 0$ be a given positive real and $\mathbb{R}_{>0}$ be the set of all positive reals. Find all functions $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f((c+1)x + f(y)) = f(x+2y) + 2cx \quad \text{for all } x, y \in \mathbb{R}_{>0}.$$

Answer: $f(x) = 2x$ for all $x > 0$.

Solution 1

We first prove that $f(x) \geq 2x$ for all $x > 0$. Suppose, for the sake of contradiction, that $f(y) < 2y$ for some positive y. Choose x such that $f((c+1)x + f(y))$ and $f(x+2y)$ cancel out, that is,

$$(c+1)x + f(y) = x + 2y \iff x = \frac{2y - f(y)}{c}.$$

Notice that $x > 0$ because $2y - f(y) > 0$. Then $2cx = 0$, which is not possible. This contradiction yields $f(y) \geq 2y$ for all $y > 0$.

Now suppose, again for the sake of contradiction, that $f(y) > 2y$ for some $y > 0$. Define the following sequence: a_0 is an arbitrary real greater than $2y$, and $f(a_n) = f(a_{n-1}) + 2cx$, so that

$$\begin{cases} (c+1)x + f(y) = a_n \\ x + 2y = a_{n-1} \end{cases} \iff x = a_{n-1} - 2y \quad \text{and} \quad a_n = (c+1)(a_{n-1} - 2y) + f(y).$$

If $x = a_{n-1} - 2y > 0$ then $a_n > f(y) > 2y$, so inductively all the substitutions make sense.

For the sake of simplicity, let $b_n = a_n - 2y$, so $b_n = (c+1)b_{n-1} + f(y) - 2y \quad (*)$. Notice that $x = b_{n-1}$ in the former equation, so $f(a_n) = f(a_{n-1}) + 2cb_{n-1}$. Telescoping yields

$$f(a_n) = f(a_0) + 2c\sum_{i=0}^{n-1} b_i.$$

One can find b_n from the recurrence equation $(*)$: $b_n = \left(b_0 + \frac{f(y)-2y}{c}\right)(c+1)^n - \frac{f(y)-2y}{c}$, and then

$$\begin{aligned} f(a_n) &= f(a_0) + 2c\sum_{i=0}^{n-1}\left(\left(b_0 + \frac{f(y)-2y}{c}\right)(c+1)^i - \frac{f(y)-2y}{c}\right) \\ &= f(a_0) + 2\left(b_0 + \frac{f(y)-2y}{c}\right)((c+1)^n - 1) - 2n(f(y)-2y). \end{aligned}$$

Since $f(a_n) \geq 2a_n = 2b_n + 4y$,

$$\begin{aligned} &f(a_0) + 2\left(b_0 + \frac{f(y)-2y}{c}\right)((c+1)^n - 1) - 2n(f(y)-2y) \geq 2b_n + 4y \\ = 2&\left(b_0 + \frac{f(y)-2y}{c}\right)(c+1)^n - 2\frac{f(y)-2y}{c}, \end{aligned}$$

which implies

$$f(a_0) + 2\frac{f(y)-2y}{c} \geq 2\left(b_0 + \frac{f(y)-2y}{c}\right) + 2n(f(y)-2y),$$

which is not true for sufficiently large n.

A contradiction is reached, and thus $f(y) = 2y$ for all $y > 0$. It is immediate that this function satisfies the functional equation.

Solution 2

After proving that $f(y) \geq 2y$ for all $y > 0$, one can define $g(x) = f(x) - 2x$, $g\colon \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$, and our goal is proving that $g(x) = 0$ for all $x > 0$. The problem is now rewritten as

$$\begin{aligned} & g((c+1)x + g(y) + 2y) + 2((c+1)x + g(y) + 2y) = g(x+2y) + 2(x+2y) + 2cx \\ \iff & g((c+1)x + g(y) + 2y) + 2g(y) = g(x+2y). \end{aligned} \tag{1}$$

This readily implies that $g(x+2y) \geq 2g(y)$, which can be interpreted as $z > 2y \implies g(z) \geq 2g(y)$, by plugging $z = x + 2y$.

Now we prove by induction that $z > 2y \implies g(z) \geq 2m \cdot g(y)$ for any positive integer $2m$. In fact, since $(c+1)x + g(y) + 2y > 2y$, $g((c+1)x + g(y) + 2y) \geq 2m \cdot g(y)$, and by (**??**),

$$g(x+2y) \geq 2m \cdot g(y) + 2g(y) = 2(m+1)g(y),$$

and we are done by plugging $z = x + 2y$ again.

The problem now is done: if $g(y) > 0$ for some $y > 0$, choose a fixed $z > 2y$ arbitrarily and and integer m such that $m > \frac{g(z)}{2g(y)}$. Then $g(z) < 2m \cdot g(y)$, contradiction.

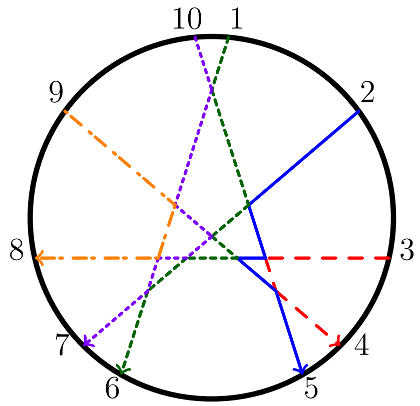
Problem 5
There are n line segments on the plane, no three intersecting at a point, and each pair intersecting once in their respective interiors. Tony and his $2n-1$ friends each stand at a distinct endpoint of a line segment. Tony wishes to send Christmas presents to each of his friends as follows:
First, he chooses an endpoint of each segment as a "sink". Then he places the present at the endpoint of the segment he is at. The present moves as follows:

- If it is on a line segment, it moves towards the sink.
- When it reaches an intersection of two segments, it changes the line segment it travels on and starts moving towards the new sink.

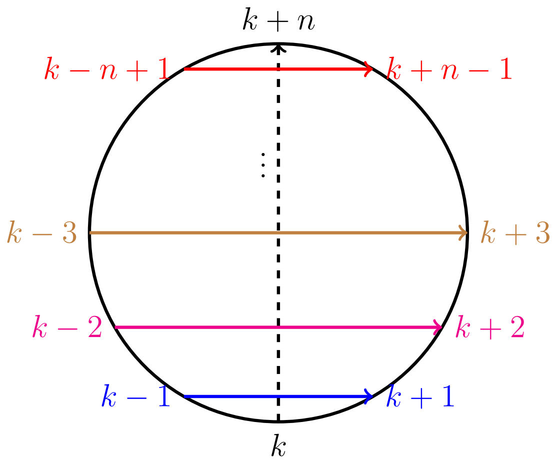
If the present reaches an endpoint, the friend on that endpoint can receive their present. Prove Tony can send presents to exactly n of his $2n-1$ friends.

Solution 1
Draw a circle that encloses all the intersection points between line segments and extend all line segments until they meet the circle, and then move Tony and all his friends to the circle. Number the intersection points with the circle from 1 to $2n$ anticlockwise, starting from Tony (Tony has number 1). We will prove that the friends eligible to receive presents are the ones on even-numbered intersection points.
First part: at most n friends can receive a present.
The solution relies on a well-known result: the n lines determine regions inside the circle; then it is possible to paint the regions with two colors such that no regions with a common (line) boundary have the same color. The proof is an induction on n: the fact immediately holds for $n = 0$, and the induction step consists on taking away one line ℓ, painting the regions obtained with $n-1$ lines, drawing ℓ again and flipping all colors on exactly one half plane determined by ℓ.
Now consider the line starting on point 1. Color the regions in red and blue such that neighboring regions have different colors, and such that the two regions that have point 1 as a vertex are red on the right and blue on the left, from Tony's point of view. Finally, assign to each red region the clockwise direction and to each blue region the anticlockwise direction. Because of the coloring, every boundary will have two directions assigned, but the directions are the same since every boundary divides regions of different colors. Then the present will follow the directions assigned to the regions: it certainly does for both regions in the beginning, and when the present reaches an intersection it will keep bordering one of the two regions it was dividing. To finish this part of the problem, consider the regions that share a boundary with the circle. The directions alternate between outcoming and incoming, starting from 1 (outcoming), so all even-numbered vertices are directed as incoming and are the only ones able to receive presents.
Second part: all even-numbered vertices can receive a present.
First notice that, since every two chords intersect, every chord separates the endpoints of each of the other $n-1$ chords. Therefore, there are $n-1$ vertices on each side of every chord, and each chord connects vertices k and $k+n$, $1 \le k \le n$.
We prove a stronger result by induction in n: let k be an integer, $1 \le k \le n$. Direct each chord from i to $i+n$ if $1 \le i \le k$ and from $i+n$ to i otherwise; in other words, the sinks are $k+1$, $k+2$, …, $k+n$. Now suppose that each chord sends a present, starting from the vertex opposite to each sink, and all presents move with the same rules. Then $k-i$ sends a present to $k+i+1$, $i = 0, 1, \ldots, n-1$ (indices taken modulo $2n$). In particular, for $i = k-1$, Tony, in vertex 1, send a present to vertex $2k$. Also, the n paths the presents make do not cross (but they may touch.) More formally, for all i, $1 \le i \le n$, if one path takes a present from $k-i$ to $k+i+1$, separating the circle into two regions, all paths taking a present from $k-j$ to $k+j+1$, $j < i$, are completely contained in one region, and all paths taking a present from

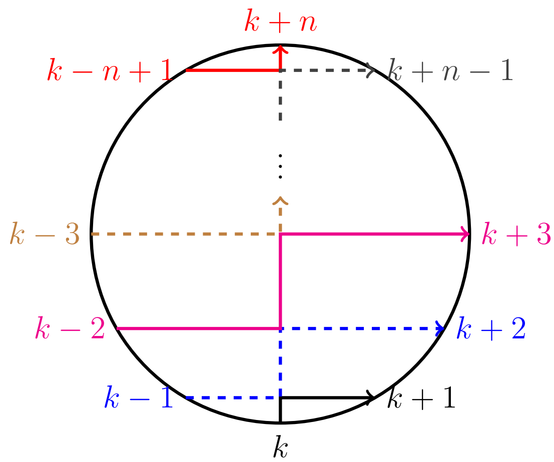
$k-j$ to $k+j+1$, $j>i$, are completely contained in the other region. For instance, possible[1] paths for $k=3$ and $n=5$ follow:



The result is true for $n=1$. Let $n>1$ and assume the result is true for less chords. Consider the chord that takes k to $k+n$ and remove it. Apply the induction hypothesis to the remaining $n-1$ lines: after relabeling, presents would go from $k-i$ to $k+i+2$, $1\le i\le n-1$ if the chord were not there.
Reintroduce the chord that takes k to $k+n$. From the induction hypothesis, the chord intersects the paths of the presents in the following order: the i-th path the chord intersects is the the one that takes $k-i$ to $k+i$, $i=1,2,\dots,n-1$.



Paths without chord $k\to k+n$



Corrected paths with chord $k\to k+n$

Then the presents cover the following new paths: the present from k will leave its chord and take the path towards $k+1$; then, for $i=1,2,\dots,n-1$, the present from $k-i$ will meet the chord from k to $k+n$, move towards the intersection with the path towards $k+i+1$ and go to $k+i+1$, as desired. Notice that the paths still do not cross. The induction (and the solution) is now complete.

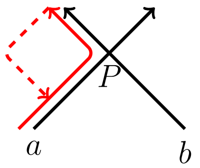
Solution 2

First part: at most n friends can receive a present.

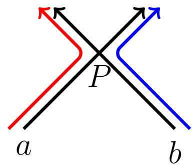
Similarly to the first solution, consider a circle that encompasses all line segments, extend the lines, and use the endpoints of the chords instead of the line segments, and prove that each chord connects vertices k and $k+n$. We also consider, even in the first part, n presents leaving from n outcoming vertices.
First we prove that a present always goes to a sink. If it does not, then it loops; let it first enter the loop at point P after turning from chord a to chord b. Therefore after it loops once,

[1]The paths do not depend uniquely on k and n; different chord configurations and vertex labelings may change the paths.

it must turn to chord b at P. But P is the intersection of a and b, so the present should turn from chord a to chord b, which can only be done in one way – the same way it came in first. This means that some part of chord a before the present enters the loop at P is part of the loop, which contradicts the fact that P is the first point in the loop. So no present enters a loop, and every present goes to a sink.



There are no loops



No two paths cross

The present paths also do not cross: in fact, every time two paths share a point P, intersection of chords a and b, one path comes from a to b and the other path comes from b to a, and they touch at P. This implies the following sequence of facts:

- Every path divides the circle into two regions with paths connecting vertices within each region.
- All n presents will be delivered to n different persons; that is, all sinks receive a present. This implies that every vertex is an endpoint of a path.
- The number of chord endpoints inside each region is even, because they are connected within their own region.

Now consider the path starting at vertex 1, with Tony. It divides the circle into two regions with an even number of vertices in their interior. Then there is an even number of vertices between Tony and the recipient of his present, that is, their vertex is an even numbered one.

Second part: all even-numbered vertices can receive a present.

The construction is the same as the in the previous solution: direct each chord from i to $i+n$ if $1 \le i \le k$ and from $i+n$ to i otherwise; in other words, the sinks are $k+1$, $k+2$, …, $k+n$. Then, since the paths do not cross, k will send a present to $k+1$, $k-1$ will send a present to $k+2$, and so on, until 1 sends a present to $(k+1)+(k-1) = 2k$.