8^{th} Iranian Geometry Olympiad

November 5, 2021

Contest problems with solutions

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This booklet is prepared by Elahe Zahiri, Mahdi Shavali, Amirmohammad Derakhshandeh and Alireza Dadgarnia.

With special thanks to Hesam Rajabzade, Mahdi Etesamifard and Morteza Saghafian.

Participating Countries

List of participated nations at the 8^{th} Iranian Geometry Olympiad:

Afghanistan	Albania	Argentina
Armenia	Austria	Bangladesh
Belarus	Bolivia	Bosnia and Herzegovina
Brazil	Bulgaria	China
Colombia	Costa Rica	Croatia
Cuba	Czech Repbulic	Dominican Republic
El Salvador	Estonia	Finland
Hong Kong	India	Iran
Ireland	Italy	Kazakhstan
Kosovo	Kyrgyzstan	Macedonia
Malaysia	Mexico	Mongolia
Nepal	Netherlands	Nicaragua
Nigeria	Pakistan	Panama
Paraguay	Philippines	Poland
Republic of Moldova	Romania	Russia
Slovakia	Slovenia	South Africa
Sweden	Syria	Tajikistan
Turkey	Turkmenistan	Ukraine
Uzbekistan	Venezuela	Vietnam

Contributing Countries

The Organizing Committee and the Scientific Committee of the IGO 2021 thank the following countries for contributing 100 problem proposals:

Czech Republic, Hong Kong, India, Iran, Mexico,
Poland, Russia, Slovakia, United Kingdom, Vietnam.

Scientific Committee


Patrik Bak (Slovakia)


Alireza Dadgarnia


Mahdi Etesamifard (chair)


Boris Frenkin (Russia)


Radek Olšák (Czech Republic)


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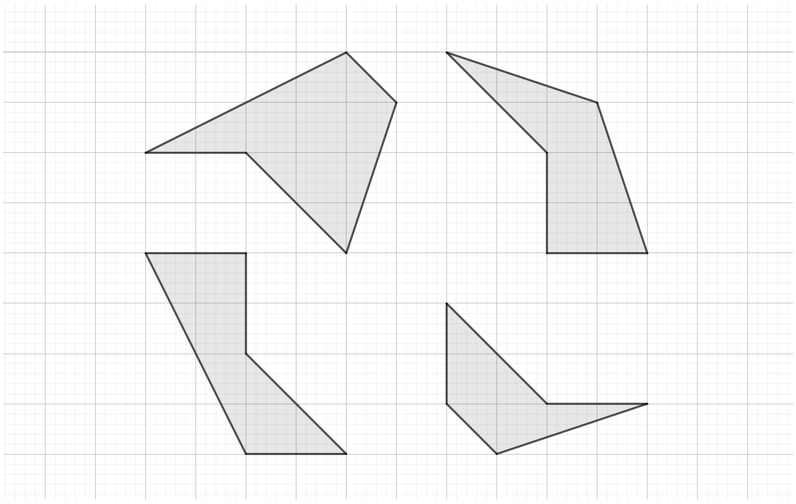
Elementary Level

Problems

Problem 1. With putting the four shapes drawn in the following figure together make a shape with at least two reflection symmetries.

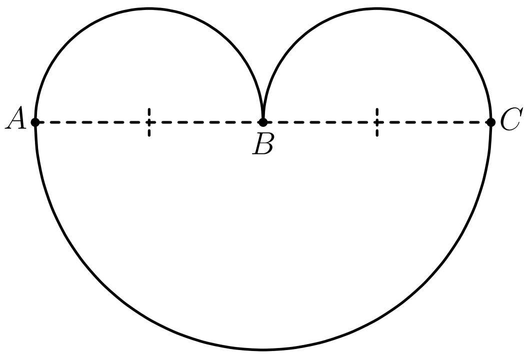


(→ p.5)

Problem 2. Points K, L, M, N lie on the sides AB, BC, CD, DA of a square $ABCD$, respectively, such that the area of $KLMN$ is equal to one half of the area of $ABCD$. Prove that some diagonal of $KLMN$ is parallel to some side of $ABCD$.

(→ p.6)

Problem 3. As shown in the following figure, a *heart* is a shape consist of three semicircles with diameters AB, BC and AC such that B is midpoint of the segment AC.
A heart ω is given. Call a pair (P, P') *bisector* if P and P' lie on ω and bisect its perimeter. Let (P, P') and (Q, Q') be bisector pairs. Tangents at points P, P', Q, and Q' to ω construct a convex quadrilateral $XYZT$. If the quadrilateral $XYZT$ is inscribed in a circle, find the angle between lines PP' and QQ'.



(→ p.8)

Problem 4. In isosceles trapezoid $ABCD$ ($AB \parallel CD$) points E and F lie on the segment CD in such a way that D, E, F and C are in that order and $DE = CF$. Let X and Y be the reflection of E and C with respect to AD and AF. Prove that circumcircles of triangles ADF and BXY are concentric.

(→ p.10)

Problem 5. Let A_1, A_2, …, A_{2021} be 2021 points on the plane, no three collinear and

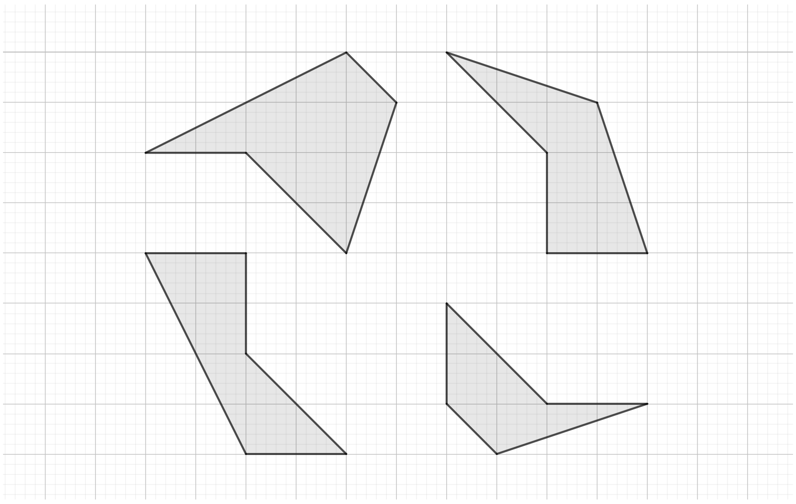
$$\angle A_1A_2A_3 + \angle A_2A_3A_4 + \cdots + \angle A_{2021}A_1A_2 = 360°,$$

in which by the angle $\angle A_{i-1}A_iA_{i+1}$ we mean the one which is less than 180° (assume that $A_{2022} = A_1$ and $A_0 = A_{2021}$). Prove that some of these angles will add up to 90°.

(→ p.13)

Solutions

Problem 1. With putting the four shapes drawn in the following figure together make a shape with at least two reflection symmetries.



Proposed by Mahdi Etesamifard - Iran

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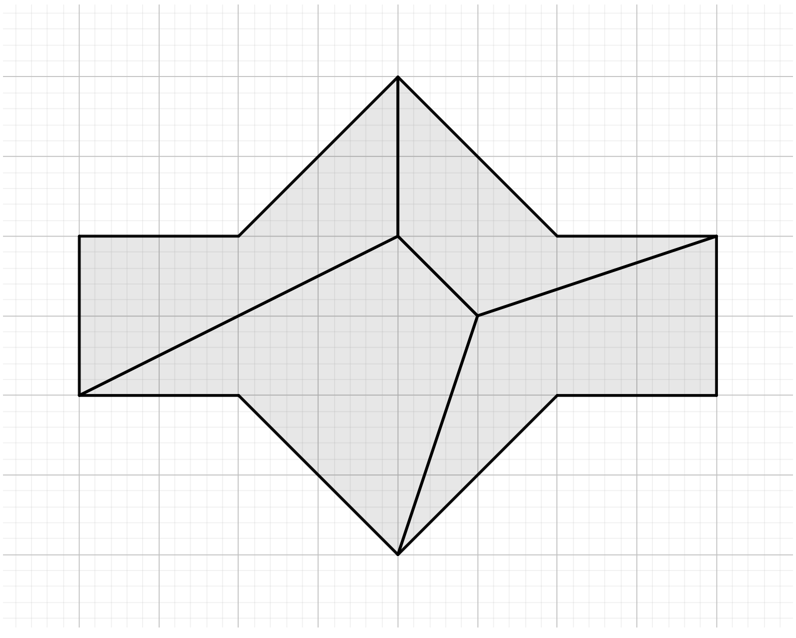
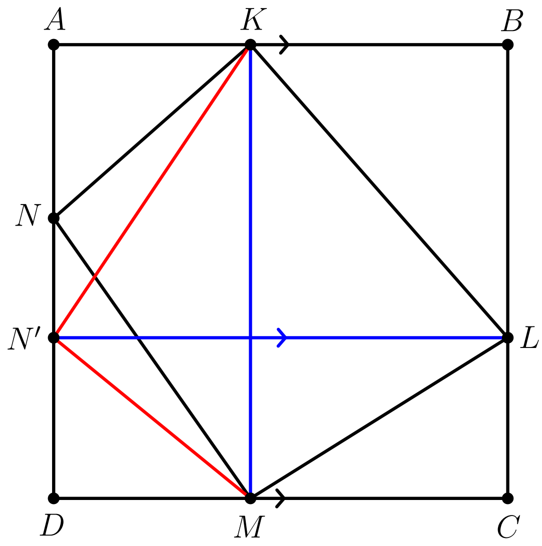
Solution. One can put the figures together like the below figure.



Problem 2. Points K, L, M, N lie on the sides AB, BC, CD, DA of a square $ABCD$, respectively, such that the area of $KLMN$ is equal to one half of the area of $ABCD$. Prove that some diagonal of $KLMN$ is parallel to some side of $ABCD$.

Proposed by Josef Tkadlec - Czech Republic

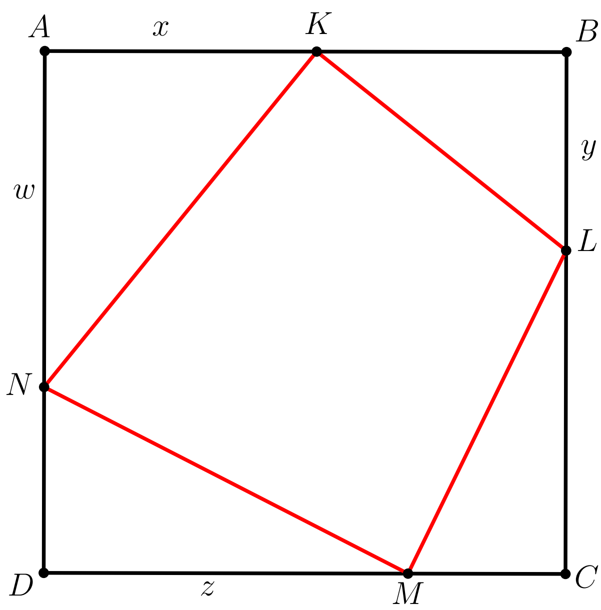
Solution 1. Let $[P]$ denote the area of a polygon P. Suppose that $LN \nparallel AB$ and let $N' \neq N$ be the point on AD such that $LN' \parallel AB$. Note that $[KLMN'] = \frac{1}{2}[ABCD]$. Thus $[KLMN] = [KLMN']$, hence $[KMN] = [KMN']$ implying that $KM \parallel NN' = AD$.



Solution 2 (Proposed Solution from Slovakia).
Denote the length as on the figure and let the side of $ABCD$ be a. The sum of the areas of the right triangles in the corners is $a^2/2$ and so

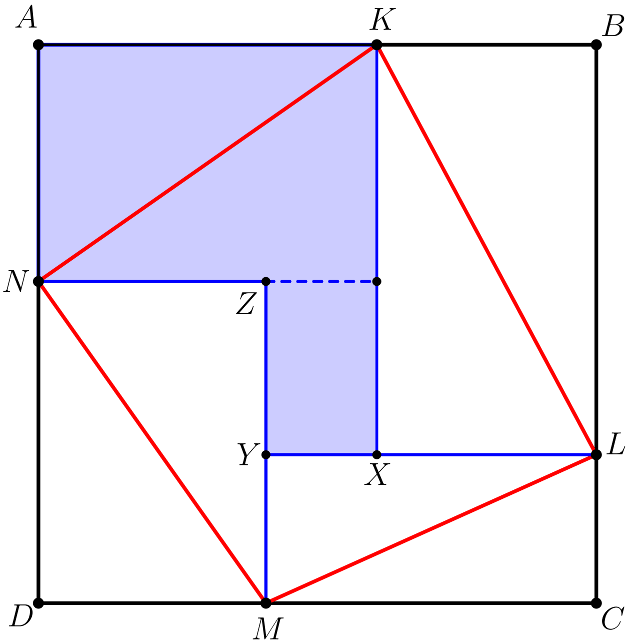
$$\frac{1}{2}wx + \frac{1}{2}(1-x)y + \frac{1}{2}(1-y)(1-z) + \frac{1}{2}z(1-w) = \frac{1}{2}a^2$$

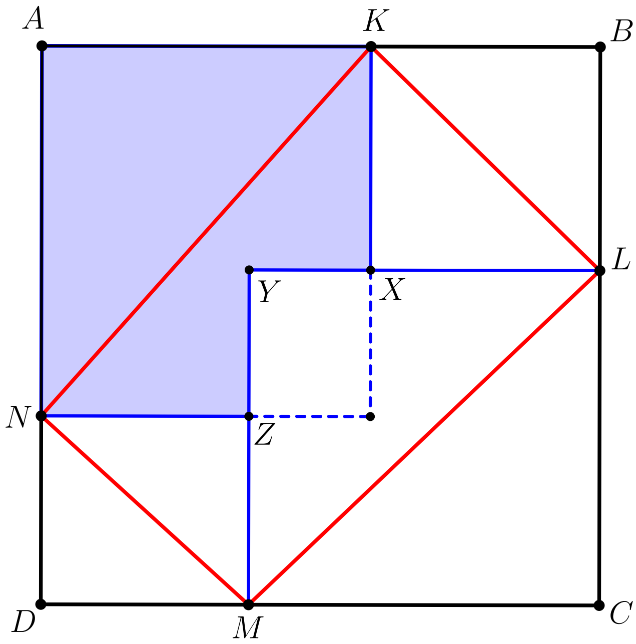
which can be modifier into $(x-z)(w-y)$, and so $x = z$ or $y = w$, which gives the desired conclusion.



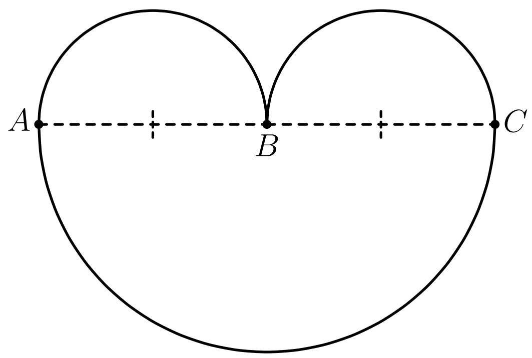
Solution 3 (Proposed Solution from Slovakia).

Without loss of generality assume that $AK > DM$. There are two cases: $AN < BL$ and $AN > BL$. In the first case, $ABCD$ can be split into three rectangles $BLXK, CMYL, DNZM$ and remaning concave hexagon $AKXYZN$. We see that the sum of areas of the right triangles is less than the area of $KLMN$. Analogously, if $AN > BL$, we will show the opposite.





Problem 3. As shown in the following figure, a *heart* is a shape consist of three semicircles with diameters AB, BC and AC such that B is midpoint of the segment AC.
A heart ω is given. Call a pair (P, P') *bisector* if P and P' lie on ω and bisect its perimeter. Let (P, P') and (Q, Q') be bisector pairs. Tangents at points P, P', Q, and Q' to ω construct a convex quadrilateral $XYZT$. If the quadrilateral $XYZT$ is inscribed in a circle, find the angle between lines PP' and QQ'.



Proposed by Mahdi Etesamifard - Iran

Solution. We prove the statement of the problem for both convex and non-convex quadrilaterals.

Lemma 1. *If a pair (P, P') is bisector, then the points P, P', and B are collinear.*

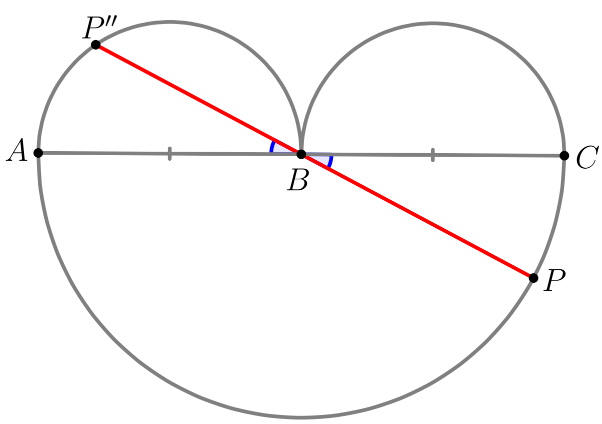
Proof. Without loss of generality suppose that P lies on the arc AC in such a way that the intersection of PB with the arc AB is P''. It's clear that the length of the arc $P''A$ is equal to

$$\frac{2\angle P''BA}{180^\circ} \times (\text{the perimeter of the semicircle with diameter } AB)$$

and length of the arc PC is equal to

$$\frac{\angle PBC}{180^\circ} \times (\text{the perimeter of the semicircle with diameter } AC)$$

We know that the perimeter of semicircle with diameter AB is $\frac{\pi \cdot AB}{2}$ and the perimeter of semicircle with diameter AC is $\frac{\pi \cdot AC}{2}$ thus the length of the arc $P''A$ is equal to $\frac{2\angle P''BA}{180^\circ} \times \frac{\pi \cdot AB}{2}$ and the length of the arc PC is equal to $\frac{\angle PBC}{180^\circ} \times \frac{\pi \cdot AC}{2}$ which are equal with each other. So (P, P'') is a bisector pair. But there is exactly one point for each P like P' such that the pair (P, P') is bisector, so $P' \equiv P''$. Hence PP' passes through B.

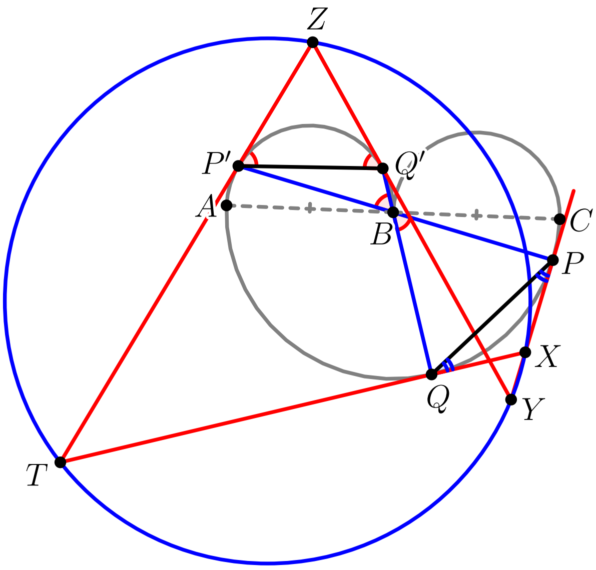


□

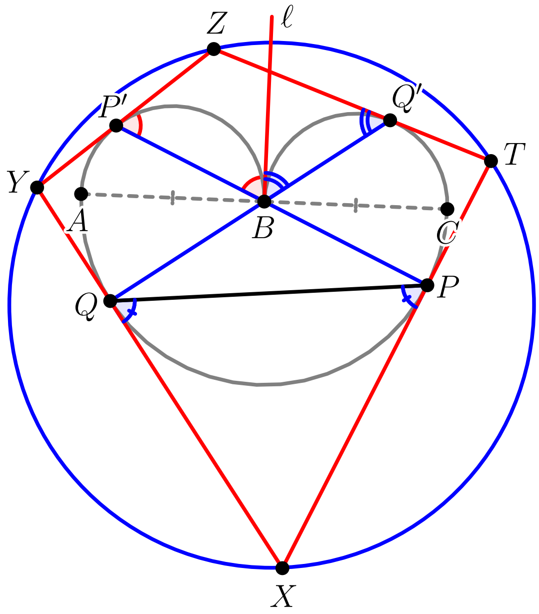
Without loss of generality suppose P and Q are on the arc AC. Now we consider these 2 cases:
Case 1. P and Q lie on the arc AC in such a way that P' and Q' both lie on the arc AB.
Notice that $\angle PBQ = 2\angle XPQ = 2\angle XQP$ and $\angle P'BQ' = \angle ZP'Q' = \angle ZQ'P'$. Therefore

$$180^\circ = \angle PXQ + \angle P'Q'X = (180^\circ - \angle PBQ) + (180^\circ - 2\angle P'BQ' = 360^\circ - 3\angle PBQ,$$

hence $\angle PBQ = 60^\circ$.



Case 1



Case 2

Case 2. P and Q lie on the arc AC in such a way that P' lies on the arc AB and Q' lies on the arc BC.
Let ℓ be the tangent line from B to the heart. Thus $\angle P'B\ell = \angle BP'Z$ and $\angle Q'B\ell = \angle BQ'Z$. So $\angle P'ZQ' = 360^\circ - 2\angle P'BQ'$. Now it's not hard to see that $\angle PBQ = 2\angle XPQ = 2\angle XQP$, hence $\angle PXQ = 180^\circ - \angle PBQ$. Thus we must have

$$180^\circ = \angle PXQ + \angle P'ZQ' = (180^\circ - \angle PBQ) + (360^\circ - 2\angle P'BQ') = 540^\circ - 3\angle PBQ,$$

so $\angle PBQ = 120^\circ$.
Hence in both cases the angle between the lines PP' and QQ' is 60°.

Problem 4. In isosceles trapezoid $ABCD$ ($AB \parallel CD$) points E and F lie on the segment CD in such a way that D, E, F and C are in that order and $DE = CF$. Let X and Y be the reflection of E and C with respect to AD and AF. Prove that circumcircles of triangles ADF and BXY are concentric.

Proposed by Iman Maghsoudi - Iran

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Solution 1. Consider point Z on AB in such a way that $AZFD$ is an isosceles trapezoid hence it is cyclic. Let O be the circumcenter of $\triangle AFD$. Since X and Y are the reflections of E and C with respect to AD and AF, and $ZBCF$ is a parallelogram, then

$$ED = XD = CF = FY = ZB$$

Suppose that AF meets CY at H. Now notice that

$$\angle OZB = \angle OZF + \angle FZB = 90^\circ - \angle ZDF + \angle BCD = 90^\circ - \angle AFD + \angle ADC$$

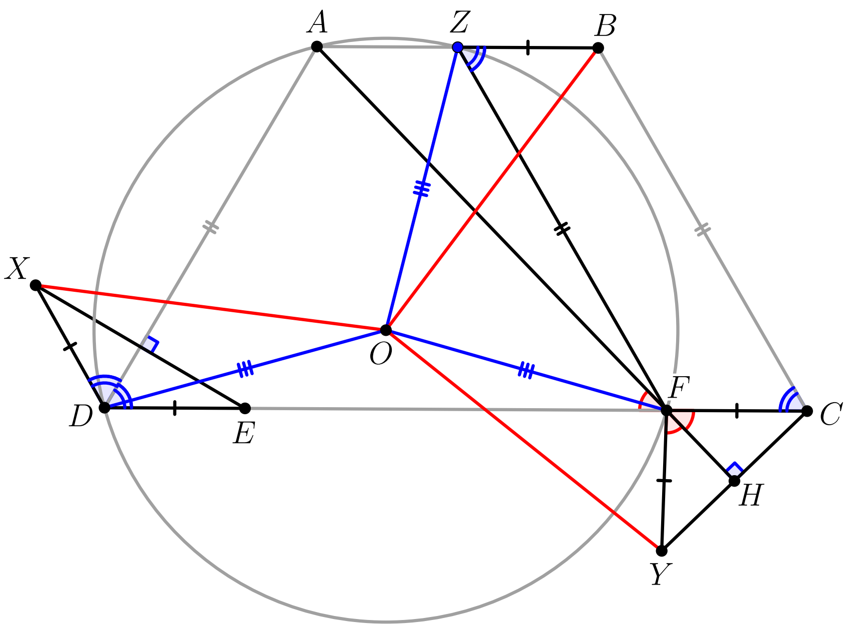
$$\angle ODX = \angle ODA + \angle ADX = 90^\circ - \angle AFD + \angle ADC$$

$$\angle OFY = 180^\circ - \angle OFA - \angle YFH = 180^\circ - (90^\circ - \angle ADC) - \angle HFC = 90^\circ - \angle AFD + \angle ADC$$

These three equality with $ZB = XD = FY$ and $OZ = OD = OF$ gives us that

$$\triangle OZB \cong \triangle ODX \cong \triangle OFY \Longrightarrow OB = OX = OY$$

Which concludes the proof.



Solution 2. Let ω be the circumcircle of AFD with center O and let this circle meets AB, XD, and YF at T, K, and L respectively. Clearly $AX = AE$, $AC = AY$ and $ABFE$ is an isosceles trapezoid, since X and Y are reflections of E and C with respect to AD and AF, and $DE = CF$. Then

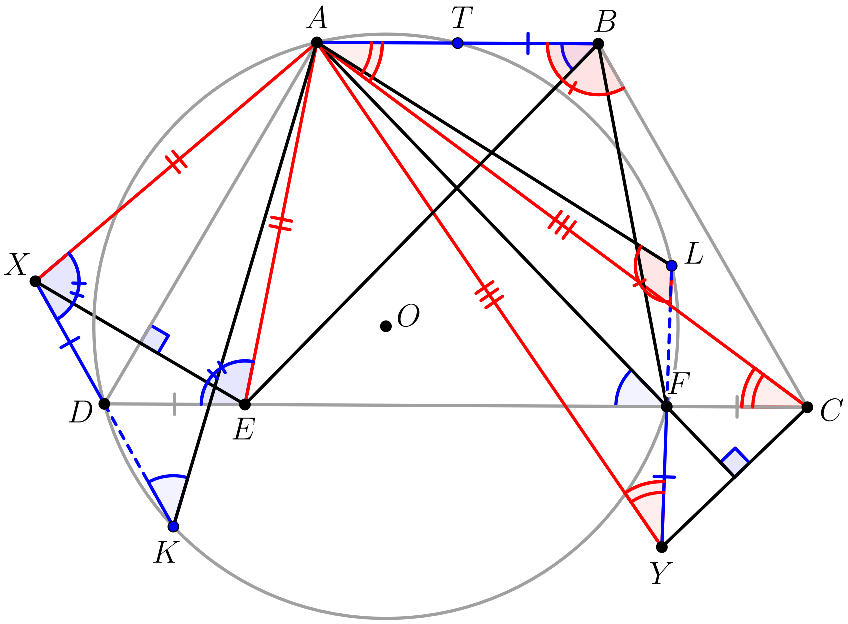
$$\angle BAE = \angle AED = \angle AXD$$

$$\angle ABE = \angle AFD = \angle AKX$$

These two equality with $AX = AE$ show that $\triangle ABE \cong \triangle AXK$ and it yields that $AB = XK$. Similarly one can show that $AB = YL$. Now notice that

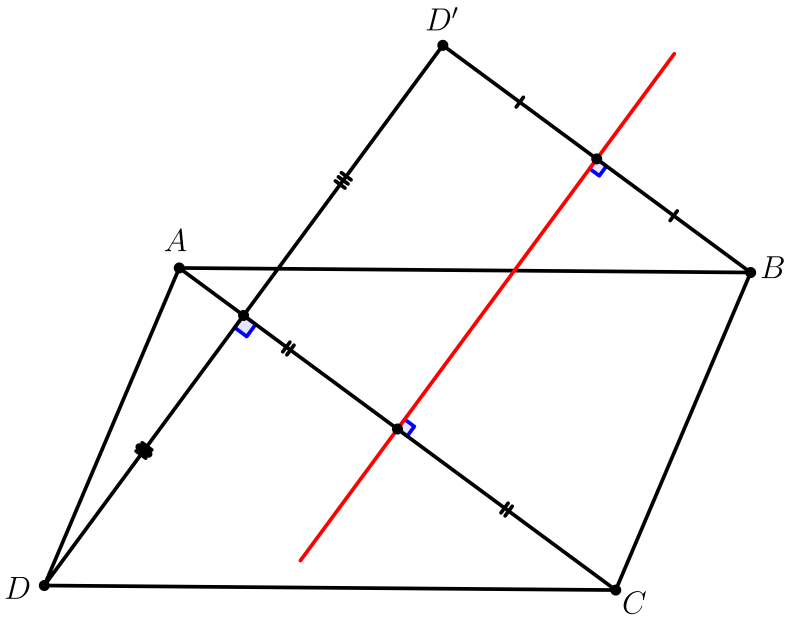
$$P_\omega^B = BT \cdot AB = OB^2 - OD^2$$
$$P_\omega^X = XD \cdot XK = OX^2 - OD^2$$
$$P_\omega^Y = YF \cdot YL = OY^2 - OD^2$$

We have already shown that $AB = XK = YL$ and $BT = XD = YF$. So the power of the points B, X, and Y with respect to ω are all equal, hence $OB = OX = OY$ and the conclusion follows.



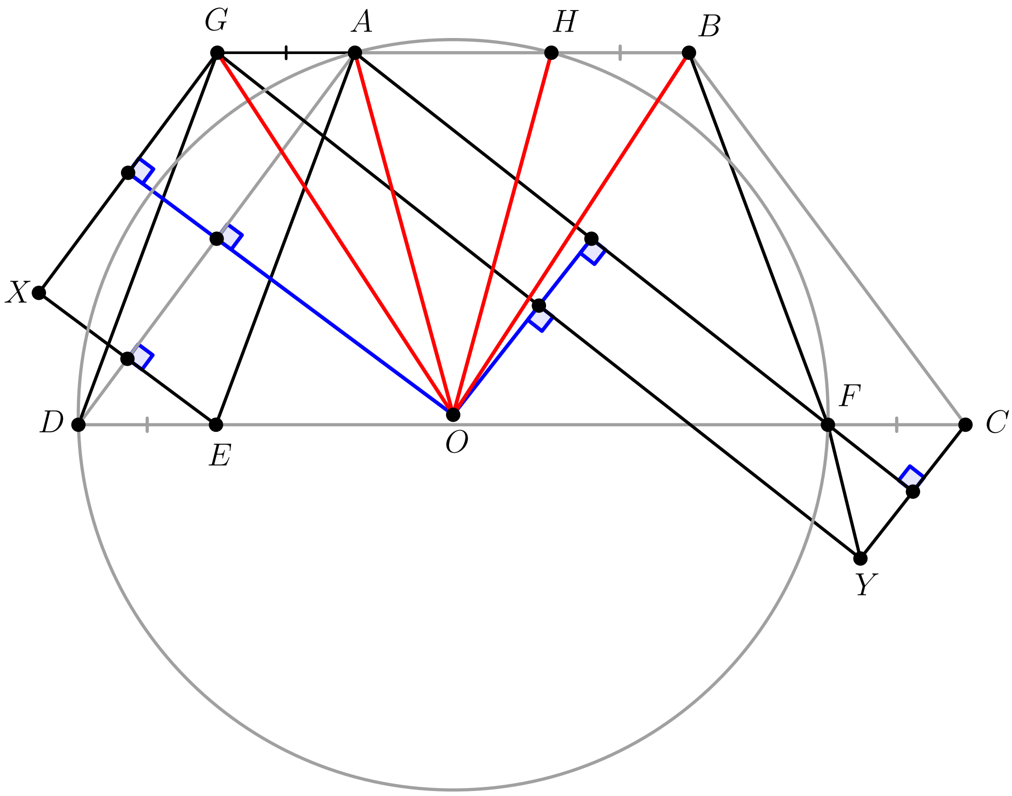
Solution 3 (Proposed Solution from Slovakia).
Lemma 1. *Let $ABCD$ be a parallelogram and D' be the reflection of D in AC. Then the perpendicular bisector of AC coincidences with the perpendicular bisector of BD' (which in case $B = D'$ degenerated to the perpendicular line to AC through B).*

Proof. Obvious when we place AC horizontally:



□

Let G be such a point on opposite ray AB such that $AG = CF$ (which equals do ED). Clearly, $DEAG$ and $FCAG$ are parallelograms. If we apply our lemma to them, we can conclude that the perpendicular bisectors of line pairs (GX, AD) and (GY, AF) coincidence, which gives that triangles GXY and ADF have a common circumcenter O.
To finish the proof, we need to show that B lies on the circle GXY . Let H be the second intersection point of circumcircle of $\triangle ADF$ and AB. Then $AHFD$ is an isosceles trapezoid and clearly $HBCF$ is a parallelogram, and so $AG = CF = BH$. Then $\triangle OAG \cong \triangle OHB$, and so $OG = OB$, therefore points X, Y, B, G are concyclic and we're done.



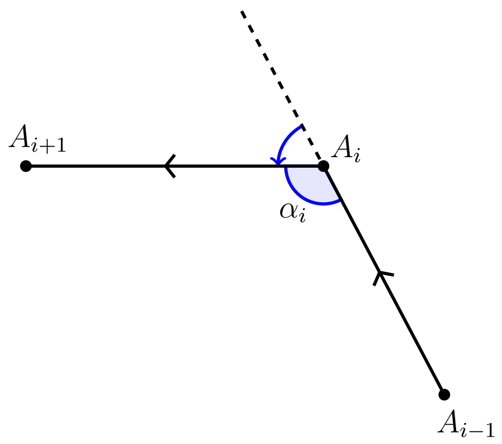
Problem 5. Let A_1, A_2, ..., A_{2021} be 2021 points on the plane, no three collinear and

$$\angle A_1A_2A_3 + \angle A_2A_3A_4 + \cdots + \angle A_{2021}A_1A_2 = 360°,$$

in which by the angle $\angle A_{i-1}A_iA_{i+1}$ we mean the one which is less than 180° (assume that $A_{2022} = A_1$ and $A_0 = A_{2021}$). Prove that some of these angles will add up to 90°.

Proposed by Morteza Saghafian - Iran

Solution. Let α_i be the angle $\angle A_{i-1}A_iA_{i+1}$ which is less than 180°.



Starting from A_1, we walk on the perimeter of the (not necessarily simple) polygon $A_1A_2 \dots A_{2021}$. As we reach a vertex A_i, we turn by angle $180° - \alpha_i$ in clockwise or counterclockwise direction. After walking one round and returning back to the edge A_1A_2, we have turned by a multiple of 360° in total. Therefore, the signed sum of turning angles is a multiple of 360°. More formally, if we define C_1 and C_2 for the set of clockwise and counterclockwise angles, then for some integer number k we have

$$360°k = \sum_{\alpha_i \in C_1} (180° - \alpha_i) - \sum_{\alpha_j \in C_2} (180° - \alpha_j)$$

But the total number of angles we have is 2021 which is an odd number, so if we cancel numbers 180 as much as possible from the above expression, we can conclude that

$$360°t + 180° = \sum_{\alpha_i \in C_1} \alpha_i - \sum_{\alpha_j \in C_2} \alpha_j$$

for some integer number t. On the other hand, by the assumption of the problem, the sum $\sum_{\alpha_i \in C_1} \alpha_i + \sum_{\alpha_j \in C_2} \alpha_j$ is equal to 360°. This implies that $\sum_{\alpha_i \in C_1} \alpha_i - \sum_{\alpha_j \in C_2} \alpha_j = \pm 180°$. Therefore, the two sums should be 90° and 270°.

Intermediate Level

Problems

Problem 1. Let ABC be a triangle with $AB = AC$. Let H be the orthocenter of ABC. Point E is the midpoint of AC and point D lies on the side BC such that $3CD = BC$. Prove that $BE \perp HD$.

(→ p.19)

Problem 2. Let $ABCD$ be a parallelogram. Points E, F lie on the sides AB, CD respectively, such that $\angle EDC = \angle FBC$ and $\angle ECD = \angle FAD$. Prove that $AB \geq 2BC$.

(→ p.20)

Problem 3. Given a convex quadrilateral $ABCD$ with $AB = BC$ and $\angle ABD = \angle BCD = 90°$. Let point E be the intersection of diagonals AC and BD. Point F lies on the side AD such that $\frac{AF}{FD} = \frac{CE}{EA}$. Circle ω with diameter DF and the circumcircle of triangle ABF intersect for the second time at point K. Point L is the second intersection of EF and ω. Prove that the line KL passes through the midpoint of CE.

(→ p.23)

Problem 4. Let ABC be a scalene acute-angled triangle with its incenter I and circumcircle Γ. Line AI intersects Γ for the second time at M. Let N be the midpoint of BC and T be the point on Γ such that $IN \perp MT$. Finally, let P and Q be the intersection points of TB and TC, respectively, with the line perpendicular to AI at I. Show that $PB = CQ$.

(→ p.24)

Problem 5. Consider a convex pentagon $ABCDE$ and a variable point X on its side CD. Suppose that points K, L lie on the segment AX such that $AB = BK$ and $AE = EL$ and that the circumcircles of triangles CXK and DXL intersect for the second time at Y. As X varies, prove that all such lines XY pass through a fixed point, or they are all parallel.

(→ p.26)

Solutions

Problem 1. Let ABC be a triangle with $AB = AC$. Let H be the orthocenter of ABC. Point E is the midpoint of AC and point D lies on the side BC such that $3CD = BC$. Prove that $BE \perp HD$.

Proposed by Tran Quang Hung - Vietnam

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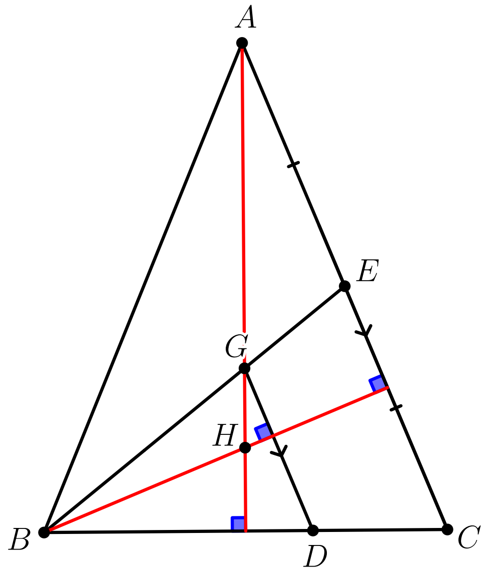
Solution. Let G be the centroid of triangle ABC. Notice that

$$\frac{CD}{BC} = \frac{1}{3} = \frac{EG}{EB},$$

hence $GD \parallel EC$. Since H is orthocenter of ABC, $BH \perp AC$, we deduce that $BH \perp GD$. Combining with $GH \perp BC$, we get that H is the orthocenter of triangle BGD, therefore $DH \perp BE$. This completes the proof.



Problem 2. Let $ABCD$ be a parallelogram. Points E, F lie on the sides AB, CD respectively, such that $\angle EDC = \angle FBC$ and $\angle ECD = \angle FAD$. Prove that $AB \geq 2BC$.

Proposed by Pouria Mahmoudkhan Shirazi - Iran

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Solution 1. First we do some angle-chasing. It's clear that $\angle DAF = \angle ECB = \angle CEB$ and $\angle EBC = \angle ADF$. Hence $\triangle DAF \sim \triangle BEC$ and it yields that $\frac{AD}{BE} = \frac{DF}{BC}$. Similarly we can show that $\frac{CF}{AD} = \frac{BC}{AE}$. Multiplying these two equality gives that

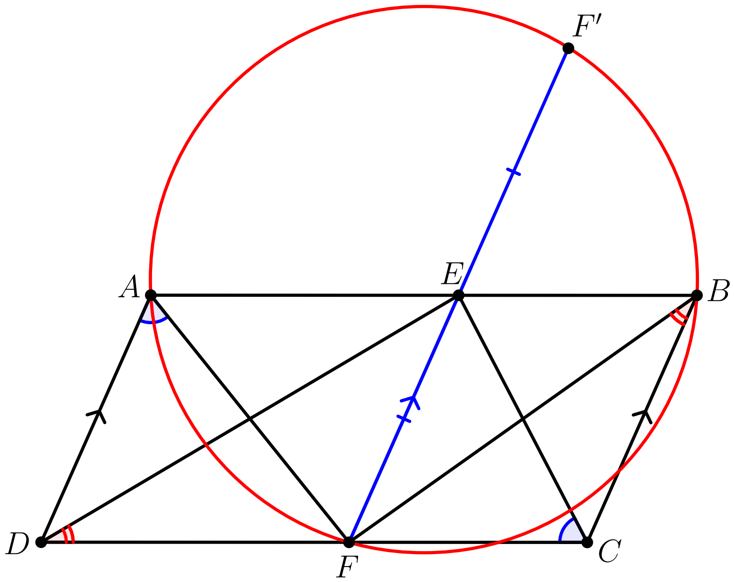
$$\frac{CF}{BE} = \frac{DF}{AE} \Longrightarrow EF \parallel AD \parallel BC.$$

Let F' be the reflection of F in E. So $BCEF'$ and $ADEF'$ are parallelograms. Thus $\angle BF'E = \angle BCE = \angle BAF$ and $\angle AF'E = \angle ADE = \angle ABF$. Summing up these two equality implies that $\angle AF'B = 180^\circ - \angle AFB$. Therefore the quadrilateral $AF'BF$ is cyclic. It means that

$$AE \cdot EB = EF \cdot EF' = EF^2 = BC^2.$$

Finally, by AM-GM inequality we have

$$BC^2 = AE \cdot EB \leq \left(\frac{AE + EB}{2}\right)^2 = \left(\frac{AB}{2}\right)^2 \Longrightarrow 2BC \leq AB.$$

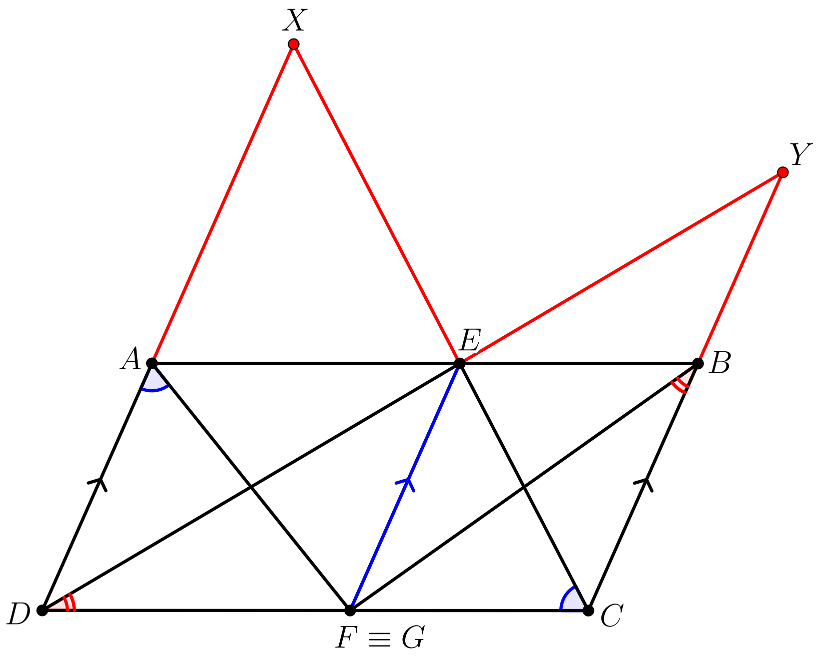


Solution 2. Let CE and DE meet AD and BC at X and Y, respectively. It's clear that the quadrilaterals $DYBF$ and $CXAF$ are cyclic. The power of the points C and D with respect to the circumcircles of these quadrilaterals gives that $CF \cdot CD = CB \cdot CY$ and $DF \cdot CD = DA \cdot DX$. Summing up these two equations implies that

$$\begin{aligned} &(CF + DF) \cdot CD = CB \cdot CY + DA \cdot DX \\ \Longrightarrow\ &CD^2 = CB \cdot CY + CB \cdot DX \\ \Longrightarrow\ &CD^2 = (CY + DX) \cdot CB \end{aligned} \tag{1}$$

Now let the line parallel to BC from E intersects CD at G. Then

$$\frac{DX}{EG} = \frac{CD}{CG} \Longrightarrow DX = \frac{CD \cdot EG}{CG} = \frac{CD \cdot BC}{CG}$$
$$\frac{CY}{EG} = \frac{CD}{DG} \Longrightarrow CY = \frac{CD \cdot EG}{DG} = \frac{CD \cdot BC}{DG}.$$



Thus

$$DX + CY = CD \cdot BC \left(\frac{1}{CG} + \frac{1}{DG} \right).$$

By applying AM-HM inequality on the last equation we get

$$\begin{aligned} CD \cdot BC \cdot \left(\frac{1}{CG} + \frac{1}{DG} \right) &\geq CD \cdot BC \cdot \frac{4}{CG + DG} \\ &= CD \cdot BC \cdot \frac{4}{CD} \\ &= 4BC. \end{aligned}$$

Now from (1) we have

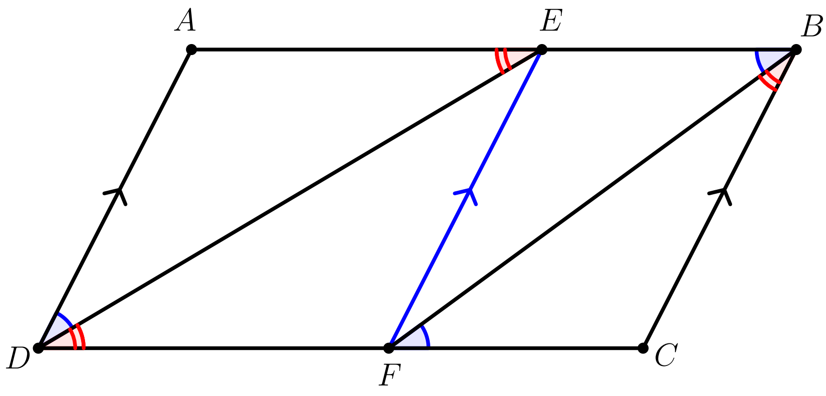
$$\begin{aligned} \frac{CD^2}{BC} = DX + CY \geq 4BC &\Longrightarrow CD^2 \geq 4BC^2 \\ &\Longrightarrow AB \geq 2BC, \end{aligned}$$

as required.

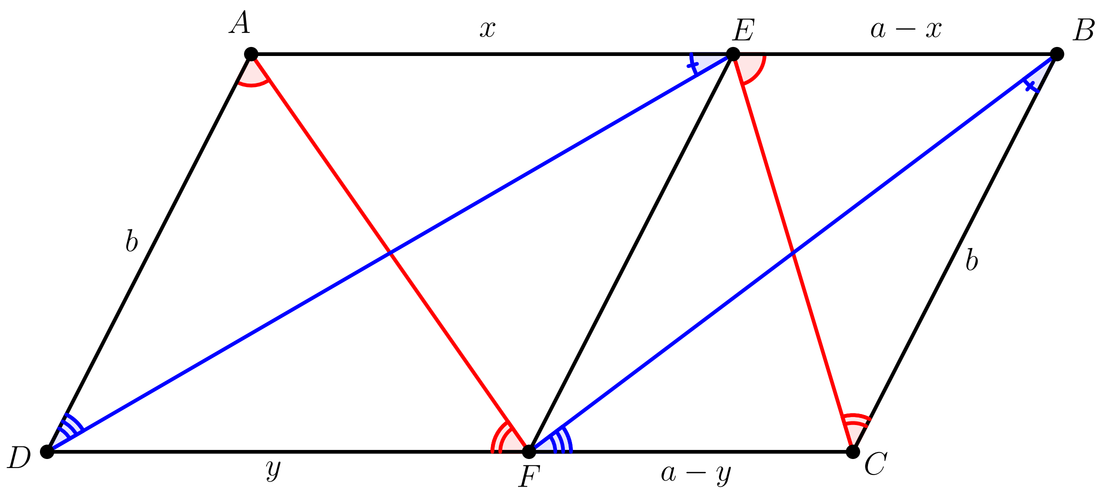
Remark. From the first solution we know that the points F and G are the same but in the second solution we do not need to prove that.

Solution 3 (Proposed Solution from Slovakia).
As in the official solution, we prove that $AD \parallel EF \parallel BC$. Along the way, we have $\triangle EAD \sim \triangle BCF$. Therefore:

$$\frac{AB}{AD} = \frac{AE + BE}{AD} = \frac{AE}{AD} + \frac{BE}{AD} = \frac{BC}{CF} + \frac{CF}{BC} \geq 2$$



Solution 4 (Proposed Solution from Slovakia).
As in every solution, we notice $\triangle ADF \sim \triangle EBC$ and $\triangle ADE \sim \triangle CFB$. Now, denote lengths as on the picture. The similarities give us $\dfrac{AD}{DF} = \dfrac{EB}{CB}$, i.e. $b^2 = (a-x)y$, and $\dfrac{AD}{AE} = \dfrac{CF}{CB}$, i.e. $b^2 = (a-y)x$. By comparing these two, we have $(a-x)y = (a-y)x$, i.e. $ay = ax$, and so $x = y$. Therefore $b^2 = (a-x)x$, which modifies as $x^2 - ax + b^2 = 0$. This quadratic equation has a discriminant of $a^2 - 4b^2$, which must be non-negative, which already gives $a \geq 2b$.



Problem 3. Given a convex quadrilateral $ABCD$ with $AB = BC$ and $\angle ABD = \angle BCD = 90°$. Let point E be the intersection of diagonals AC and BD. Point F lies on the side AD such that $\dfrac{AF}{FD} = \dfrac{CE}{EA}$. Circle ω with diameter DF and the circumcircle of triangle ABF intersect for the second time at point K. Point L is the second intersection of EF and ω. Prove that the line KL passes through the midpoint of CE.

Proposed by Mahdi Etesamifard and Amir Parsa Hosseini - Iran

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Solution. Let K' be a point such that the triangles ABC and $AK'D$ are spirally similar. We claim that K and K' are the same point. Since $\frac{AE}{EC} = \frac{DF}{FA}$, the triangles $FK'D$ and EBA are also similar. Then $\angle FK'D = \angle ABE = 90°$, so to prove the claim it suffices to show that A, B, F, and K' lie on a circle. From the other hand

$$\angle DEC = 90° - \angle BAC = 90° - \angle BCA = \angle DCE, \tag{1}$$

hence $DC = DE$. Also since ABC and $AK'D$ are spirally similar, we have $\triangle ABK' \sim \triangle ACD$. Therefore

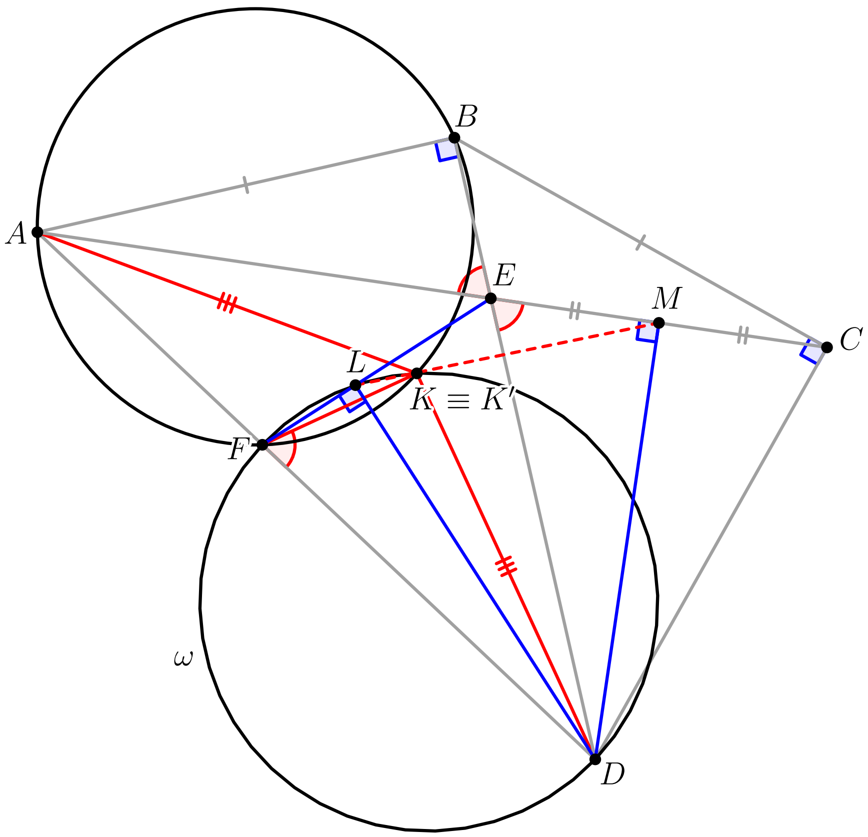
$$\angle ABK' = \angle ACD \stackrel{(1)}{=} \angle DEC = \angle AEB = \angle K'FD,$$

so $ABK'F$ is cyclic and the claim is proved.

Now we are ready to prove the statement of problem. Let M be the midpoint of CE. Notice that $\angle DME = 90°$, since $DC = DE$. It yields that the points E, L, D, and M lie on a circle. Then

$$\angle MLD = \angle MED = \angle AEB = \angle KFD = \angle KLD,$$

which implies the desired collinearity.



Problem 4. Let ABC be a scalene acute-angled triangle with its incenter I and circumcircle Γ. Line AI intersects Γ for the second time at M. Let N be the midpoint of BC and T be the point on Γ such that $IN \perp MT$. Finally, let P and Q be the intersection points of TB and TC, respectively, with the line perpendicular to AI at I. Show that $PB = CQ$.

Proposed by Patrik Bak - Slovakia

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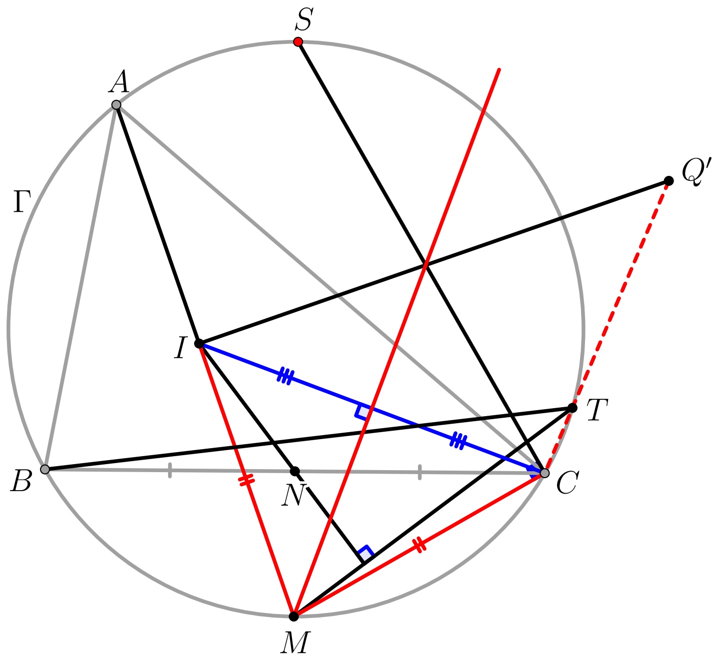
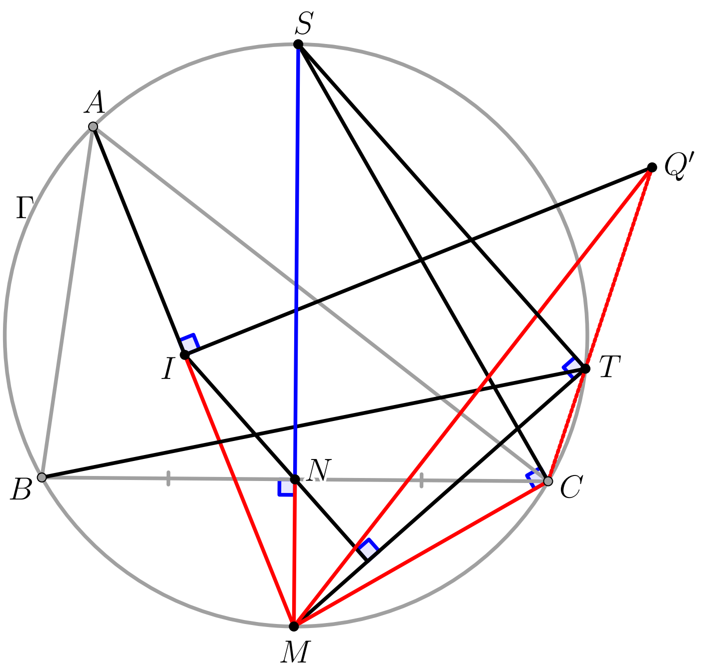
Solution 1. Let S be the midpoint of arc BAC of Γ and let Q' be the reflection of S in the perpendicular bisector of IC. We will prove that $Q = Q'$, which would mean that $SI = CQ$. Analogously, we would have $SI = PB$, and so the proof would be finished.



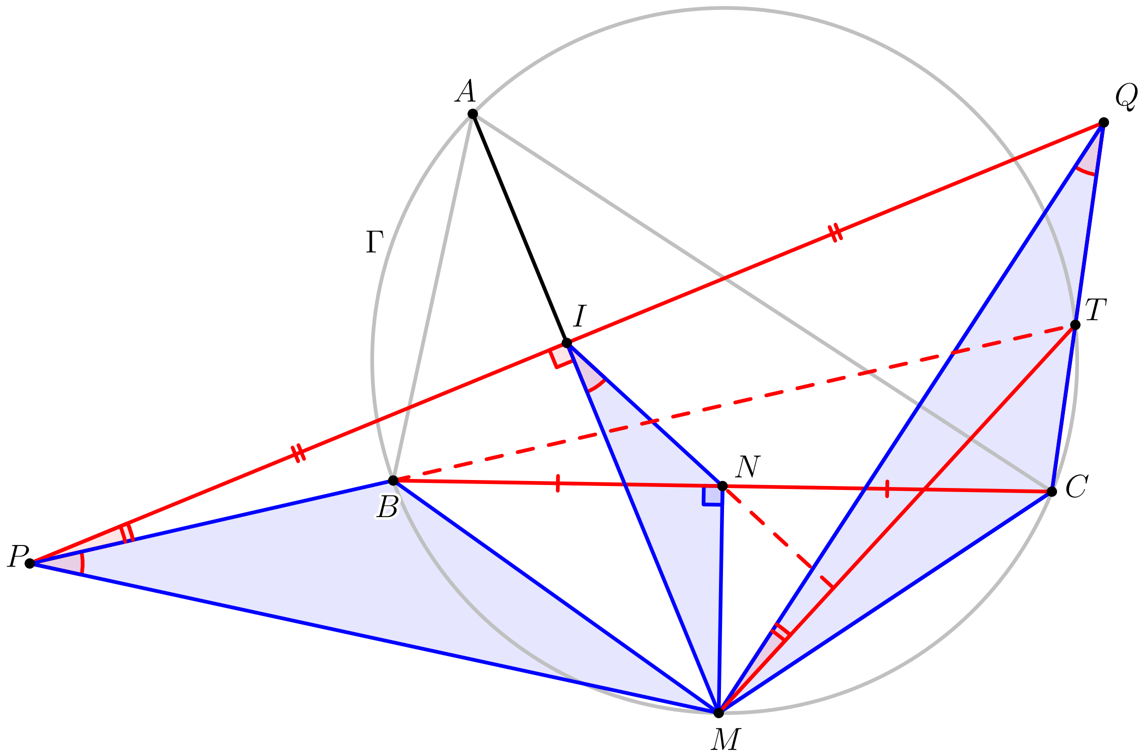
It is well-known that M is the circumcenter of BIC, so the perpendicular bisector of CI passes through M. Since MS is a diameter of Γ, we have $SC \perp MC$, and by the symmetry subsequently $Q'I \perp MI$. Now, it is enough to show that Q', C, T are collinear.



We have $\angle IQ'M = \angle MSC = \angle NCM$. Notice also $MN \perp NC$. Together, triangles MIQ', MNC are directly similar, which gives that triangles MIN, $MQ'C$ are also directly similar.
Since $IN \perp MT$ and $ST \perp MT$, we have $ST \parallel IN$, so $\angle MCT = \angle MST = \angle SNI$. This together with $\angle INM = \angle Q'CM$ finally gives us that points T, C, and Q' are collinear, which concludes the proof.

Solution 2 (Proposed Solution from Slovakia). We define P and Q as the points on the line perpendicular to AI at I such that $\triangle MBC$ and $\triangle MPQ$ are spirally similar. The equality $MB = MC$ then implies $MP = MQ$. Since N and I are the midpoints of BC and PQ, respectively, spiral similarity gives that all triangles $\triangle MPB, \triangle MIN$ and $\triangle MQC$ are similar. The similarity gives $\angle MPB = \angle MQC$, which means that if we define T as the intersection point of lines CQ and PB, then P, T, Q, M are concyclic, and so $180° - \angle CTB = \angle QTP = \angle QMP = \angle CMB$, therefore T lies on Γ. To finish our proof, we need to show that $IN \perp MT$:

$$\begin{aligned}\angle MIN + \angle TMI &= \angle MIN + \angle TMQ + \angle QMI \\ &= \angle MPB + \angle TPQ + \angle IMP \\ &= \angle MPI + \angle IMP = 90°\end{aligned}$$



Problem 5. Consider a convex pentagon $ABCDE$ and a variable point X on its side CD. Suppose that points K, L lie on the segment AX such that $AB = BK$ and $AE = EL$ and that the circumcircles of triangles CXK and DXL intersect for the second time at Y. As X varies, prove that all such lines XY pass through a fixed point, or they are all parallel.

Proposed by Josef Tkadlec - Czech Republic

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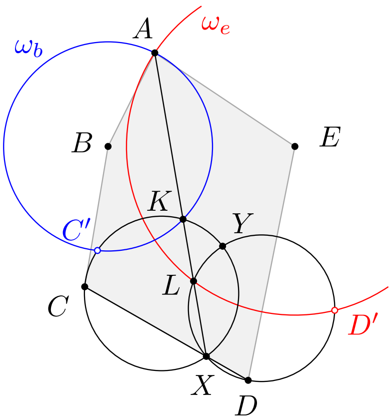
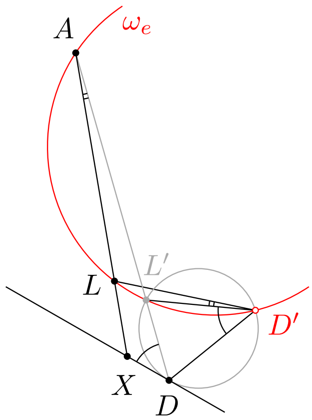
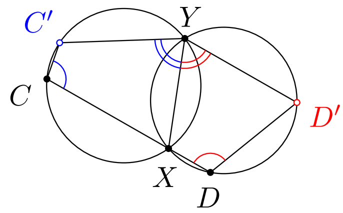
Solution. Let ω_b, ω_e be the circles centered at B, E and passing through A (note that $K \in \omega_b$ and $L \in \omega_e$). By (XYZ) we denote the circumcircle of triangle XYZ. Let $\angle(p, q)$ denote the directed angle between lines p, q.
First we show that all the circles (DXL) pass through a fixed point D' on ω_e. Motivated by the limiting case $X = D$, let $L' = AD \cap \omega_e$ and let D' be the intersection of ω_e and the circle through L' and D tangent to CD. Then D' is the fixed point: Indeed, for any X and the corresponding L we have

$$\begin{aligned}\angle(D'L, D'D) &= \angle(D'L, D'L') + \angle(D'L', D'D) = \angle(AL, AL') + \angle(DL', DX)\\ &= \angle(AX, AD) + \angle(DA, DX) = \angle(XA, XD) = \angle(XL, XD),\end{aligned}$$

hence (XLD) passes through D'. Similarly, all circles (CXK) pass through a fixed point C' on ω_b.







Now the problem can be conveniently rephrased with respect to a (fixed) quadrilateral $C'CDD'$. Note that the angle $\angle(YC', YD')$ is fixed: Indeed,

$$\angle(YC', YD') = \angle(YC', YX) + \angle(YX, YD') = \angle(CC', CX) + \angle(DX, DD') = \angle(CC', DD').$$

We conclude by distinguishing two cases.

1. $Y \in C'D'$:

   Since $\angle(YC', YX) = \angle(CC', CX)$ is fixed, all lines XY form the same angle with line $C'D'$ and are thus parallel.

2. $Y \notin C'D'$:

   Since $\angle(YC', YX) = \angle(CC', CX)$ is fixed, lines XY all intersects $(YC'D')$ at a fixed point.

Advanced Level

Problems

Problem 1. Acute-angled triangle ABC with circumcircle ω is given. Let D be the midpoint of AC, E be the foot of altitude from A to BC, and F be the intersection point of AB and DE. Point H lies on the arc BC of ω (the one that does not contain A) such that $\angle BHE = \angle ABC$. Prove that $\angle BHF = 90°$.

(→ p.31)

Problem 2. Two circles Γ_1 and Γ_2 meet at two distinct points A and B. A line passing through A meets Γ_1 and Γ_2 again at C and D respectively, such that A lies between C and D. The tangent at A to Γ_2 meets Γ_1 again at E. Let F be a point on Γ_2 such that F and A lie on different sides of BD, and $2\angle AFC = \angle ABC$. Prove that the tangent at F to Γ_2, and lines BD and CE are concurrent.

(→ p.32)

Problem 3. Consider a triangle ABC with altitudes AD, BE, and CF, and orthocenter H. Let the perpendicular line from H to EF intersects EF, AB and AC at P, T and L, respectively. Point K lies on the side BC such that $BD = KC$. Let ω be a circle that passes through H and P, that is tangent to AH. Prove that circumcircle of triangle ATL and ω are tangent, and KH passes through the tangency point.

(→ p.34)

Problem 4. 2021 points on the plane in the convex position, no three collinear and no four concyclic, are given. Prove that there exist two of them such that every circle passing through these two points contains at least 673 of the other points in its interior.
(A finite set of points on the plane are in convex position if the points are the vertices of a convex polygon.)

(→ p.38)

Problem 5. Given a triangle ABC with incenter I. The incircle of triangle ABC is tangent to BC at D. Let P and Q be points on the side BC such that $\angle PAB = \angle BCA$ and $\angle QAC = \angle ABC$, respectively. Let K and L be the incenter of triangles ABP and ACQ, respectively. Prove that AD is the Euler line of triangle IKL.
(The Euler line of a triangle is the line going through the circumcenter and orthocenter of that triangle.)

(→ p.39)

Solutions

Problem 1. Acute-angled triangle ABC with circumcircle ω is given. Let D be the midpoint of AC, E be the foot of altitude from A to BC, and F be the intersection point of AB and DE. Point H lies on the arc BC of ω (the one that does not contain A) such that $\angle BHE = \angle ABC$. Prove that $\angle BHF = 90^\circ$.

Proposed by Harris Leung - Hong Kong

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Solution. Note that $DC = DE = DA$ since $\angle CEA = 90^\circ$. Thus

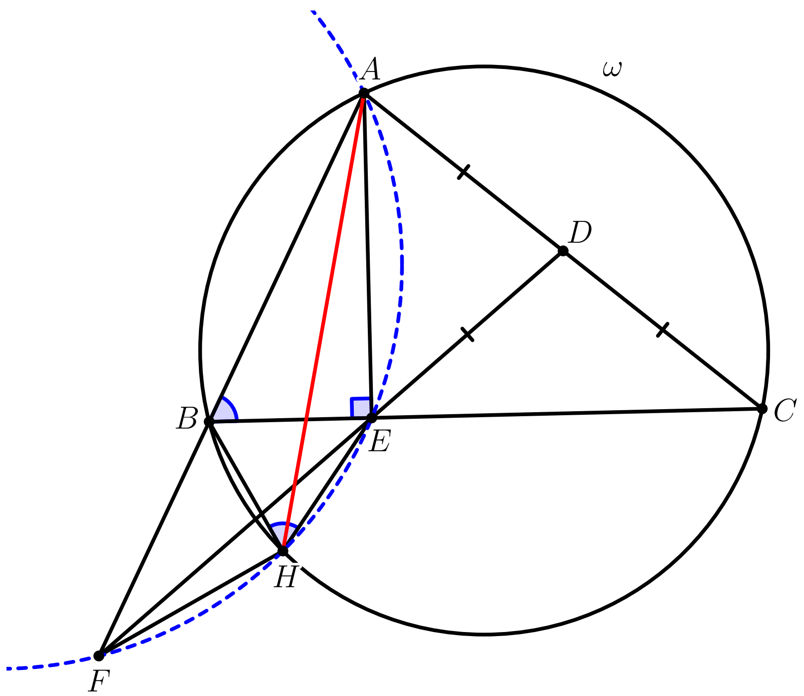
$$\angle EFA = \angle DEA - \angle FAE = (90^\circ - \angle C) - (90^\circ - \angle B) = \angle B - \angle C,$$

and

$$\angle EHA = \angle BHE - \angle BHA = \angle B - \angle C = \angle EFA.$$

This implies A, E, H, F are concyclic. Therefore

$$\angle BHF = \angle AHF - \angle AHB = \angle AEF - \angle C = (90^\circ + \angle C) - \angle C = 90^\circ.$$

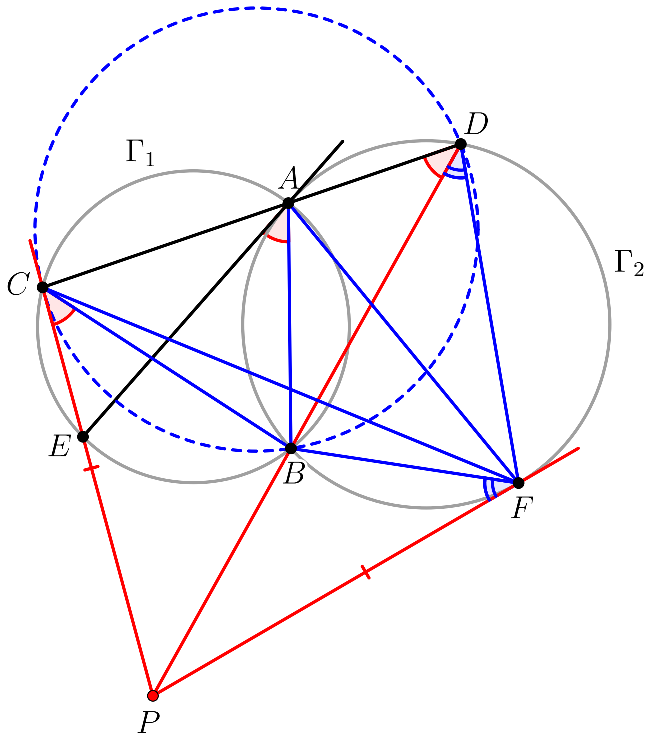


Remark. When $AB > AC$, F lies on the other side, so some of the arguments should be modified. Though the result still holds.

Problem 2. Two circles Γ_1 and Γ_2 meet at two distinct points A and B. A line passing through A meets Γ_1 and Γ_2 again at C and D respectively, such that A lies between C and D. The tangent at A to Γ_2 meets Γ_1 again at E. Let F be a point on Γ_2 such that F and A lie on different sides of BD, and $2\angle AFC = \angle ABC$. Prove that the tangent at F to Γ_2, and lines BD and CE are concurrent.

Proposed by Tak Wing Ching - Hong Kong

Solution 1.



Clearly, the point F is uniquely determined. We redefine the point F as follows. Let BD meet CE at P, and let F be the contact point of the tangent from P to Γ_2 that lies on different side of BD as A. It suffices to prove that $\angle AFC = \frac{1}{2}\angle ABC$. By (XYZ) we denote the circumcircle of triangle XYZ.

First, since $\angle ECB = \angle EAB = \angle ADB$, the line CP is tangent to (CBD). Therefore

$$PC = \sqrt{PB \cdot PD} = PF.$$

Then

$$\begin{aligned}\angle FBC &= \angle BFP + \angle FPC + \angle PCB \\ &= \angle BDF + (180^\circ - 2\angle CFP) + \angle CDB \\ &= \angle CDF + 180^\circ - 2\angle CFP.\end{aligned}$$

(Note that $\angle FBC$ may refer to a reflex angle in some configurations.) It follows that

$$\begin{aligned}\angle ABC &= \angle FBC - \angle FBA \\ &= (\angle CDF + 180^\circ - 2\angle CFP) - (180^\circ - \angle ADF) \\ &= 2\angle CDF - 2\angle CFP \\ &= 2\angle AFP - 2\angle CFP \\ &= 2\angle AFC,\end{aligned}$$

and this completes the proof.

Solution 2 (Proposed Solution from Czech republic).

We apply inversion centered at A.

inverted problem: Let $C'B'D'$ be a triangle. On sides $C'B'$ and $C'D'$ are points E' and A respectively, such that $AE' \parallel D'B'$. Point F' lies on $D'B'$ such that $2\angle F'C'A = \angle B'C'A$. Denote $\omega 1$ circle passing through A and F' that is touching $D'B'$ at F'. Denote $\omega 2$ circumcircle $AB'D'$ and $\omega 3$ circumcircle $C'AE'$. Prove that these three circles pass through fixed point different from A.

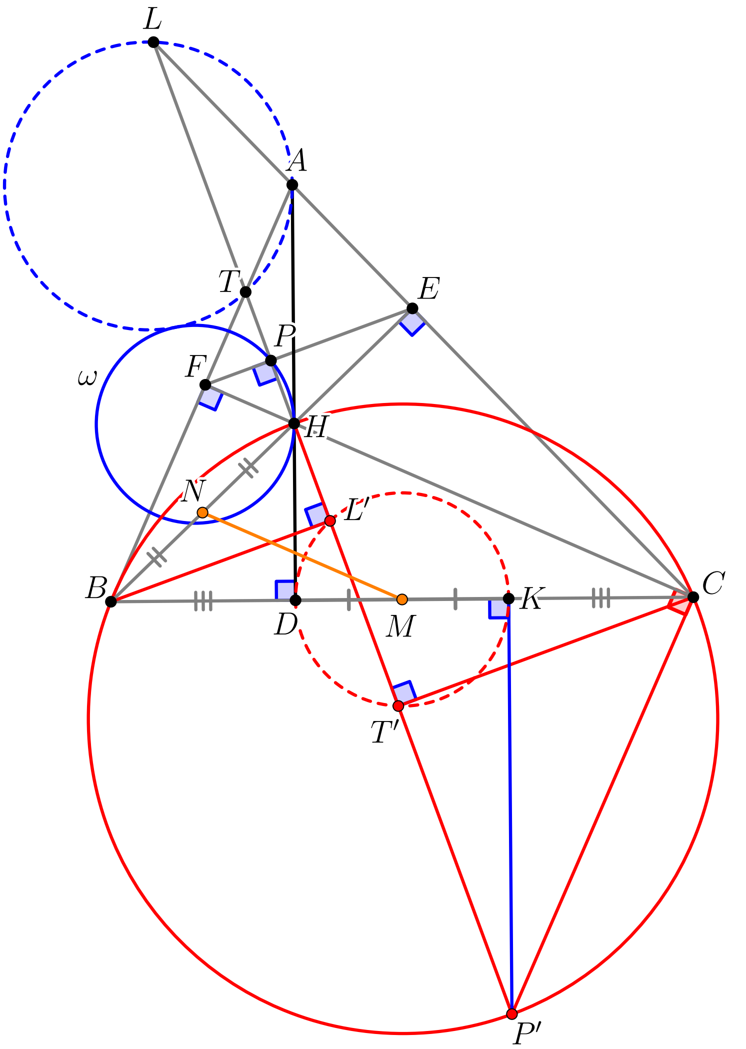
From statement we get that $C'F'$ is angle bisector of $\angle B'C'D'$. Denote G' intersection of external bisector of $\angle B'C'D'$ with $C'D'$. Denote Ω circle with diameter $G'F'$. Denote ω_4 circumcircle of (C', B', D'). We will prove that Ω is perpendicular to all three circles ω_1, ω_2 and ω_3. For ω_1 it's immediate as they have perpendicular diameters. Then because (G, B', F', D') is harmonic, we have that $\omega_2 \perp \Omega$ and $\omega_4 \perp \Omega$. And from homothethy we have that ω_3 and ω_4 are parallel at A. Hence $\omega_3 \perp \Omega$. Denote S center of Ω. From perpendicular circles we have that S has the same power to all three circles ω_1, ω_2 and ω_3. But so does point A. Hence these circles form a pencil, hence they pass through another fixed point.

Remark. the point P cannot be defined if Γ_1 passes through the centre of Γ_2, since in that case CE and BD are parallel. Indeed, in that case the three lines are parallel to each other, so the assertion of this question should be changed to 'concurrent or parallel'.

Problem 3. Consider a triangle ABC with altitudes AD, BE, and CF, and orthocenter H. Let the perpendicular line from H to EF intersects EF, AB and AC at P, T and L, respectively. Point K lies on the side BC such that $BD = KC$. Let ω be a circle that passes through H and P, that is tangent to AH. Prove that circumcircle of triangle ATL and ω are tangent, and KH passes through the tangency point.

Proposed by Mahdi Etesamifard - Iran

Solution 1. Perform an inversion centered at H with radius $-AH \cdot HD$. The images of the points are denoted by primes. It's clear that $F' \equiv C$ and $E' \equiv B$ so P' lies on the circumcircle of triangle HBC and $\angle HCP' = 90°$. Also T' and L' lie on the line HP' such that $\angle CT'H = \angle BL'H = 90°$, since circumcircle of triangles EHD and FHD passes through T' and L', respectively. Note that $ACP'B$ is a parallelogram, so by symmetry one can show that $P'K \perp BC$. Therefore $P'K$ is the image of ω under the inversion and it suffices to show that $P'K$ touch the circumcircle of triangle $DT'L'$.



Let M and N be the midpoints of BC and BH. Then

$$\angle L'DH = \angle L'BH = \angle FEH = \angle FAH.$$

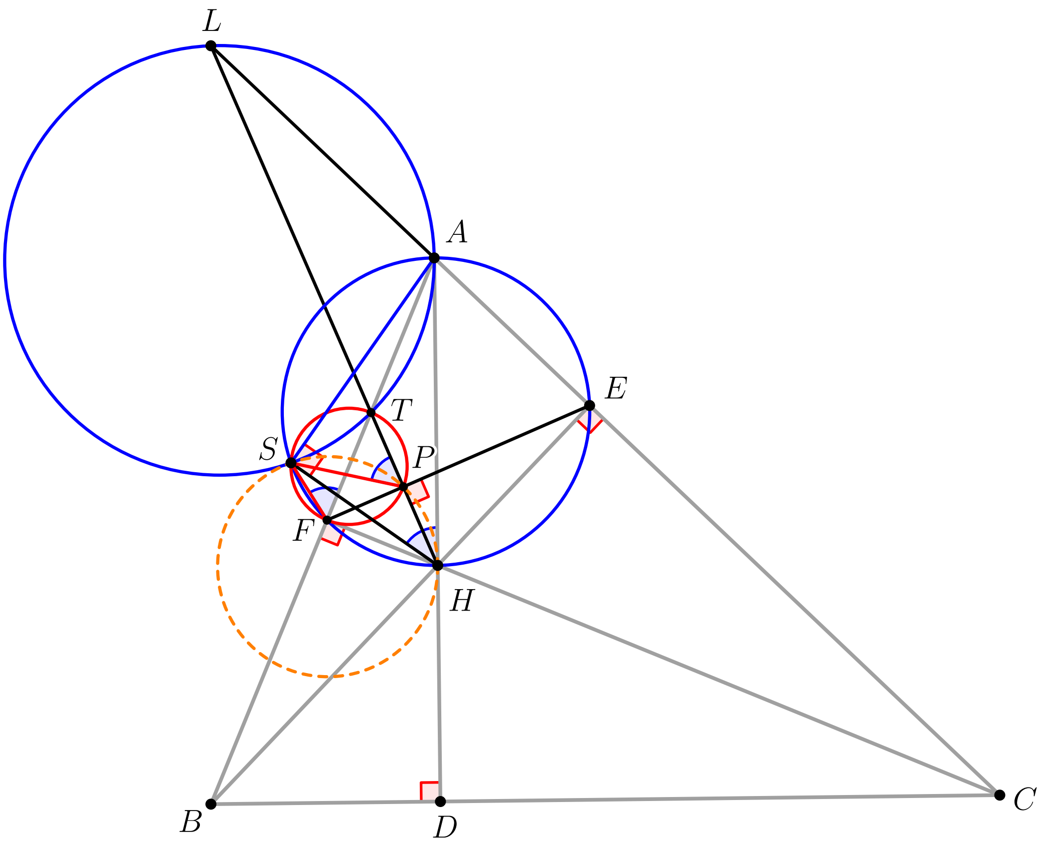
It implies that $DL' \parallel AB$. Also $MN \perp AB$ so MN must be the perpendicular bisector of DL', since $NL' = ND$. This means $ML' = MD$. Similarly, one can show that $MT' = MD$. Also it is clear that $MD = MK$, therefore the quadrilateral $DL'KT'$ is inscribed in a circle centered at M. Now since $\angle MKP' = 90°$ the result follows.

Solution 2 (Proposed Solution from Slovakia).

Tangency part. Consider the spiral similarity which maps LE to TF. Since $A = LE \cap TF$, its center S lies on the circumcircles of $\triangle ATL$ and $\triangle AEF$. Moreever, S is also the center of the spiral similarity mapping TL to EF . Since $P = TL \cap EF$, S lies on the circumcircle of $\triangle PTF$. A simple angle chasing then gives:

$$\angle AHS = \angle AFS = \angle TFS = \angle TPS$$

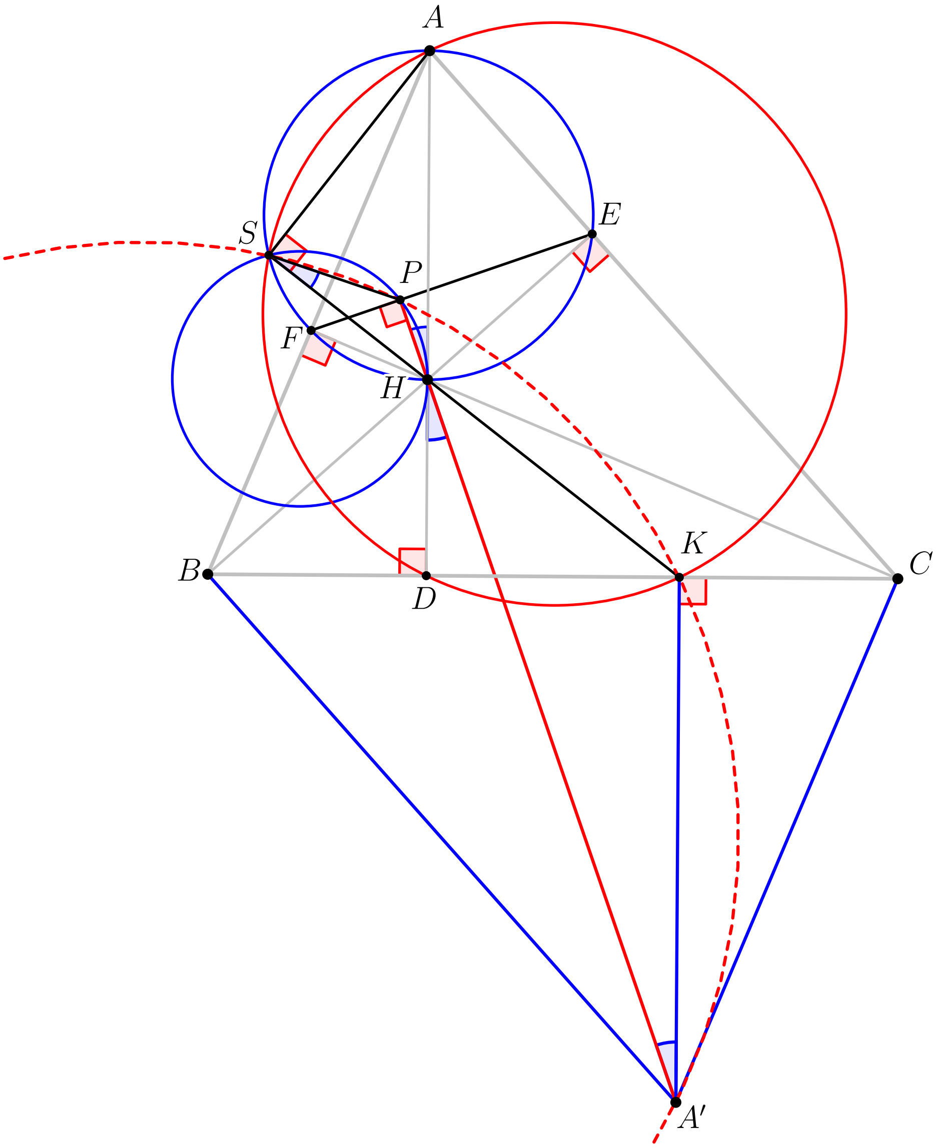
which means AH is tangent to the circumcircle of $\triangle SPH$.
(Note that the proven tangency holds in a general situation: E and F can be arbitrary points on AC and AB, P can be an arbitrary point on EF , and H can be an arbitrary point on the circumcircle of $\triangle AEF$.)



To prove the tangency of the circumcircles of $ALTS$ and $\triangle SPH$, we will show that the needed condition is the equality of angles $\angle ASH$ and $\angle TPF$, which in our situation is trivial (as they are both right). The criterion for tangency is that $\angle ASH$ equals to the sum of angle of arc (AS) of the circumcircle of $ATSL$ and the angle of arc (SH) of the circumcircle of $\triangle SPH$. This sum equals to:

$$\angle STF + \angle SPT = \angle STF + \angle SFT = \angle TPF$$

Collinearity. First of all, we will redefine the tagency point S as the second intersection point of KH and the circumcircle of $AFHE$. We will be finished when we prove $\angle HSP = \angle AHP$, which subsequently means that AH and the circumcircle of $\triangle SPH$ are tangent.
Let A' be the point for which $ABA'C$ is a paralellogram. Then B, A', C, H are concyclic and lie on the circle with a diameter $A'H$.
Note that A', H, P are collinear. This is a simple angle chasing exercise, but can also be seen like this: In triangle $\triangle HBC$, HA' is a line through the circumcenter, whereas in triangle $\triangle HEF$, HP is a line through the orthocenter. Since triangles $\triangle HBC$ and $\triangle HEF$ are isogonal, lines HA' and HP must be the same.

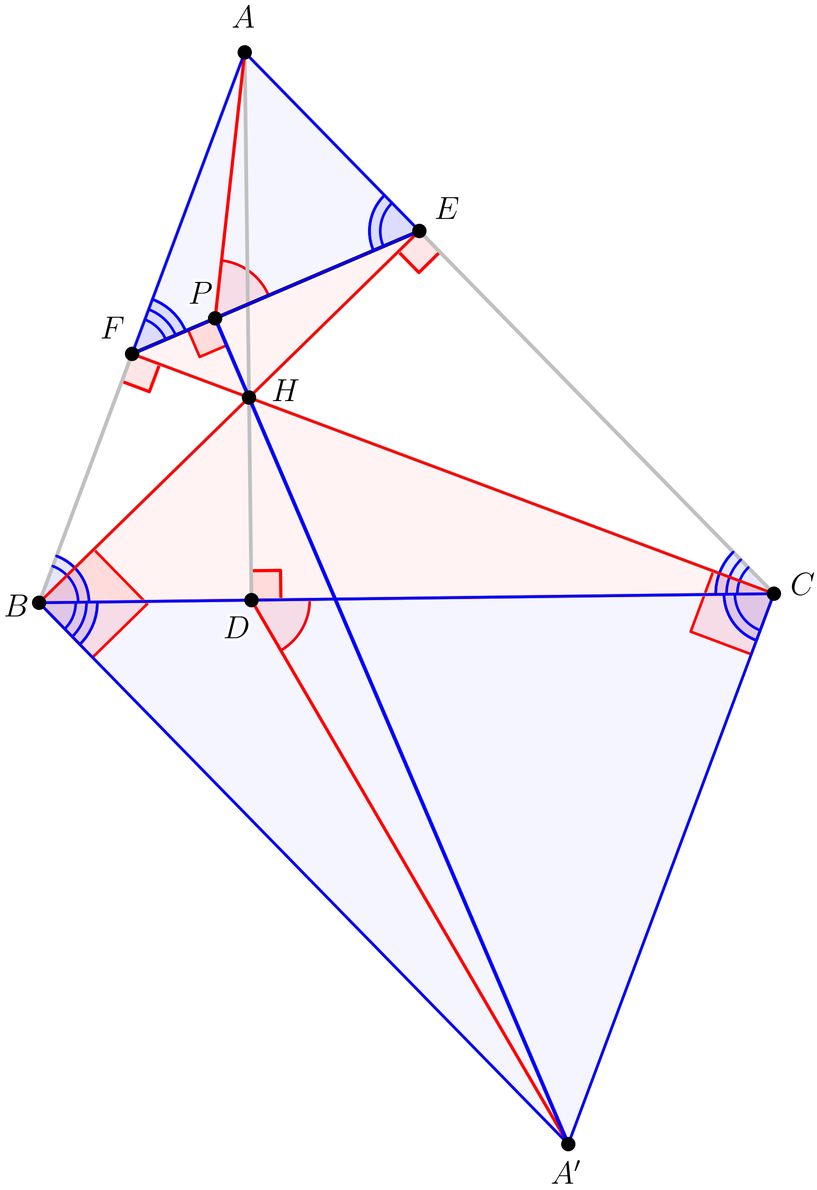


Clearly $A'K \parallel AD$, so we have $\angle AHP = \angle DHA' = \angle KA'H = \angle KA'P$. To prove $\angle AHP = \angle HSP$, we just need to show that points S, P, K, A' are concyclic. Notice A, S, D, K are concyclic due to right angles $\angle ASK$ and $\angle ADK$. Therefore, we need just to prove that A, P, D, A' are concyclic, then we would have

$$HS \cdot HK = HA \cdot HD = HP \cdot HA'$$

whih would show that S, P, K, A' are concyclic.

Notice we have got rid of point S. We now need to show that $\angle APA' = \angle ADA'$. Since $\angle EPH = \angle ADC = 90°$, we only need to prove $\angle APE = \angle A'DC$. This will be easy: Triangles $\triangle HBC$ and $\triangle HEF$ are similar and in this similarity, point D corresponds to P ,which gives $BD : CD = FP : PE$. Also, triangles $\triangle AEF$ and $\triangle A'CB$ are similar, and so in this similarity, points P and D correspond, which gives $\angle APE = \angle A'DC$, which was needed to show.


A
E
P
F
H
B
D
C
A′

Problem 4. 2021 points on the plane in the convex position, no three collinear and no four concyclic, are given. Prove that there exist two of them such that every circle passing through these two points contains at least 673 of the other points in its interior.
(A finite set of points on the plane are in convex position if the points are the vertices of a convex polygon.)

Proposed by Morteza Saghafian - Iran

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Solution. Call the points $P_1, P_2, \dots, P_{2021}$. We need two lemmas for the statement.

Lemma 1. *Call a triangle* ***good*** *if its circumcircle covers all the other points. All the good triangles form a triangulation of the 2021-gon $P_1P_2 \dots P_{2021}$.*

Proof. Note that on every side of this polygon we can construct exactly one good triangle by simply selecting the smallest angle formed by some other vertex. Now start from a good triangle say $P_iP_jP_k$ and pick one of its sides say P_iP_j. Note that all the vertices on the same side of P_iP_j as P_k subtend a larger angle with P_i and P_j as the endpoints, while all the vertices X on the other side fulfill $\angle P_iXP_j + \angle P_iP_kP_j > 180°$. Taking point P_l so that $\angle P_iP_lP_j$ is minimal and P_l, P_k are on the different sides of P_iP_j will also create $P_iP_lP_j$ as a good triangle, because:

- For all points X on the same side of P_iP_j as P_l, $\angle P_iXP_j$ is larger than or equal to $\angle P_iP_lP_j$.
- Among all of angles $\angle P_iXP_j$ so that X and P_l are on different sides of P_iP_j, $\angle P_iP_kP_j$ is the smallest angle and hence $\angle P_iXP_j + \angle P_iP_lP_j > 180°$.

Continuing this process we reach a triangulation of the 2021-gon and call it T. Also note that any triangle from the triangulation uniquely determines the rest. Suppose now there exists a triangle not belonging to the previous triangulation but still being good. Go ahead and apply the same procedure as in the previous case and suppose we reach a triangulation T'. Fix a certain side of the 2021-gon say the side P_1P_2. Note that this side is part of only one good triangle and hence T, T' share a triangle however then we have $T = T'$ and hence we are done. □

Lemma 2. *In the above triangulation T of $P_1P_2 \dots P_{2021}$, there exists a drawn diagonal with at least 673 points on each side.*

Proof. Consider a regular 2021-gon $Q_1Q_2 \dots Q_{2021}$ and draw the diagonals between Q_is of similar indices as in T. Obviously, we get a triangulation T' of $Q_1Q_2 \dots Q_{2021}$. Now let O, the circumcenter of $Q_1Q_2 \dots Q_{2021}$, be in the one of the triangles in T'. Then this triangle, say $Q_iQ_jQ_k$ is an acute-angled triangle (because its circumcenter lies in its interior). Let Q_iQ_j be its longest side. So the angle $\angle Q_iQ_kQ_j$ is acute and at least $60°$ and this means there are at least 673 of other Q_ls on each side of Q_iQ_j.
Now moving back to $P_1P_2 \dots P_{2021}$, the diagonal P_iP_j has the desired property. □

Finally notice that the two points P_i, P_j in Lemma 2 satisfy the statement of the problem since every circle passing through these two points covers all the points on either one side of P_iP_j.

Problem 5. Given a triangle ABC with incenter I. The incircle of triangle ABC is tangent to BC at D. Let P and Q be points on the side BC such that $\angle PAB = \angle BCA$ and $\angle QAC = \angle ABC$, respectively. Let K and L be the incenter of triangles ABP and ACQ, respectively. Prove that AD is the Euler line of triangle IKL.
(The Euler line of a triangle is the line going through the circumcenter and orthocenter of that triangle.)

Proposed by Le Viet An - Vietnam

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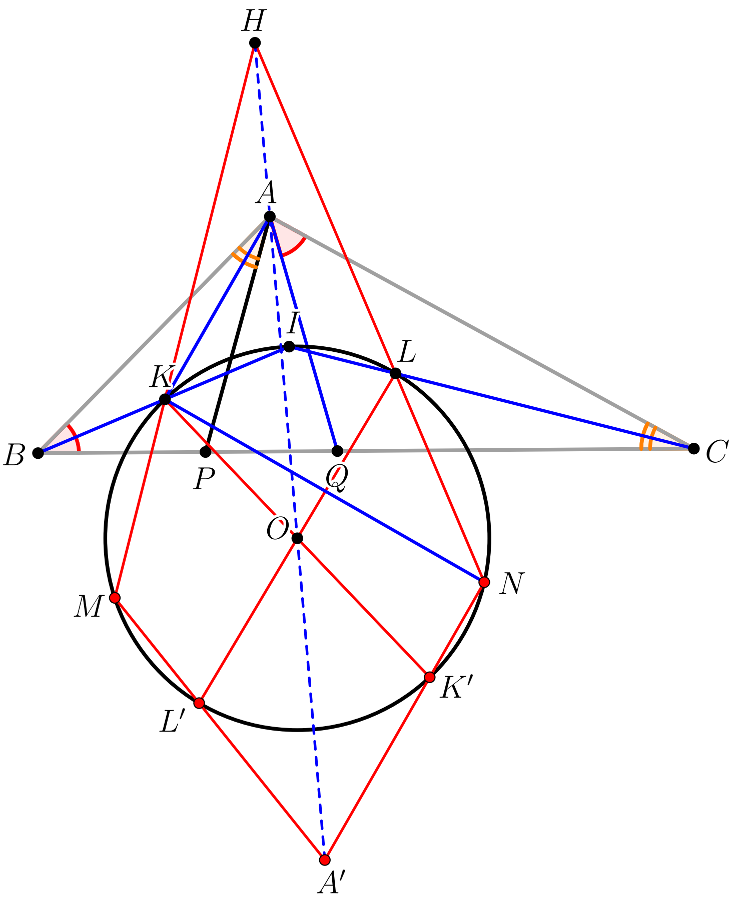
Solution. Let Γ, O and H be the circumcircle, circumcenter and orthocenter of triangle IKL, respectively.

Claim 1. *A lies on OH.*

Proof. Suppose that Γ meets HK, HL, OK, and OL at M, N, K', and L', respectively. Also ML' intersects NK' at A'. Applying Pascal's theorem to the hexagon $MKK'NLL'$ implies that the points H, O, and A' are collinear.
It is easy to see that $\angle AKI = \frac{\angle ABK + \angle BAK}{2} = \frac{\angle B + \angle C}{2}$ and

$$\angle IKN = \angle ILH = 90^\circ - (180^\circ - \angle BIC) = 90^\circ - \frac{\angle B + \angle C}{2}.$$

Hence $\angle AKN = \angle AKI + \angle IKN = 90^\circ$. It yields that $AK \parallel A'K'$. Similarly, one can show that $AL \parallel A'L'$. Combining with $KL \parallel K'L'$ (by symmetry about center O), follows that two triangles AKL and $A'K'L'$ are homothetic. Therefore AA', KK', and LL' are concurrent and this proves the claim.



□

Claim 2. *D lies on OH.*

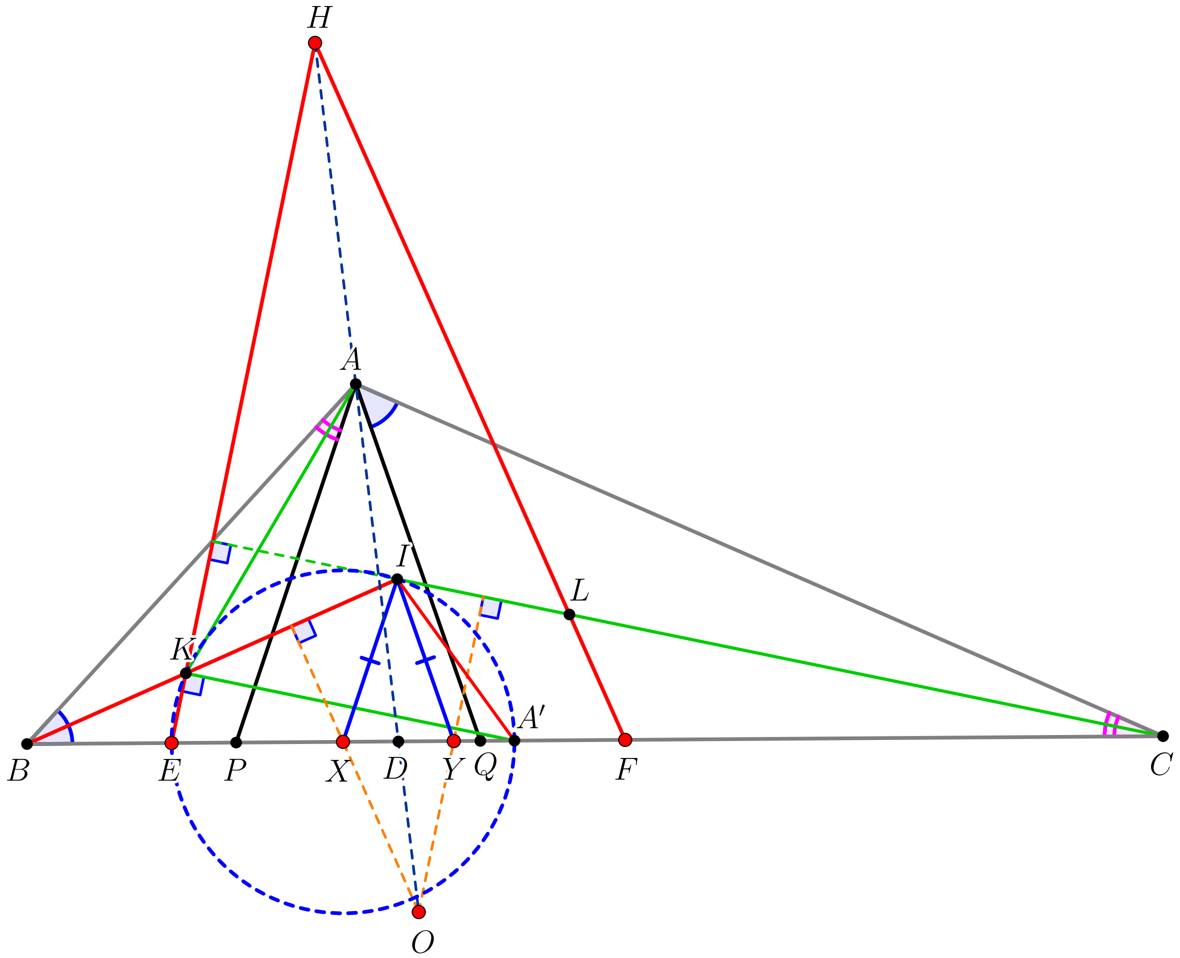
Proof. Let HK and HL intersect BC at E and F. Point A' is the symmetric point to A with respect to BI. Then

$$\angle KA'B = \angle KAB = \frac{1}{2}\angle BAP = \frac{1}{2}\angle C = \angle ICB.$$

It follows that $KA' \parallel CI$, so $\angle A'KE = 90°$. Now notice that $\angle A'IK = \angle AIK = 90° + \frac{\angle C}{2}$. On the other hand, note that $KE \perp CI$ so $\angle A'EK = 90° - \angle ICE = 90° - \frac{\angle C}{2}$. Hence $\angle A'IK + \angle A'EK = 180°$ that implies quadrilateral $A'IKE$ is inscribed.
Let X be the midpoint of $A'E$. Since $\angle A'IE = 90°$, X is the circumcenter of triangle EKI. It yields that

$$\angle IXD = \angle IXA' = 2\angle IKA' = 2\angle IKA = 2(\angle KAB + \angle KBA) = \angle C + \angle B.$$

Similarly, if Y be the circumcenter of triangle ILF, one can show that Y lies on BC and $\angle IYD = \angle B + \angle C$. So the triangle IXY is isosceles at I. It yields that $XE = XI = YI = YF$. From the other hand, $ID \perp XY$ that implies D is the midpoint of XY and EF. Finally note that OX and OY are perpendicular bisectors of IK and IL, so $OX \parallel HF$ and $OY \parallel HE$. It implies that triangles OXY and HFE are similar. Now note that DO and DH are medians of this triangles so $\angle HDE = \angle ODY$ and the conclusion follows.



□

These two claims together yields the desired result.