EGMO | 2014

European Girls' Mathematical Olympiad

Antalya · Turkey

Language: **English**

Day: **1**

Saturday, April 12, 2014

Problem 1. Determine all real constants t such that whenever a, b, c are the lengths of the sides of a triangle, then so are $a^2 + bct$, $b^2 + cat$, $c^2 + abt$.

Problem 2. Let D and E be points in the interiors of sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$. Let the lines CD and BE meet at F. Prove that the incentre I of triangle ABC, the orthocentre H of triangle DEF and the midpoint M of the arc BAC of the circumcircle of triangle ABC are collinear.

Problem 3. We denote the number of positive divisors of a positive integer m by $d(m)$ and the number of distinct prime divisors of m by $\omega(m)$. Let k be a positive integer. Prove that there exist infinitely many positive integers n such that $\omega(n) = k$ and $d(n)$ does not divide $d(a^2 + b^2)$ for any positive integers a, b satisfying $a + b = n$.

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Language: **English**

Day: 2

Sunday, April 13, 2014

Problem 4. Determine all integers $n \geq 2$ for which there exist integers $x_1, x_2, \ldots, x_{n-1}$ satisfying the condition that if $0 < i < n$, $0 < j < n$, $i \neq j$ and n divides $2i + j$, then $x_i < x_j$.

Problem 5. Let n be a positive integer. We have n boxes where each box contains a non-negative number of pebbles. In each move we are allowed to take two pebbles from a box we choose, throw away one of the pebbles and put the other pebble in another box we choose. An initial configuration of pebbles is called *solvable* if it is possible to reach a configuration with no empty box, in a finite (possibly zero) number of moves. Determine all initial configurations of pebbles which are not solvable, but become solvable when an additional pebble is added to a box, no matter which box is chosen.

Problem 6. Determine all functions $f \colon \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f(y^2 + 2xf(y) + f(x)^2) = (y + f(x))(x + f(y))$$

for all real numbers x and y.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

1. Determine all real constants t such that whenever a, b, c are the lengths of the sides of a triangle, then so are a^2+bct, b^2+cat, c^2+abt.

Proposed by S. Khan, UNK

The answer is the interval $[2/3, 2]$.

Solution 1.

If $t<2/3$, take a triangle with sides $c=b=1$ and $a=2-\epsilon$. Then $b^2+cat+c^2+abt-a^2-bct = 3t-2+\epsilon(4-2t-\epsilon) \le 0$ for small positive ϵ; for instance, for any $0<\epsilon<(2-3t)/(4-2t)$.

On the other hand, if $t>2$, then take a triangle with sides $b=c=1$ and $a=\epsilon$. Then $b^2+cat+c^2+abt-a^2-bct = 2-t+\epsilon(2t-\epsilon) \le 0$ for small positive ϵ; for instance, for any $0<\epsilon<(t-2)/(2t)$.

Now assume that $2/3 \le t \le 2$ and $b+c>a$. Then using $(b+c)^2 \ge 4bc$ we obtain

$$\begin{aligned} b^2+cat+c^2+abt-a^2-bct &= (b+c)^2+at(b+c)-(2+t)bc-a^2 \\ &\ge (b+c)^2+at(b+c)-\frac{1}{4}(2+t)(b+c)^2-a^2 \\ &\ge \frac{1}{4}(2-t)(b+c)^2+at(b+c)-a^2 . \end{aligned}$$

As $2-t\ge 0$ and $t>0$, this last expression is an increasing function of $b+c$, and hence using $b+c>a$ we obtain

$$b^2+cat+c^2+abt-a^2-bct > \frac{1}{4}(2-t)a^2+ta^2-a^2 = \frac{3}{4}\left(t-\frac{2}{3}\right)a^2 \ge 0$$

as $t\ge 2/3$. The other two inequalities follow by symmetry.

Solution 2.

After showing that t must be in the interval $[2/3,2]$ as in **Solution 1**, we let $x=(c+a-b)/2$, $y=(a+b-c)/2$ and $z=(b+c-a)/2$ so that $a=x+y$, $b=y+z$, $c=z+x$. Then we have:

$$b^2+cat+c^2+abt-a^2-bct = (x^2+y^2-z^2+xy+xz+yz)t+2(z^2+xz+yz-xy)$$

Since this linear function of t is positive both at $t=2/3$ where

$$\frac{2}{3}(x^2+y^2-z^2+xy+xz+yz)+2(z^2+xz+yz-xy) = \frac{2}{3}((x-y)^2+4(x+y)z+2z^2) > 0$$

and at $t=2$ where

$$2(x^2+y^2-z^2+xy+xz-yz)+2(z^2+xz+yz+xy) = 2(x^2+y^2)+4(x+y)z > 0 ,$$

it is positive on the entire interval $[2/3,2]$.

Solution 3.

After the point in **Solution 2** where we obtain

$$b^2 + cat + c^2 + abt - a^2 - bct = (x^2 + y^2 - z^2 + xy + xz + yz)t + 2(z^2 + xz + yz - xy)$$

we observe that the right hand side can be rewritten as

$$(2 - t)z^2 + (x - y)^2 t + (3t - 2)xy + z(x + y)(2 + t).$$

As the first three terms are non-negative and the last term is positive, the result follows.

Solution 4.

First we show that t must be in the interval $[2/3, 2]$ as in **Solution 1**. Then:

Case 1: If $a \geq b, c$, then $ab + ac - bc > 0$, $2(b^2 + c^2) \geq (b + c)^2 > a^2$ and $t \geq 2/3$ implies:

$$\begin{aligned} b^2 + cat + c^2 + abt - a^2 - bct &= b^2 + c^2 - a^2 + (ab + ac - bc)t \\ &\geq (b^2 + c^2 - a^2) + \frac{2}{3}(ab + ac - bc) \\ &\geq \frac{1}{3}(3b^2 + 3c^2 - 3a^2 + 2ab + 2ac - 2bc) \\ &\geq \frac{1}{3}\left[(2b^2 + 2c^2 - a^2) + (b - c)^2 + 2a(b + c - a)\right] \\ &> 0 \end{aligned}$$

Case 2: If $b \geq a, c$, then $b^2 + c^2 - a^2 > 0$. If also $ab + ac - bc \geq 0$, then $b^2 + cat + c^2 + abt - a^2 - bct > 0$. If, on the other hand, $ab + ac - bc \leq 0$, then since $t \leq 2$, we have:

$$\begin{aligned} b^2 + cat + c^2 + abt - a^2 - bct &\geq b^2 + c^2 - a^2 + 2(ab + ac - bc) \\ &\geq (b - c)^2 + a(b + c - a) + a(b + c) \\ &> 0 \end{aligned}$$

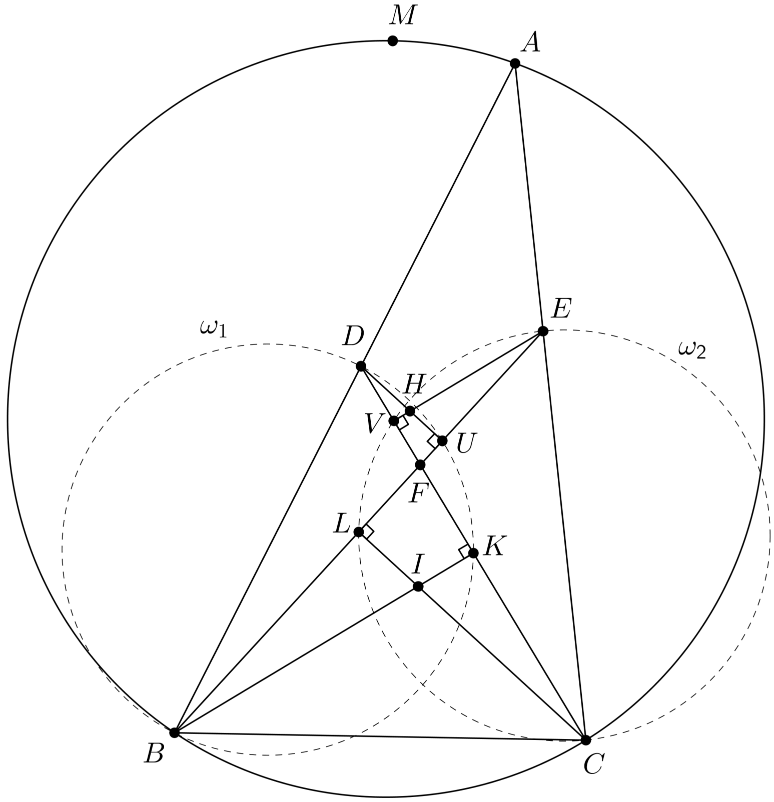
By symmetry, we are done.

2. Let D and E be two points on the sides AB and AC, respectively, of a triangle ABC, such that $DB = BC = CE$, and let F be the point of intersection of the lines CD and BE. Prove that the incenter I of the triangle ABC, the orthocenter H of the triangle DEF and the midpoint M of the arc BAC of the circumcircle of the triangle ABC are collinear.

Proposed by Danylo Khilko, UKR

Solution 1.

As $DB = BC = CE$ we have $BI \perp CD$ and $CI \perp BE$. Hence I is orthocenter of triangle BFC. Let K be the point of intersection of the lines BI and CD, and let L be the point of intersection of the lines CI and BE. Then we have the power relation $IB \cdot IK = IC \cdot IL$. Let U and V be the feet of the perpendiculars from D to EF and E to DF, respectively. Now we have the power relation $DH \cdot HU = EH \cdot HV$.



Let ω_1 and ω_2 be the circles with diameters BD and CE, respectively. From the power relations above we conclude that IH is the radical axis of the circles ω_1 and ω_2.

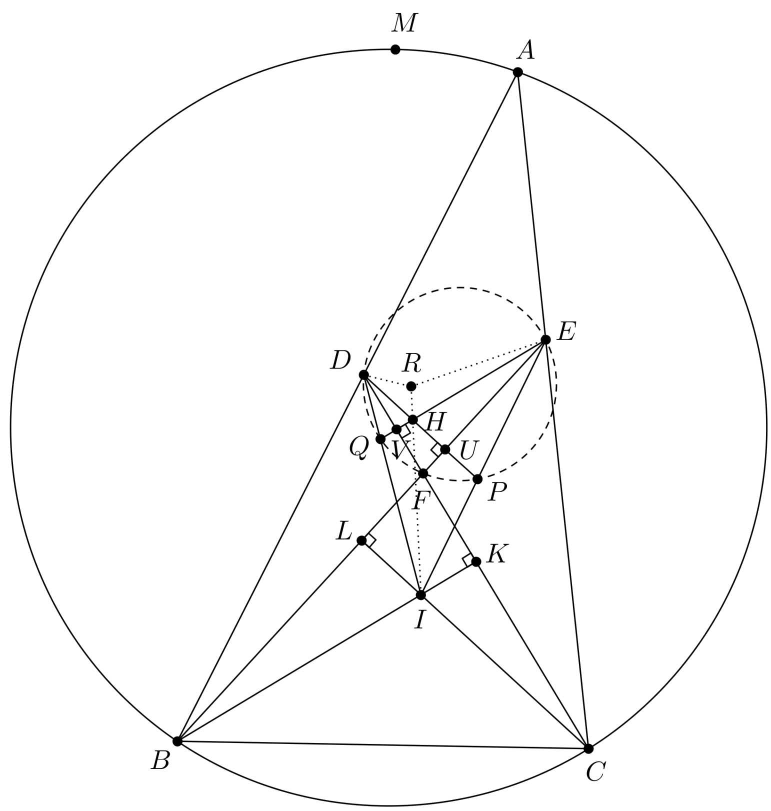
Let O_1 and O_2 be centers of ω_1 and ω_2, respectively. Then $MB = MC$, $BO_1 = CO_2$ and $\angle MBO_1 = \angle MCO_2$, and the triangles MBO_1 and MCO_2 are congruent. Hence $MO_1 = MO_2$. Since radii of ω_1 and ω_2 are equal, this implies that M lies on the radical axis of ω_1 and ω_2 and M, I, H are collinear.

Solution 2.

Let the points K, L, U, V be as in **Solution 1**. Le P be the point of intersection of DU and EI, and let Q be the point of intersection of EV and DI.

Since $DB = BC = CE$, the points CI and BI are perpendicular to BE and CD, respectively. Hence the lines BI and EV are parallel and $\angle IEB = \angle IBE = \angle UEH$. Similarly, the lines CI and DU are parallel and $\angle IDC = \angle ICD = \angle VDH$. Since $\angle UEH = \angle VDH$, the points D, Q, F, P, E are concyclic. Hence $IP \cdot IE = IQ \cdot ID$.

Let R be the second point intersection of the circumcircle of triangle HEP and the line HI. As $IH \cdot IR = IP \cdot IE = IQ \cdot ID$, the points D, Q, H, R are also concyclic. We have $\angle DQH = \angle EPH = \angle DFE = \angle BFC = 180° - \angle BIC = 90° - \angle BAC/2$. Now using the concylicity of D, Q, H, R, and E, P, H, R we obtain $\angle DRH = \angle ERH = \angle 180° - (90° - \angle BAC/2) = 90° + \angle BAC/2$. Hence R is inside the triangle DEH and $\angle DRE = 360° - \angle DRH - \angle ERH = 180° - \angle BAC$ and it follows that the points A, D, R, E are concyclic.



As $MB = MC$, $BD = CE$, $\angle MBD = \angle MCE$, the triangles MBD and MCE are congruent and $\angle MDA = \angle MEA$. Hence the points M, D, E, A are concyclic. Therefore the points M, D, R, E, A are concylic. Now we have $\angle MRE = 180° - \angle MAE = 180° - (90° + \angle BAC/2) = 90° - \angle BAC/2$ and since $\angle ERH = 90° + \angle BAC/2$, we conclude that the points I, H, R, M are collinear.

Solution 3.

Suppose that we have a coordinate system and $(b_x, b_y), (c_x, c_y), (d_x, d_y), (e_x, e_y)$ are the coordinates of the points B, C, D, E, respectively. From $\overrightarrow{BI}\cdot\overrightarrow{CD} = 0$, $\overrightarrow{CI}\cdot\overrightarrow{BE} = 0$, $\overrightarrow{EH}\cdot\overrightarrow{CD} = 0$, $\overrightarrow{DH}\cdot\overrightarrow{BE} = 0$ we obtain $\overrightarrow{IH}\cdot(\overrightarrow{B} - \overrightarrow{C} - \overrightarrow{E} + \overrightarrow{D}) = 0$. Hence the slope of the line IH is $(c_x + e_x - b_x - d_x)/(b_y + d_y - c_y - e_y)$.

Assume that the x-axis lies along the line BC, and let $\alpha = \angle BAC$, $\beta = \angle ABC$, $\theta = \angle ACB$. Since $DB = BC = CE$, we have $c_x - b_x = BC$, $e_x - d_x = BC - BC\cos\beta - BC\cos\theta$, $b_y = c_y = 0$, $d_y - e_y = BC\sin\beta - BC\sin\theta$. Therefore the slope of IH is $(2 - \cos\beta - \cos\theta)/(\sin\beta - \sin\theta)$.

Now we will show that the slope of the line MI is the same. Let r and R be the inradius and circumradius of the triangle ABC, respectively. As $\angle BMC = \angle BAC = \alpha$ and $BM = MC$, we have

$$m_y - i_y = \frac{BC}{2}\cot\left(\frac{\alpha}{2}\right) - r \quad\text{and}\quad m_x - i_x = \frac{AC - AB}{2}$$

where (m_x, m_y) and (i_x, i_y) are the coordinates of M and I, respectively. Therefore the slope of MI is $(BC\cot(\alpha/2) - 2r)/(AC - AB)$.

Now the equality of these slopes follows using

$$\frac{BC}{\sin\alpha} = \frac{AC}{\sin\beta} = \frac{AB}{\sin\theta} = 2R,$$

hence

$$BC\cot\left(\frac{\alpha}{2}\right) = 4R\cos^2\left(\frac{\alpha}{2}\right) = 2R(1 + \cos\alpha)$$

and

$$\frac{r}{R} = \cos\alpha + \cos\beta + \cos\theta - 1$$

as

$$\frac{BC\cot(\alpha/2) - 2r}{AC - AB} = \frac{2R(1 + \cos\alpha) - 2r}{2R(\sin\beta - \sin\theta)} = \frac{2 - \cos\beta - \cos\theta}{\sin\beta - \sin\theta}$$

giving the collinearity of the points I, H, M.

Solution 4.

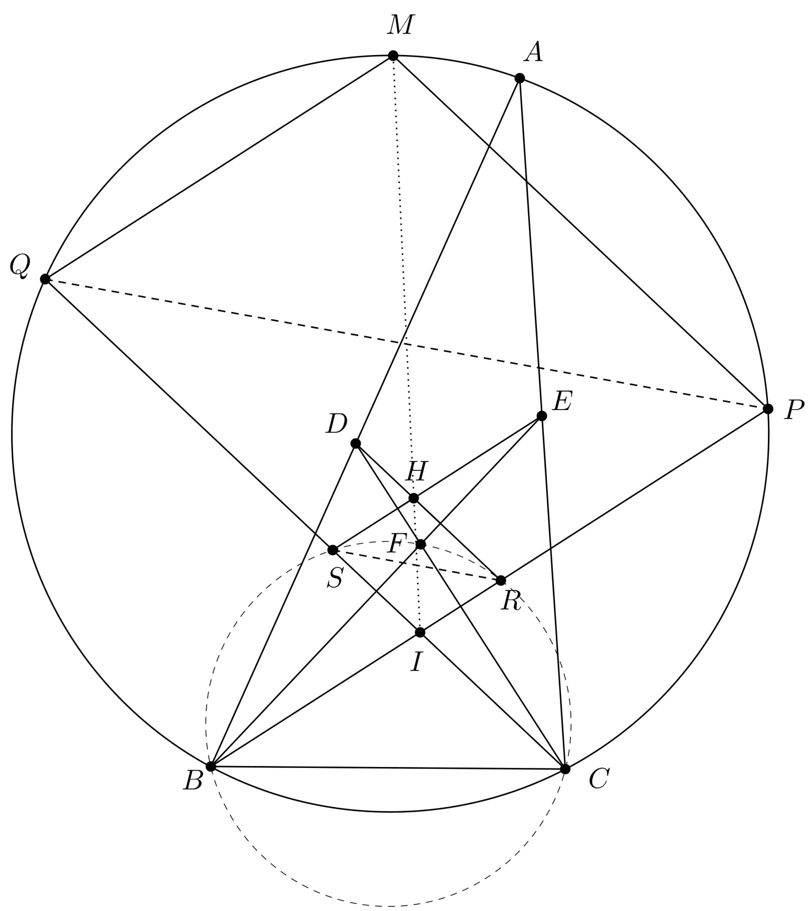
Let the bisectors BI and CI meet the circumcircle of ABC again at P and Q, respectively. Let the altitude of DEF belonging to D meet BI at R and the one belonging to E meet CI at S.

Since BI is angle bisector of the iscosceles triangle CBD, BI and CD are perpendicular. Since EH and DF are also perpendicular, HS and RI are parallel. Similarly, HR and SI are parallel, and hence $HSIR$ is a parallelogram.

On the other hand, as M is the midpoint of the arc BAC, we have $\angle MPI = \angle MPB = \angle MQC = \angle MQI$, and $\angle PIQ = (\widehat{PA} + \widehat{CB} + \widehat{AQ})/2 = (\widehat{PC} + \widehat{CB} + \widehat{BQ})/2 = \angle PMQ$. Therefore $MPIQ$ is a parallelogram.

Since CI is angle bisector of the iscosceles triangle BCE, the triangle BSE is also isosceles. Hence $\angle FBS = \angle EBS = \angle SEB = \angle HEF = \angle HDF = \angle RDF = \angle FCS$ and B, S, F, C are concyclic. Similarly, B, F, R, C are concyclic. Therefore B, S, R, C are concyclic. As B, Q, P, C are also concyclic, SR an QP are parallel.

Now it follows that $HSIR$ and $MQIP$ are homothetic parallelograms, and therefore M, H, I are collinear.



3. We denote the number of positive divisors of a positive integer m by $d(m)$ and the number of distinct prime divisors of m by $\omega(m)$. Let k be a positive integer. Prove that there exist infinitely many positive integers n such that $\omega(n) = k$ and $d(n)$ does not divide $d(a^2 + b^2)$ for any positive integers a, b satisfying $a + b = n$.

Proposed by JPN

Solution.

We will show that any number of the form $n = 2^{p-1}m$ where m is a positive integer that has exactly $k - 1$ prime factors all of which are greater than 3 and p is a prime number such that $(5/4)^{(p-1)/2} > m$ satisfies the given condition.

Suppose that a and b are positive integers such that $a + b = n$ and $d(n) \mid d(a^2 + b^2)$. Then $p \mid d(a^2 + b^2)$. Hence $a^2 + b^2 = q^{cp-1}r$ where q is a prime, c is a positive integer and r is a positive integer not divisible by q. If $q \geq 5$, then

$$2^{2p-2}m^2 = n^2 = (a + b)^2 > a^2 + b^2 = q^{cp-1}r \geq q^{p-1} \geq 5^{p-1}$$

gives a contradiction. So q is 2 or 3.

If $q = 3$, then $a^2 + b^2$ is divisible by 3 and this implies that both a and b are divisible by 3. This means $n = a + b$ is divisible by 3, a contradiction. Hence $q = 2$.

Now we have $a + b = 2^{p-1}m$ and $a^2 + b^2 = 2^{cp-1}r$. If the highest powers of 2 dividing a and b are different, then $a + b = 2^{p-1}m$ implies that the smaller one must be 2^{p-1} and this makes 2^{2p-2} the highest power of 2 dividing $a^2 + b^2 = 2^{cp-1}r$, or equivalently, $cp - 1 = 2p - 2$, which is not possible. Therefore $a = 2^t a_0$ and $b = 2^t b_0$ for some positive integer $t < p - 1$ and odd integers a_0 and b_0. Then $a_0^2 + b_0^2 = 2^{cp-1-2t}r$. The left side of this equality is congruent to 2 modulo 4, therefore $cp - 1 - 2t$ must be 1. But then $t < p - 1$ gives $(c/2)p = t + 1 < p$, which is not possible either.

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European Girls' Mathematical Olympiad

Antalya - Turkey

Problems and Solutions

Day 2

4. Determine all integers $n \geq 2$ for which there exist integers $x_1, x_2, \ldots, x_{n-1}$ satisfying the condition that if $0 < i < n$, $0 < j < n$, $i \neq j$ and n divides $2i + j$, then $x_i < x_j$.

Proposed by Merlijn Staps, NLD

The answer is that $n = 2^k$ with $k \geq 1$ or $n = 3 \cdot 2^k$ where $k \geq 0$.

Solution 1.

Suppose that n has one of these forms. For an integer i, let x_i be the largest integer such that 2^{x_i} divides i. Now assume that $0 < i < n$, $0 < j < n$, $i \neq j$, n divides $2i + j$ and $x_i \geq x_j$. Then the highest power of 2 dividing $2i + j$ is 2^{x_j} and therefore $k \leq x_j$ and $2^k \leq j$. Since $0 < j < n$, this is possible only if $n = 3 \cdot 2^k$ and either $j = 2^k$ or $j = 2^{k+1}$. In the first case, $i \neq j$ and $x_i \geq x_j$ imply $i = 2^{k+1}$ leading to the contradiction $3 \cdot 2^k = n \mid 2i + j = 5 \cdot 2^k$. The second case is not possible as $i \neq j$ and $x_i \geq x_j$ now imply $i \geq 2^{k+2} > n$.

Now suppose that n does not have one of these forms and $x_1, x_2, \ldots, x_{n-1}$ satisfying the given condition exist. For any positive integer m, let a_m be the remainder of the division of $(-2)^m$ by n. Then none of a_m is 0 as n is not a power of 2. Also $a_m \neq a_{m+1}$ for any $m \geq 1$ as $a_m = a_{m+1}$ would lead to n dividing $3 \cdot 2^m$. Moreover n divides $2a_m + a_{m+1}$. Hence we must have $x_{a_1} < x_{a_2} < x_{a_3} < \ldots$ which is not possible as a_m's can take on only finitely many values.

Solution 2.

Let $E = \{n/3, n/2, 2n/3\} \cap \{1, 2, \ldots, n-1\}$, $D = \{1, 2, \ldots, n-1\} \setminus E$, and let $f : D \to \{1, 2, \ldots, n-1\}$ be the function sending i in D to the unique $f(i)$ in $\{1, 2, \ldots, n-1\}$ such that $f(i) \equiv -2i \pmod{n}$.

Then the condition of the problem is that $x_i < x_{f(i)}$ for each i in D. Since D is a finite set, the integers $x_1, x_2, \ldots, x_{n-1}$ exist if and only if for each i in D there exists a positive integer $k(i)$ such that $f^{k(i)}(i)$ belongs to E. This can be seen as follows:

- If $f^k(i)$ does not belong to E for any $k > 0$ for some i, then there exists $k_2 > k_1 > 0$ such that $f^{k_1}(i) = f^{k_2}(i)$, leading to the contradiction $x_{f^{k_1}(i)} < x_{f^{k_2}(i)} = x_{f^{k_1}(i)}$.
- On the other hand, if such $k(i)$ exists for each i in D, and if $k_0(i)$ denotes the smallest such, then the condition of the problem is satisfied by letting $x_i = -k_0(i)$ for i in D, and $x_i = 0$ for i in E.

In other words, the integers $x_1, x_2, \ldots, x_{n-1}$ exist if and only if for each i in D there exists a positive integer $k(i)$ such that $(-2)^{k(i)} i \equiv n/3, n/2$ or $2n/3 \pmod{n}$. For $i = 1$, this implies that $n = 2^k$ with $k \geq 1$ or $n = 3 \cdot 2^k$ with $k \geq 0$. On the other hand, if n has one of these forms, letting $k(i) = k$ does the trick for all i in D.

Solution 3.

Suppose that $x_1, x_2, \ldots, x_{k-1}$ satisfy the condition of the problem for $n = k$. Let $y_{2i} = x_i$ for $1 \leq i \leq k-1$ and choose y_{2i-1} for $1 \leq i \leq k$ to be less than $\min\{x_1, x_2, \ldots, x_{k-1}\}$. Now suppose that for $n = 2k$ we have $0 < i < n$, $0 < j < n$, $i \neq j$, n divides $2i+j$. Then j is even. If i is also even, then $0 < i/2 < k$, $0 < j/2 < k$ and k divides $2(i/2) + (j/2)$; hence $y_i = x_{i/2} < x_{j/2} = y_j$. On the other hand, if i is odd, then $y_i < \min\{x_1, x_2, \ldots, x_{k-1}\} \leq x_{j/2} = y_j$. Therefore, $y_1, y_2, \ldots, y_{2k-1}$ satisfy the condition of the problem for $n = 2k$.

Since the condition is vacuous for $n = 2$ and $n = 3$, it follows that $x_1, x_2, \ldots, x_{n-1}$ satisfying the condition exist for all $n = 2^k$ with $k \geq 1$ and $n = 3 \cdot 2^k$ with $k \geq 0$.

Now suppose that $x_1, x_2, \ldots, x_{n-1}$ satisfying the condition of the problem exist for $n = 2^k m$ where k is a nonnegative integer and $m > 3$ is an odd number. Let $b_0 = 2^k$ and let b_{i+1} be the remainder of the division of $(-2)b_i$ by n for $i \geq 0$. No terms of this sequence is 0 and no two consecutive terms are both equal to b_1 as $m > 3$. On the other hand, as $(-2)^{\phi(m)} \equiv 1 \pmod{m}$, we have $b_{\phi(m)} \equiv (-2)^{\phi(m)} 2^k \equiv 2^k \equiv b_0 \pmod{n}$, and hence $b_{\phi(m)} = b_0$. Since $2b_i + b_{i+1}$ is divisible by n for all $i \geq 0$, we have $x_{b_0} < x_{b_1} < \cdots < x_{b_{\phi(m)}} = x_{b_0}$, a contradiction.

5. Let n be a positive integer. We have n boxes where each box contains a non-negative number of pebbles. In each move we are allowed to take two pebbles from a box we choose, throw away one of the pebbles and put the other pebble in another box we choose. An initial configuration of pebbles is called *solvable* if it is possible to reach a configuration with no empty box, in a finite (possibly zero) number of moves. Determine all initial configurations of pebbles which are not solvable, but become solvable when an additional pebble is added to a box, no matter which box is chosen.

Proposed by Dan Schwarz, ROU

The answer is any configuration with $2n-2$ pebbles which has even numbers of pebbles in each box.

Solution 1. Number the boxes from 1 through n and denote a configuration by $x = (x_1, x_2, \dots, x_n)$ where x_i is the number of pebbles in the ith box. Let

$$D(x) = \sum_{i=1}^{n} \left\lfloor \frac{x_i - 1}{2} \right\rfloor$$

for a configuration x. We can rewrite this in the form

$$D(x) = \frac{1}{2}N(x) - n + \frac{1}{2}O(x)$$

where $N(x)$ is the total number of pebbles and $O(x)$ is the number of boxes with an odd number of pebbles for the configuration x.

Note that a move either leaves D the same (if it is made into a box containing an even number of pebbles) or decreases it by 1 (if it is made into a box with an odd number of pebbles). As D is nonnegative for any configuration which does not have any empty boxes, it is also nonnegative for any solvable configuration. On the other hand, if a configuration has nonnegative D, then making $m_i = \lfloor (x_i-1)/2 \rfloor$ moves from the ith box into m_i empty boxes for each i with $m_i > 0$ fills all boxes as $D(x) \geq 0$ means $\sum_{m_i>0} m_i \geq$ (number of empty boxes).

As $N(x)$ and $O(x)$ have the same parity, a configuration x is solvable exactly when $O(x) \geq 2n - N(x)$, and unsolvable exactly when $O(x) \leq 2n-2-N(x)$. In particular, any configuration with $2n-1$ pebbles is solvable, and a configuration with $2n-2$ pebbles is unsolvable if and only if all boxes contain even numbers of pebbles.

Suppose that x' is obtained from x by adding a pebble in some box. Then $O(x') = O(x) + 1$ or $O(x') = O(x) - 1$. If x is unsolvable and x' is solvable, then we must have $O(x) \leq 2n - 2 - N(x)$ and $O(x') \geq 2n - N(x') = 2n - 1 - N(x)$, and hence $O(x') = O(x) + 1$. That is, the pebble must be added to a box with an even number of pebbles. This can be the case irrespective of where the pebble is added only if all boxes contain even numbers of pebbles, and $0 = O(x) \leq 2n - 2 - N(x)$ and $1 = O(x') \geq 2n - 1 - N(x)$; that is, $N(x) = 2n - 2$.

Solution 2. Let x be a configuration and $\tilde{x}$ be another configuration obtained from x by removing two pebbles from a box and depositing them in another box.

Claim 1: $\tilde{x}$ is solvable if and only if x is solvable.

Let us call two configurations *equivalent* if they have the same total number of pebbles and parities of the number of pebbles in the corresponding boxes are the same. (It does not matter whether we consider this equivalence for a fixed ordering of the boxes or up to permutation.) From *Claim 1* it follows that two equivalent configurations are both solvable or both unsolvable. In particular, any configuration with $2n-1$ or more pebbles is solvable, because it is equivalent to a configuration with no empty boxes.

Let us a call a configuration with all boxes containing two or fewer pebbles *scant*. Every unsolvable configuration is equivalent to a scant configuration.

Claim 2: A scant configuration is solvable if and only if it contains no empty boxes.

By *Claim 1* and *Claim 2*, addition of a pebble to a scant unsolvable configuration makes it solvable if and only if the configuration has exactly one empty box and the pebble is added to the empty box or to a box containing two pebbles. Hence, the addition of a pebble makes an unsolvable scant configuration into a solvable configuration irrespective of where it is added if and only if all boxes have even numbers of pebbles and exactly one of them is empty. Therefore, the addition of a pebble makes an unsolvable configuration into a solvable one irrespective of where the pebble is added if and only if the configuration has $2n-2$ pebbles and all boxes have even numbers of pebbles.

Proof of Claim 1: Suppose that the two pebbles were moved from box B in x to box $\tilde{B}$ in $\tilde{x}$, and x is solvable. Then we perform exactly the same sequence of moves for $\tilde{x}$ as we did for x except that instead of the first move that is made out of B we make a move from $\tilde{B}$ (into the same box), and if there was no move from B, then at the end we make a move from $\tilde{B}$ to B in case B is now empty.

Proof of Claim 2: Any move from a scant configuration either leaves the number of empty boxes the same and the resulting configuration is also scant (if it is made into an empty box), or increases the number of empty boxes by one (if it is made into a nonempty box). In the second case, if the move was made into a box containing one pebble, then the resulting configuration is still scant. On the other hand, if it is made into a box containing two pebbles, then the resulting configuration is equivalent to the scant configuration which has one pebble in the box the move was made into and exactly the same number of pebbles in all other boxes as the original configuration. Therefore, any sequence of move from a scant configuration results in a configuration with more or the same number of empty boxes.

6. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$f\left(y^2 + 2xf(y) + f(x)^2\right) = (y + f(x))(x + f(y))$$

for all real numbers x and y.

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The answer is the functions $f(x) = x$, $f(x) = -x$ and $f(x) = \dfrac{1}{2} - x$.

Solution.

It can be easily checked that the functions $f(x) = x$, $f(x) = -x$ and $f(x) = \dfrac{1}{2} - x$ satisfy the given condition. We will show that these are the only functions doing so.

Let $y = -f(x)$ in the original equation to obtain

$$f(2f(x)^2 + 2xf(-f(x))) = 0$$

for all x. In particular, 0 is a value of f. Suppose that u and v are such that $f(u) = 0 = f(v)$. Plugging $x = u$ or v and $y = u$ or v in the original equations we get $f(u^2) = u^2$, $f(u^2) = uv$, $f(v^2) = uv$ and $f(v^2) = v^2$. We conclude that $u^2 = uv = v^2$ and hence $u = v$. So there is exactly one a mapped to 0, and

$$f(x)^2 + xf(-f(x)) = \frac{a}{2} \tag{*}$$

for all x.

Suppose that $f(x_1) = f(x_2) \neq 0$ for some x_1 and x_2. Using (*) we obtain $x_1 f(-f(x_1)) = x_2 f(-f(x_2)) = x_2 f(-f(x_1))$ and hence either $x_1 = x_2$ or $f(x_1) = f(x_2) = -a$. In the second case, letting $x = a$ and $y = x_1$ in the original equation we get $f(x_1^2 - 2a^2) = 0$, hence $x_1^2 - 2a^2 = a$. Similarly, $x_2^2 - 2a^2 = a$, and it follows that $x_1 = x_2$ or $x_1 = -x_2$ in this case.

Using the symmetry of the original equation we have

$$f(f(x)^2 + y^2 + 2xf(y)) = (x + f(y))(y + f(x)) = f(f(y)^2 + x^2 + 2yf(x)) \tag{**}$$

for all x and y. Suppose $f(x)^2 + y^2 + 2xf(y) \neq f(y)^2 + x^2 + 2yf(x)$ for some x and y. Then by the observations above, $(x + f(y))(y + f(x)) \neq 0$ and $f(x)^2 + y^2 + 2xf(y) = -(f(y)^2 + x^2 + 2yf(x))$. But these conditions are contradictory as the second one can be rewritten as $(f(x) + y)^2 + (f(y) + x)^2 = 0$.

Therefore from (**) now it follows that

$$f(x)^2 + y^2 + 2xf(y) = f(y)^2 + x^2 + 2yf(x) \tag{***}$$

for all x and y. In particular, letting $y = 0$ we obtain $f(x)^2 = (f(0) - x)^2$ for all x. Let $f(x) = s(x)(f(0) - x)$ where $s : \mathbf{R} \to \{1, -1\}$. Plugging this in (***) gives

$$x(ys(y) + f(0)(1 - s(y)) = y(xs(x) + f(0)(1 - s(x)))$$

for all x and y. So $s(x) + f(0)(1 - s(x))/x$ must be constant for $x \neq 0$.

If $f(0) = 0$ it follows that $s(x)$ is constant for $x \neq 0$, and therefore either $f(x) = x$ for all x or $f(x) = -x$ for all x. Suppose that $f(0) \neq 0$. If $s(x)$ is -1 for all $x \neq 0$, then $-1 + 2f(0)/x$ must be constant for all $x \neq 0$, which is not possible. On the other hand, if there exist nonzero x and y such that $s(x) = -1$ and $s(y) = 1$, then $-1 + 2f(0)/x = 1$. That is, there can be only one such x, that x is $f(0)$, and hence $f(x) = f(0) - x$ for all x. Putting this back in the original equation gives $2f(0)^2 = f(0)$ and hence $f(0) = 1/2$. We are done.

Remark:

The following line of reasoning or a variant of it can be used between (*) and (***):

Suppose that $f(x_1) = f(x_2) \neq 0$ for some x_1 and x_2. Then from (*) it follows that $x_1 f(-f(x_1)) = x_2 f(-f(x_2)) = x_2 f(-f(x_1))$ and hence either $x_1 = x_2$ or $f(x_1) = f(x_2) = -a$. In the second case, using (*) again we obtain $a^2 = a/2$ and therefore $a = 1/2$. Now letting $x = 1/2$ in the original equation gives $f(y^2 + f(y)) = y(f(y) + 1/2)$ for all y. From this letting $y = 0$ we obtain $f(0) = 1/2$, and letting $f(y) = -1/2$ we obtain $f(y^2 - 1/2) = 0$ and $y^2 = 1$. To summarize, $f(x_1) = f(x_2) \neq 0$ implies either $x_1 = x_2$ or $x_1, x_2 \in \{1, -1\}$ and $f(1) = f(-1) = -1/2$, $f(1/2) = 0$, $f(0) = 1/2$.

Using the symmetry of the original equation we have

$$f(f(x)^2 + y^2 + 2xf(y)) = (x + f(y))(y + f(x)) = f(f(y)^2 + x^2 + 2yf(x)) \qquad (**)$$

for all x and y. Let $y = 0$. Then

$$f(f(x)^2 + 2xf(0)) = f(f(0)^2 + x^2)$$

for all x. If $f(x)^2 + 2xf(0) \neq f(0)^2 + x^2$ for some x, then by the observation above we must have $f(1/2) = 0$, $f(0) = 1/2$ and $f(x)^2 + 2xf(0) = -(f(0)^2 + x^2)$. We can rewrite this as $f(x)^2 + (f(0) + x)^2 = 0$ to obtain $x = 1/2$ and $f(0) = -x = -1/2$, which contradicts $f(0) = 1/2$. So we conclude that $f(x)^2 + 2xf(0) = f(0)^2 + x^2$ for all x. This implies $f(x)^2 = (f(0) - x)^2$ for all x. In particular, the second case considered above is not possible as $(f(0) - 1)^2 = f(1) = f(-1) = (f(0) + 1)^2$ means $f(0) = 0$, contradicting $f(0) = 1/2$. Therefore f is injective and from (**) now it follows that

$$f(x)^2 + y^2 + 2xf(y) = f(y)^2 + x^2 + 2yf(x) \qquad (***)$$

for all x and y.