Медитеранска математичка олимпијада

29.04.2018 година

Задача 1. Целиот број $a \geq 1$ го нарекуваме *интересен*, ако за секој природен $n \geq 1$ бројот $a^{n+2} + 3a^2 + 1$ е сложен. Докажи дека множеството $\{1, 2, 3, ..., 2018\}$ содржи најмалку 500 интересни броеви.

Задача 2. Нека $a_1, a_2, ..., a_n$, $n \geq 2$ се реални броеви такви што $0 \leq a_i \leq \frac{\pi}{2}$. Докажи дека

$$(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\sin a_i})(1+\prod_{i=1}^{n}(\sin a_i)^{1/n}) \leq 1.$$

Кога важи знак за равенство?

Задача 3. Определи го најголемиот природен број N за кој постои $6 \times N$ табела T за која се исполнети следниве својства:

i) Секоја колона во некој редослед ги содржи броевите 1, 2, 3, 4, 5 и 6.

ii) За секои две колони $i \neq j$ постои ред r така што $T(r,i) = T(r,j)$.

iii) За секои две колони $i \neq j$ постои ред s така што $T(s,i) \neq T(s,j)$.

Забелешка. Со $T(m,k)$ е означен елементот кој се наоѓа во пресекот на $m-$тиот ред и $k-$тата колона.

Задача 4. Даден е остроаголен $\triangle ABC$. Нека правите AE и AF, ($E, F \in BC$) се симетрични во однос на симетралата на $\measuredangle A$. Правите AE и AF по вторпат ја сечат опишаната кружница околу $\triangle ABC$ во точките M и N, соодветно. Точките P и R припаѓаат на полуправите AB и AC, соодветно и притоа важи $\measuredangle AER = \measuredangle C$ и $\measuredangle PEA = \measuredangle B$. Нека $L = AE \cap PR$ и $D = BC \cap LN$. Докажи дека

$$\frac{1}{MN} + \frac{1}{EF} = \frac{1}{ED}.$$

Секоја задача се вреднува по 7 поени.

Време за работа 4:30

Користењето на калкулатор не е дозволено.

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Problem 1.

An integer $a \geq 1$ is called *Aegean*, if none of the numbers $a^{n+2} + 3a^n + 1$ with $n \geq 1$ is prime. Prove that there are at least 500 Aegean integers in the set $\{1, 2, \ldots, 2018\}$.

Solution

We identify two infinite families of Aegean integers a. The first family consists of the integers of the form $a \equiv 1 \pmod 5$, as then all $n \geq 1$ satisfy

$$(a^2+3)a^n + 1 \equiv (1^2+3) \cdot 1^n + 1 \equiv 5 \equiv 0 \pmod 5.$$

Consequently $a = 5b + 1$ is Aegean for $b = 1, \ldots, 403$.

The second family consists of the integers of the form $a \equiv -1 \pmod{15}$. Indeed if $n = 2k+1$ is odd, then $a \equiv -1 \pmod 3$ implies

$$(a^2+3)a^n + 1 \equiv ((-1)^2+3)(-1)^{2k+1} + 1 \equiv -4 + 1 \equiv 0 \pmod 3.$$

On the other hand if $n = 2k$ is even, then $a \equiv -1 \pmod 5$ implies

$$(a^2+3)a^n + 1 \equiv ((-1)^2+3)(-1)^{2k} + 1 \equiv 4 + 1 \equiv 0 \pmod 5.$$

This yields that $a = 15c - 1$ is Aegean for $c = 1, \ldots, 134$.

Altogether, these two (disjoint) families yield at least $403 + 134 = 537$ Aegean integers in the range $\{1, 2, \ldots, 2018\}$.

Problem 2.

Let $a_1, a_2, \ldots, a_n$ be $n \geq 2$ real numbers such that $0 \leq a_i \leq \pi/2$. Prove that

$$\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\sin a_i}\right)\left(1+\prod_{i=1}^{n}(\sin a_i)^{1/n}\right) \leq 1.$$

Solution. First, we write the inequality claimed in the equivalent form

$$\sum_{i=1}^{n}\frac{1}{1+\sin a_i} \leq \frac{n}{1+\prod_{i=1}^{n}(\sin a_i)^{1/n}},$$

and using induction, we will prove it for all $n = 2^j$, where j is a positive integer. Indeed, for $j = 1$ the inequality claimed is

$$\frac{1}{1+\sin a_1} + \frac{1}{1+\sin a_2} \leq \frac{2}{1+\sqrt{\sin a_1 \, \sin a_2}}$$

or

$$2\,(1+\sin a_1)(1+\sin a_2) \geq (2+\sin a_1+\sin a_2)\,(1+\sqrt{\sin a_1 \, \sin a_2}),$$

$$\sin a_1 + \sin a_2 + 2\sin a_1 \sin a_2 \geq (2+\sin a_1+\sin a_2)\,\sqrt{\sin a_1 \, \sin a_2},$$

$$(\sin a_1 + \sin a_2)\,(1-\sqrt{\sin a_1 \, \sin a_2}) - 2\,\sqrt{\sin a_1 \, \sin a_2}\,(1+\sqrt{\sin a_1 \, \sin a_2})$$

from which

$$(\sqrt{\sin a_1} - \sqrt{\sin a_2})^2\,(1-\sqrt{\sin a_1 \, \sin a_2}) \geq 0$$

follows and the inequality holds.

Assume that it holds

$$\sum_{i=1}^{2^j}\frac{1}{1+\sin a_i} \leq \frac{2^j}{1+\sqrt[2^j]{\sin a_1 \, \sin a_2 \cdots \sin a_{2^j}}}$$

Then, for 2^{j+1} we have

$$\begin{aligned}
\sum_{i=1}^{2^{j+1}}\frac{1}{1+\sin a_i} &= \sum_{i=1}^{2^j}\left(\frac{1}{1+\sin a_{2i-1}} + \frac{1}{1+\sin a_{2i}}\right) \\
&\leq 2\sum_{i=1}^{2^j}\frac{1}{1+\sqrt{\sin a_{2i-1} \, \sin a_{2i}}} \\
&\leq \frac{2^{j+1}}{1+\sqrt[2^j]{\sqrt{\sin a_1 \, \sin a_2}\,\sqrt{\sin a_3 \, \sin a_4}\ldots\sqrt{\sin a_{2^{j+1}-1} \, \sin a_{2^{j+1}}}}} \\
&= \frac{2^{j+1}}{1+\sqrt[2^{j+1}]{\sin a_1 \, \sin a_2 \cdots \sin a_{2^{j+1}}}}
\end{aligned}$$

Thus, by PMI the inequality holds for $n = 2^j$.

Finally, we will use Backward induction. That is, we prove $P(k) \Rightarrow P(k-1)$ for all $k \geq 3$. Putting

$$\sin a_k = \sqrt[k-1]{\sin a_1 \, \sin a_2 \cdots \sin a_{k-1}},$$

we have

$$\sum_{i=1}^{k} \frac{1}{1+\sin a_i} = \sum_{i=1}^{k-1} \frac{1}{1+\sin a_i} + \frac{1}{1+\sqrt[k-1]{\sin a_1 \, \sin a_2 \cdots \sin a_{k-1}}}$$

$$\leq \frac{k}{1+\sqrt[k]{\sin a_1 \cdots \sin a_{k-1} \cdot \sqrt[k-1]{\sin a_1 \, \sin a_2 \cdots \sin a_{k-1}}}}$$

$$= \frac{k}{1+\sqrt[k-1]{\sin a_1 \, \sin a_2 \cdots \sin a_{k-1}}}$$

from which

$$\sum_{i=1}^{k-1} \frac{1}{1+\sin a_i} \leq \frac{k-1}{1+\sqrt[k-1]{\sin a_1 \, \sin a_2 \cdots \sin a_{k-1}}}$$

follows. Equality holds when $a_1 = a_2 = \ldots = a_n$, and we are done.

Problem 3.

Determine the largest integer N, for which there exists a $6 \times N$ table T that has the following properties:

(i) Every column contains the numbers $1, 2, \ldots, 6$ in some ordering.

(ii) For any two columns $i \neq j$, there exists a row r such that $T(r,i) = T(r,j)$.

(iii) For any two columns $i \neq j$, there exists a row s such that $T(s,i) \neq T(s,j)$.

Solution

We show that $N = 5! = 120$ is the largest such integer. The lower bound construction is as follows. For every permutation of the integers $1, \ldots, 5$ create a corresponding column whose first 5 entries agree with the permutation and whose last entry (in the 6th row) equals 6.

The upper bound argument is as follows. Consider a $6 \times N$ table T with the desired properties. For each of its columns c and for every integer $x = 1, 2, \ldots, 6$ we define a new column c_x that consists of the 6entries

$$T(1,c) + x, \quad T(2,c) + x, \quad T(3,c) + x, \quad T(4,c) + x, \quad T(5,c) + x, \quad T(6,c) + x.$$

Now consider two columns i and j, and two integers x and y with $1 \leq x, y \leq 6$, and assume that the columns i_x and j_y agree componentwise modulo 6. By condition (ii) there exists a row r such that $T(r,i) = T(r,j)$. This means

$$T(r,i) + x \;=\; i_x(r) \;\equiv\; j_y(r) \;=\; T(r,j) + y \;=\; T(r,i) + y \pmod 6,$$

which implies $x \equiv y \bmod 6$ and hence $x = y$. If $i \neq j$, then by condition (iii) there exists a row s such that $T(s,i) \neq T(s,j)$. By using $x = y$ this then would imply the contradiction

$$T(s,i) + x \;=\; i_x(s) \;\equiv\; j_y(s) \;=\; T(s,j) + y \;=\; T(s,j) + x \;\not\equiv\; T(s,i) + x \pmod 6.$$

Hence whenever two columns i_x and j_y agree componentwise modulo 6, then $i = j$ and $x = y$ must hold. This implies that the $6N$ columns c_x with $c \in T$ and $x = 1, 2, \ldots, 6$ must be pairwise distinct. By condition (i), these pairwise distinct objects correspond to pairwise distinct permutations of $1, 2, \ldots, 6$. Therefore $6N \leq 6!$, so that $N \leq 5!$.

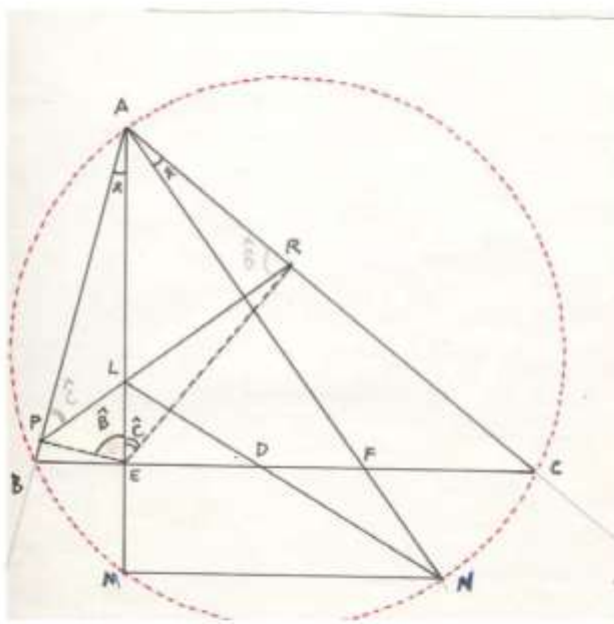
Problem 4.

ABC is an acute triangle. *AE* and *AF* are isogonal cevians, where $E \in BC$ and $F \in BC$. The straight lines *AE* and *AF* intersect again the circumcircle of *ABC* at points *M* and *N*, respectively. In the rays *AB* and *AC* we get points *P* and *R* such that $\measuredangle PEA = \measuredangle B$ and $\measuredangle AER = \measuredangle C$. Let $L = AE \cap PR$ and $D = BC \cap LN$. Prove, with reasons, that

$$\frac{1}{MN} + \frac{1}{EF} = \frac{1}{ED}.$$

Solution 1

Consider the following diagram:



The following couples of triangles are clearly similar:

$\triangle AEF$ and $\triangle AMN$, because a general property of isogonal show that MN and BC are parallel; then we have $\frac{AE}{AM} = \frac{EF}{MN}$ (1). By the same rason, $\triangle LED$ and $\triangle LMN$ are also similar, and so we have $\frac{LE}{LM} = \frac{ED}{MN}$ (2).

The following couples of triangles are similar, too:

$$\triangle APE \text{ and } \triangle ABE \Rightarrow AE^2 = AP \cdot AB$$

$$\triangle APL \text{ and } \triangle ABM \Rightarrow \frac{AP}{AM} = \frac{AL}{AB} \Rightarrow AM \cdot AL = AP \cdot AB = AE^2$$

And so we get

$$\frac{AM}{AE} = \frac{AE}{AL} \ (3).$$

Then using (2),

$$\frac{1}{MN} = \frac{LE}{LM \cdot ED}.$$

And using (1),

$$\frac{1}{EF} = \frac{AM}{AE \cdot MN} = \frac{AM}{AE} \cdot \frac{LE}{LM \cdot ED}.$$

Therefore,

$$\frac{1}{MN} + \frac{1}{EF} = \frac{1}{ED} \cdot \left[\frac{LE}{LM} \left(1 + \frac{AM}{AE} \right) \right],$$

and so we need just to prove that the last bracket equals 1. To this, we will use (3):

$$\left[\frac{LE}{LM} \left(1 + \frac{AM}{AE} \right) \right] =$$

$$\left[\frac{LE}{LM} \left(1 + \frac{AM}{AE} \right) \right] = \frac{LE}{LM} \left(1 + \frac{AE}{AL} \right) = \frac{(AE - AL)(AE + AL)}{(AM - AL) \cdot AL} = \frac{AE^2 - AL^2}{AM \cdot AL - AL^2} =$$

$$= \frac{AE^2 - AL^2}{AE^2 - AL^2}$$, and we are done.■

Observation

PE and BE are antiparallel with respect to AM and AB, so P and B are homologous in the inversion of pole A and power AE^2. The same reasoning applies to PL and BM. This means that $AP \cdot AB = AL \cdot AM = AE^2$, and continue as in the featured solution.