10^{TH} EUROPEAN MATHEMATICAL CUP

11^{th} December 2021 - 19^{th} December 2021

Junior Category

Problems and Solutions

Problem 1. We say that a quadruple of nonnegative real numbers (a, b, c, d) is *balanced* if

$$a + b + c + d = a^2 + b^2 + c^2 + d^2.$$

Find all positive real numbers x such that

$$(x - a)(x - b)(x - c)(x - d) \geqslant 0$$

for every balanced quadruple (a, b, c, d).

(Ivan Novak)

First Solution. We'll call any $x \in \langle 0, \infty\rangle$ satisfying the problem's condition *great*. Let (a, b, c, d) be a balanced quadruple. Without loss of generality let $a \geqslant b \geqslant c \geqslant d$. We can rewrite the equation $a^2 + b^2 + c^2 + d^2 = a + b + c + d$ as

$$\left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 + \left(c - \frac{1}{2}\right)^2 + \left(d - \frac{1}{2}\right)^2 = 1,$$

which implies $(a - \frac{1}{2})^2 \leqslant 1$, meaning that $a \leqslant \frac{3}{2}$.

6 points.

If we take $x \geqslant \frac{3}{2}$, the values of $x - a$, $x - b$, $x - c$ and $x - d$ are all nonnegative. Thus, any $x \geqslant \frac{3}{2}$ is great.

1 point.

If we take $(a, b, c, d) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, then for any $x \in \langle \frac{1}{2}, \frac{3}{2}\rangle$ we have $(x - a)(x - b)(x - c)(x - d) < 0$. Thus, no $x \in \langle \frac{1}{2}, \frac{3}{2}\rangle$ is great.

2 points.

If we take $(a, b, c, d) = (1, 0, 0, 0)$, then for any $x \in \langle 0, 1\rangle$ we have $(x - a)(x - b)(x - c)(x - d) < 0$. Thus, no $x \in \langle 0, 1\rangle$ is great.

1 point.

We conclude that a number x is great if and only if $x \geqslant \frac{3}{2}$.

Second Solution. Here we present another way to conclude that all $x \geqslant \frac{3}{2}$ satisfy the condition. As in the first solution, we call any $x \in \langle 0, \infty\rangle$ which satisfies the problem's condition *great*, and without loss of generality let (a, b, c, d) be a balanced quadruple satisfying $a \geqslant b \geqslant c \geqslant d$. We notice that for all $y \in \mathbb{R}$ we have

$$\left(y - \frac{1}{2}\right)^2 \geqslant 0 \implies y^2 \geqslant y - \frac{1}{4}.$$

Applying this inequality to b, c and d separately and summing the inequalities we get the following:

$$\begin{cases} b^2 \geqslant b - \frac{1}{4} \\ c^2 \geqslant c - \frac{1}{4} \\ d^2 \geqslant d - \frac{1}{4} \end{cases} \implies b^2 + c^2 + d^2 \geqslant b + c + d - \frac{3}{4}.$$

3 points.

Using the equality $b^2 + c^2 + d^2 = a + b + c + d - a^2$ transforms the inequality above into

$$a \geqslant a^2 - \frac{3}{4} \implies 1 \geqslant \left(a - \frac{1}{2}\right)^2,$$

which implies $a \leqslant \frac{3}{2}$,

3 points.

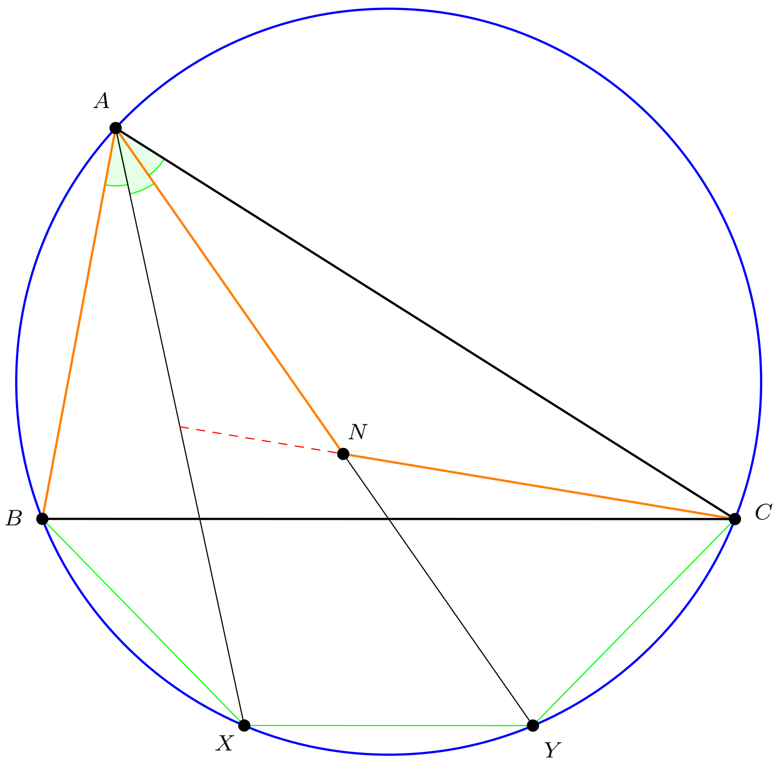
meaning that all $x \geqslant \frac{3}{2}$ are great.

1 point.

The rest of the solution is the same as the previous solution.

Problem 2. Let ABC be an acute-angled triangle such that $|AB| < |AC|$. Let X and Y be points on the minor arc BC of the circumcircle of ABC such that $|BX| = |XY| = |YC|$. Suppose that there exists a point N on the segment $\overline{AY}$ such that $|AB| = |AN| = |NC|$. Prove that the line NC passes through the midpoint of the segment $\overline{AX}$.

(Ivan Novak)



First Solution. Let the line CN intersect the circumcircle of ABC at $T \neq C$.

Since $|BX| = |XY| = |YC|$, we have $\sphericalangle BAX = \sphericalangle XAY = \sphericalangle YAC$. Denote that angle by φ. Furthermore, since $|AN| = |NC|$, we have $\sphericalangle NAC = \sphericalangle NCA = \varphi$, which means that $|AT| = |CY|$.

1 point.

Furthermore, from $|AB| = |AN|$ and $\sphericalangle BAX = \sphericalangle XAY$ it follows that AX is the perpendicular bisector of $\overline{BN}$, so $|XN| = |BX|$. Since $|BX| = |CY| = |AT|$, we have $|XN| = |AT|$.

3 points.

Now $ATBX$ is an isosceles trapezoid because $|AT| = |BX|$ and because it's cyclic, which means that $|AB| = |TX|$, which, combined with the fact that $|AB| = |AN|$, yields $|TX| = |AN|$.

3 points.

Therefore, from $|TX| = |AN|$ and $|XN| = |AT|$ we have that triangles ATX and ANX are congruent, as well as triangles ATN and XTN. Therefore, $\sphericalangle ATX = \sphericalangle ANX$ and $\sphericalangle TAN = \sphericalangle TXN$, which means that $ATXN$ is a parallelogram. Now, as NT and AX are diagonals of a parallelogram, NT passes through the midpoint of $\overline{AX}$, which proves the claim.

3 points.

Second Solution. Let l be the line parallel to AB through C, and let P be the intersection of l and AY. Let α denote the angle $\angle BAC$.

From $|AN| = |NC|$ it follows that $\angle ABX = \angle XAN = \angle NAC = \angle NCA = \frac{\alpha}{3}$. Then it's easy to see that $\angle CNP = \angle NAC + \angle NCA = \frac{2\alpha}{3}$.

1 point.

Similarly, $\angle NPC = \angle PAB = \frac{2\alpha}{3}$ by definition of P.

2 points.

Thus, the triangle CNP is isosceles. Therefore, we conclude that $|CN| = |PC|$.

1 point.

However, $|CN| = |AN| = |AB|$, which implies $|AB| = |PC|$. Since AB and PC are parallel, this means that $ABPC$ is a parallelogram.

2 points.

This implies that AY is the A-median of triangle ABC. Since AX is isogonal to AY with respect to $\sphericalangle BAC$, we conclude that AX is the A-symmedian in the triangle ABC.

1 point.

From a well-known lemma, it now follows that CB is the C-symmedian in the triangle AXC. Note that

$$\angle BCX = \angle BAX = \angle NAC = \angle NCA.$$

We now see that CN and CB are isogonal with respect to $\angle ACX$. Hence, CN is the C-median of triangle ACX and we are done.

3 points.

Third Solution. Denote the angles of ABC by α, β and γ in a standard way, so that $\alpha = \angle BAC$, $\beta = \angle CBA$ and $\gamma = \angle ACB$.

Note that $\angle BCX = \frac{\alpha}{3}$ and $\angle WCA = \angle NCA = \angle NAC = \frac{\alpha}{3}$ since NCA is isosceles. Thus, $\angle WCX = \angle BCA = \gamma$. Also, note that $\angle WAC = \frac{2\alpha}{3}$.

From sine law in triangle ANC, we have $\frac{|AN|}{\sin \frac{\alpha}{3}} = \frac{|AC|}{\sin \frac{2\alpha}{3}}$. Using the fact that $\frac{|AB|}{|AC|} = \frac{\sin\gamma}{\sin\beta}$ and $|AB| = |AN|$, we get the equality

$$\sin\gamma \sin\frac{2\alpha}{3} = \sin\beta \sin\frac{\alpha}{3}. \quad (1)$$

3 points.

Let W denote the intersection of AX and NC. From sine law in triangle AWC, we have

$$\frac{|AW|}{|WC|} = \frac{\sin\frac{\alpha}{3}}{\sin\frac{2\alpha}{3}}.$$

1 point.

From sine law in WXC, we have

$$\frac{|WX|}{|WC|} = \frac{\sin\gamma}{\sin\beta}.$$

1 point.

From these two equalities we get

$$\frac{|AW|}{|WX|} = \frac{\sin\beta \sin\frac{\alpha}{3}}{\sin\gamma \sin\frac{2\alpha}{3}},$$

which equals 1 by (1). Thus, $|AW| = |WX|$.

5 points.

Notes on marking:

- In the second solution, proving that it suffices to prove that AY is the A-median of ABC (last **4 points**) is worth at most **2 points** if a student doesn't prove the parts before that.

Problem 3. Let ℓ be a positive integer. We say that a positive integer k is *nice* if $k! + \ell$ is a square of an integer. Prove that for every positive integer $n \geqslant \ell$, the set $\{1, 2, \ldots, n^2\}$ contains at most $n^2 - n + \ell$ nice integers.

(Theo Lenoir)

Solution. We claim that for every $k \geqslant \ell + 1$, at most one number among $k^2 - 1$ and k^2 is nice.

1 point.

Suppose for the sake of contradiction that both $k^2 - 1$ and k^2 are nice for some $k \geqslant \ell + 1$.
Let $u = \sqrt{(k^2-1)! + \ell}$ and $v = \sqrt{(k^2)! + \ell}$. Then

$$v^2 - \ell = (k^2)! = k^2(u^2 - \ell),$$

which can be rearranged into

$$(ku)^2 - v^2 = (k^2 - 1)\ell. \quad (1)$$

2 points.

Note that this implies $ku > v$ and, furthermore,

$$(ku)^2 - v^2 = (ku - v)(ku + v) \geqslant ku + v > ku > k\sqrt{(k^2-1)!} = \sqrt{(k^2)!}$$

Furthermore, we have the following bounds:

$$(k^2)! \geqslant k^2(k^2-1)(k^2-2) > k^2(k^2-1)(k-1)^2 > (k^2-1)^2(k-1)^2 \geqslant (k^2-1)^2\ell^2,$$

where we used the fact that $k^2 - 2 > (k-1)^2 = k^2 - 2k + 1$ for $k \geqslant 2$ and the assumption $\ell \leqslant k - 1$. But this implies that the left hand side in (1) is greater than the right hand side, which is a contradiction.

7 points.

Thus, there is at least one integer which is not good among $\{k^2 - 1, k^2\}$ for every $k \in \{\ell + 1, \ldots, n\}$, which means there are at least $n - \ell$ integers which aren't good. Thus, the claim is proven.

Partial solution. This is a sketch of a partial solution using analytic number theory. This is not a solution to the original problem, but it provides a better asymptotic bound on the number of nice integers. This solution is worth **5 points** in total.
We first solve the case where ℓ is not a perfect square. Let p be a prime such that $\nu_p(\ell)$ is odd. Then for every $k \geqslant 2p$, we have $\nu_p(k! + \ell) = \nu_p(\ell)$, which is odd. Hence, every $k \geqslant 2p$ is not nice, so there are at most 2ℓ nice numbers and $2\ell \leqslant n^2 - n + \ell$ for $\ell \geqslant 2$.

1 point.

Now consider the case when ℓ is a square. Then $\ell + 1$ is not a perfect square. Note that $(p-2)! \equiv 1 \pmod{p}$ for every prime number p, and thus $(p-2)! + \ell \equiv \ell + 1 \pmod{p}$. If we pick p to be a prime such that $\ell + 1$ is not a quadratic residue modulo p, we conclude that $(p-2)! + \ell$ is not a square.

1 point.

Note that, by quadratic reciprocity, $\ell + 1$ being a quadratic residue modulo p for $p > \ell + 1$ depends only on the remainder of p modulo $8(\ell+1)$, and since $\ell + 1$ is not a square, there must exist a class of residues modulo $8(\ell + 1)$ such that $\ell + 1$ is not a quadratic residue modulo primes from that class.

1 point.

By Dirichlet's theorem on arithmetic progressions, the number of primes less than some N which are from a given class of residues modulo $8(\ell+1)$ is asymptotically

$$\frac{1}{\varphi(8(\ell+1))}\pi(N),$$

where $\pi(N)$ is the number of primes less than N and φ is the Euler's Totient function. By Prime number theorem, $\pi(N)$ is asymptotically $N/\log(N)$. Hence, for large n, the number of integers less than n^2 which are not nice is at least

$$c \cdot \frac{n^2}{\log(n^2)},$$

where $c > 0$ is some constant. For n large enough, this is obviously bigger than $n - \ell$.

2 points.

Notes on marking:

- In the second solution, if a contestant isn't rigorous enough with the bounds in the end, they shouldn't get more than **1 point** for the last part.
- Points from the second solution are not additive with the points from the first solution.

Problem 4. Let n be a positive integer. Morgane has coloured the integers $1, 2, \ldots, n$. Each of them is coloured in exactly one colour. It turned out that for all positive integers a and b such that $a < b$ and $a + b \leqslant n$, at least two of the integers among a, b and $a + b$ are of the same colour. Prove that there exists a colour that has been used for at least $2n/5$ integers.

(Vincent Juge)

First Solution. Throughout the solution, instead of colourings, we will consider partitions, and 'being coloured in the same colour' will be interpreted as 'being in the same element of a partition', and the colours will be interpreted as the blocks of partitions.

Let A denote the first colour that appears, i.e. the block that contains 1. Also, let B denote the block which contains the first integer not in A.

We shall prove that either A or B has at least $2n/5$ elements.

Let C be the union of all blocks other than A and B, and let b be the smallest element of B.

Lemma 1. *For any x, if $x \in C$ then $x - 1 \in A$ and either $x + 1 \in A$ or $x = n$.*

Proof. We'll actually prove a stronger claim: If $x \in C$, then $x-1, x-2, \ldots, x-(b-1) \in A$ and $x+1, x+2, \ldots, x+(b-1) \in A$ if they're not greater than n.

1 point.

For the sake of contradiction, consider the least x for which this claim doesn't hold. But then $x - j \notin C$ for any $j < b$ since otherwise $x - j$ would be the least counterexample since $(x - j) + j \in C$. Since $j \in A$ for $j < b$, we conclude that $x - j \in A$ for all $j < b$.

Now, for any $i < b$, we have $b \in B$ and $x + i - b \in A$, which implies $x + i \in A \cup B$. We also have $x \in C$ and $i \in A$, which implies $x + i \in C \cup A$. Hence, $x + i \in A$. But then $x + 1, \ldots, x + b - 1$ are all in A, a contradiction.

We conclude that the stronger claim holds for every x. Thus, the lemma is proven.

1 point.

□

Lemma 2. There do not exist x and y such that $x \in B$, $x + 1 \in B$, $y \in C$, $y + 2 \in C$.

Proof. Assume that there exist such x and y and assume that such x is minimal. Note that then $x - 1 \notin B$ and $1 \in A$, so $x - 1 \in A$.

We now distinguish two cases, depending on whether $x > y$ or not.

- If $x > y$, consider the integer $r = x - y = (x + 1) - (y + 1)$. Since $x \in B$ and $y \in C$, $r \in B \cup C$. Since $x + 1 \in B$ and $y + 1 \in A$, $r \in B \cup A$. Hence, $r \in B$. Similarly, consider $r + 1 = (x + 1) - y = x - (y - 1)$. Since $x + 1 \in B$ and $y \in C$, $r + 1 \in B \cup C$. Since $x \in B$ and $y - 1 \in A$, $r + 1 \in A \cup B$. Hence, $r + 1 \in B$.
- If $x < y$, consider the integer $r = (y + 1) - x = (y + 2) - (x + 1)$. Since $y + 1 \in A$ and $x \in B$, $r \in A \cup B$. Since $y + 2 \in C$ and $x + 1 \in B$, e $r \in C \cup B$. We conclude that $r \in B$.

  However, $y \in C$ and $y + 1 - x \in B$ implies that the integer $x - 1 = y - (y + 1 - x)$ is either in B or C, but we've already proven that $x - 1 \in A$. Thus, we've reached a contradiction.

  3 points.

  However, since $r = x - y < x$, this contradicts the minimality of x and we've reached a contradiction again.

  3 points.

□

Now we've proved that there either doesn't exist $x \in B$ such that $x + 1 \in B$, or that there doesn't exist $y \in C$ such that $y + 2 \in C$.

In the first case, for every $x \in B$, we have $x - 1 \notin B$ and $1 \in A$. Hence, $x - 1 \in A$. But then $|A| \geqslant n/2$, since $x \mapsto x - 1$ is an injective function from $B \cup C$ to A.

1 point.

In the second case, for every $y \in C$, both $y + 1$ and $y - 1$ are either in A or greater than n, while $y + 2$ and $y - 2$ are not in A. Thus, $y \mapsto y - 1$ and $y \mapsto y + 1$ are injective functions from C to A and $A \cup \{n + 1\}$ respectively, and their images are disjoint. Additionally noting that $1 \in A$ and $1 \neq y - 1, y + 1$ for any $y \in C$, we conclude that $|C| \leqslant \frac{|A|}{2}$. But then $n = |A| + |B| + |C| \leqslant \frac{3|A|+2|B|}{2}$, so either $|A|$ or $|B|$ must be greater than or equal to $\frac{2n}{5}$.

1 point.

Second Solution. We give an alternative proof of **Lemma 1**.

Assume for the sake of contradiction that there are two consecutive integers x and $x+1$ such that both are in C. Let x be the smallest integer with that property. Consider the integer $(x+1)-b = x-(b-1)$. Since $x+1 \in C$ and $b \in B$, we have either $x+1-b \in C$ or $x+1-b \in B$. Since $x \in C$ and $b-1 \in A$, we have either $x+1-b \in C$ or $x+1-b \in A$. We conclude that $x+1-b \in C$.

Furthermore, considering $x-b = (x+1-b)-1$, we have $x \in C$, $b \in B$, $x+1-b \in C$ and $1 \in A$. We conclude that $x-b \in C$, but then $x-b$ and $x+1-b$ are both in C, contradicting the minimality of x. We conclude that there are no consecutive integers in C.

1 point.

Now, for all $x \in C$, the integers $x-1$ and $x+1$ must either be equal to $n+1$ or belong to A, since $1 \in A$, $x-1 \notin C$ and $x+1 \notin C$.

1 point.

□

We also give a slightly different proof of **Lemma 2.**

Proof. We first prove that if x and $x+1$ are both in B and are greater than $y \in C$, then $x-y+1$ and $x-y$ are both in B. Note that $y+1 \in A$ and $y-1 \in A$ by **Lemma 1.**

Note that $x-y = (x+1)-(y+1)$, so $x-y \in B \cup C$ and $x-y \in B \cup A$, which implies $x-y \in B$. Similarly, considering $(x+1)-y = x-(y-1)$, we can conclude that $x-y+1 \in B$.

2 points.

To now prove **Lemma 2**, it suffices to consider the case $x < y$, where x and y are from the statement of the lemma.

1 point.

We deal with that case in the same way as in the first solution.

3 points.

□

The remainder of the solution is the same.

2 points.

Notes on marking:

- Lemma 2 is worth **6 points**. If a contestant states Lemma 2 and they don't prove any of its two subcases, they should get **1 point** for Lemma 2.
- In the second solution, proving the first part of Lemma 2 is worth less than the case $x > y$ in the first solution, because one can prove this part without stating Lemma 2. Proving this part in the context of Lemma 2 is worth **3 points**, and without the context of Lemma 2 it's worth **2 points**.

10$^{\text{TH}}$ European Mathematical Cup

11th December 2021 - 19th December 2021

Senior Category

Problems and Solutions

Problem 1. Alice drew a regular 2021-gon in the plane. Bob then labelled each vertex of the 2021-gon with a real number, in such a way that the labels of consecutive vertices differ by at most 1. Then, for every pair of non-consecutive vertices whose labels differ by at most 1, Alice drew a diagonal connecting them. Let d be the number of diagonals Alice drew. Find the least possible value that d can obtain.

(Ivan Novak)

First Solution. Consider the following labelling of the vertices, where the i-th number of the 2021-tuple below is the label of the i-th vertex:

$$(0.5,\ 1.5,\ 2.5, \ldots, 1009.5,\ 1010.5,\ 1010,\ 1009,\ 1008, \ldots, 2,\ 1).$$

It's easy to see that in this case, Alice will draw 2018 diagonals, those connecting the vertices whose pairs of labels are $\{1.5, 1\}, \{2.5, 2\}, \ldots, \{1009.5, 1009\}$ and $\{1.5, 2\}, \{2.5, 3\}, \ldots, \{1009.5, 1010\}$.

3 points.

We now prove that 2018 is the minimum amount of diagonals Alice could have drawn. Call any labelling of a convex n-gon which satisfies the condition that consecutive vertices have labels which differ by at most 1 a *sweet* labelling, and also call the corresponding n-gon sweet.

We will prove by mathematical induction that for every $n \geqslant 3$, in any sweet labelling of an n-gon, there are at least $n-3$ pairs of nonconsecutive vertices whose labels differ by at most 1. The claim is obvious for $n = 3$. Suppose that the claim is true for some positive integer n.

Consider a sweet labelling of some $n+1$-gon P. Consider a vertex v with the maximum label, L. Then both of its neighbouring vertices have labels in the set $[L-1, L]$, which means that their labels differ by at most 1.

1 point.

Then the n-gon P' obtained from P by erasing v and connecting its neighbouring vertices is also sweet.

2 points.

Applying the inductive hypothesis on it, there are at least $n-3$ pairs of nonconsecutive vertices of P' whose labels differ by at most 1. Adding the pair of neighbours of v, we conclude that P has at most $n-2$ pairs of such vertices. This completes the step of the induction.

4 points.

Second Solution. The example which achieves the desired bound is the same as in the previous solution.

3 points.

Let the labels of the vertices of the 2021-gon be $v_1, \ldots, v_{2021}$, where we assume the labels to be ordered so that $v_1 \leqslant \ldots \leqslant v_{2021}$. Note that this is not necessarily the order in which the values appear on the 2021-gon.

We claim that $|v_i - v_{i+2}| \leqslant 1$, for all $i = 1, 2, \ldots, 2019$.

Assume for the sake of contradiction that there exists an $i \in \{1, 2, \ldots, 2019\}$ for which $v_{i+2} - v_i > 1$. Start a circular walk around the 2021-gon, going from the vertex which has the value v_1, visiting all of the vertices one by one, and returning back to the starting vertex. Doing so visits the values $v_1, \ldots, v_{2021}$ in a certain permuted order, starting and ending on v_1.

We look at the first time during the walk when we step on a value whose index is greater than or equal to $i+2$. Let this index be $j \geqslant i+2$. Let's say that on the previous step, we were on value v_b, where $b \leqslant i+1$. Note that if $b \leqslant i$, then $v_j - v_b \geqslant v_{i+2} - v_i > 1$, so it must be the case that $b = i+1$. Next, we look at the first time we return to an index which is smaller than or equal to i. Such an index must exist since we eventually return back to v_1, and we'll denote it by k. A similar argument as for v_j shows that in the step before reaching v_k, we must have been on index b. This is a contradiction as no vertex can be visited more than once, except for the one we started with.

5 points.

We now have $|v_i - v_j| \leqslant 1$, for all $i = 3, \ldots, 2019$ and $j \in \{i-1, i-2, i+1, i+2\}$, $|v_1 - v_i| \leqslant 1$ for $j \in \{2, 3\}$ and $|v_j - v_{2021}| \leqslant 1$ for $j \in \{2019, 2020\}$, which gives at least $\frac{1}{2}(2+3+4\cdot 2017+3+2) - 2021 = 2018$ diagonals.

2 points.

Third Solution. The example which achieves the desired bound is the same as in the previous solution.

3 points.

Note that

We label the vertices v_1, v_2, …, v_{2021}, and, respectively, their labels x_1, x_2, …, x_{2021} and view the indices modulo 2021. We'll say vertices (v_i, v_j) are a *nigh* pair if their labels differ by at most 1.

Without loss of generality, let 1 and g be the indices among $\{1, 2, \ldots, 2021\}$ of the vertices with the smallest and greatest label, respectively. Without loss of generality we can also assume $g \leqslant 1011$ since we can otherwise mirror the 2021-gon. Additionally, we will assume that $g \geqslant 4$, and the cases $g = 3$ and $g = 2$ are dealt with separately. We make use of the following lemmas.

Lemma 1. *For every $1 < k < g$, there are at least two indices $g \leqslant i, j \leqslant 2022$ such that (v_k, v_i) and (v_k, v_j) are pairs of nigh vertices. Similarly, for every $g < k < 2022$, there are at least two indices $1 \leqslant i, j \leqslant g$ such that (v_k, v_i) and (v_k, v_j) are pairs of nigh vertices.*

Proof. As the statement is obviously symmetric, we will only prove the first half. Let m and M be the smallest and the biggest label, respectively. If $x_k \leqslant m+1$, v_{2021} and v_1 satisfy the condition of the lemma. Similarly, if $x_k \geqslant M-1$, v_g and v_{g+1} satisfy the condition of the lemma. Otherwise, there are at least two indices within the desired range with labels in $[x-1, x+1]$ due to the Intermediate Value Theorem. Namely, if we imagine jumping along the vertices from v_1 do v_g, at the vertex v_1 the label is less than $x-1$ and at the vertex v_g the label is greater than $x+1$. Then at some point in between we must have been at a vertex whose label is from $[x-1, x)$ and at a vertex whose label is from $[x, x+1)$. $\square$

4 points.

Lemma 2. At most one of the vertices v_{2021} and v_2 and at most one of the vertices v_{g-1} and v_{g+1} can be elements of three nigh pairs.

Proof. Clearly, due to Lemma 1, both v_{2021} and v_2 are elements of at least three nigh pairs. Assume both vertices are elements of exactly three nigh pairs. Assume that $x_{2021} \leqslant x_2$. It follows that $x_2 - 1 \leqslant x_1 \leqslant x_{2020} \leqslant x_{2021} + 1 \leqslant x_2 + 1$, therefore, v_{2020}, v_{2021}, v_1 and v_3 are four vertices forming nigh pairs with v_2, a contradiction. We treat the other cases analogously. $\square$

2 points.

Finally, due to Lemma 1, each vertex not neighbouring with g or 1 forms at least 4 nigh pairs. The vertices v_1 and v_g form at least 2 nigh pairs and at most two of their neighbouring vertices form 3 nigh pairs. The minimal number of nigh pairs is therefore $\frac{1}{2} \cdot (2017 \cdot 4 + 2 \cdot 3 + 2 \cdot 2) = 4039$.

1 point.

Assume $g \leqslant 3$. If $g = 2$, all the pairs of vertices are nigh as their labels are all in the set $[x_1, x_2]$. If $g = 3$, each vertex forms a nigh pair with at least one of v_1 or v_3. Without loss of generality, more than half of the remaining vertices form nigh pairs with v_1. But then all of those vertices are nigh as well, so there are at least $\binom{1015}{2} > 4039$ nigh pairs, making this case suboptimal as well.

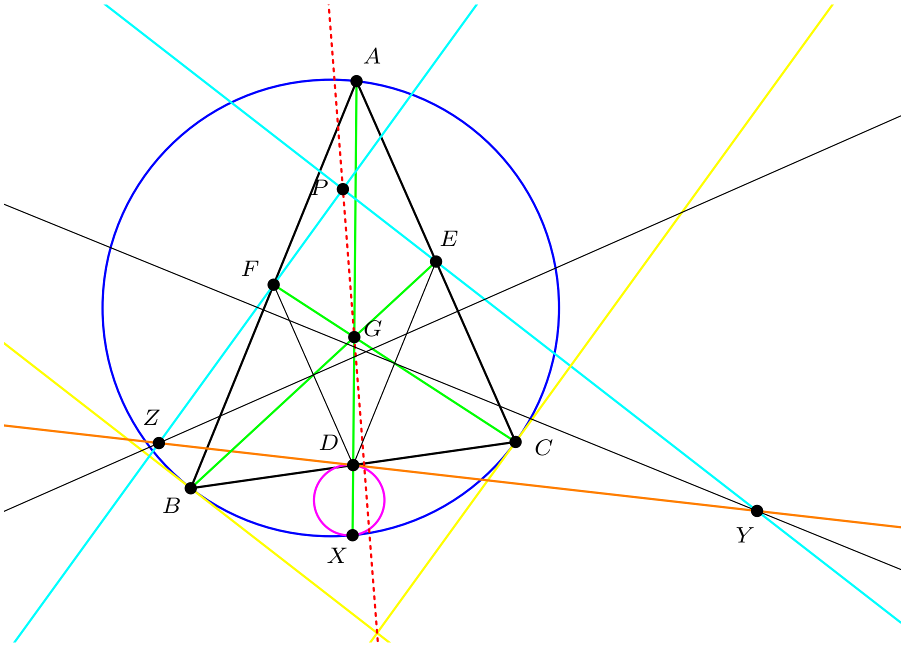
0 points.

Notes on marking:

- Points from different solutions are not additive. Student's score should be the maximum of points scored over all solutions.
- Miscounting the number d in the optimal example or making a similar minor mistake in that part should be awarded **2 points** out of possible **3** for that part of the solution.
- In the first solution, just stating the idea of induction is worth no points on its own.
- In the third solution, dealing with the case $g \leqslant 3$ is worth no points on its own. However, a contestant who doesn't comment these cases can get at most **9 points** for their solution.

Problem 2. Let ABC be a triangle and let D, E and F be the midpoints of sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively. Let $X \neq A$ be the intersection of AD with the circumcircle of ABC. Let Ω be the circle through D and X, tangent to the circumcircle of ABC. Let Y and Z be the intersections of the tangent to Ω at D with the perpendicular bisectors of segments $\overline{DE}$ and $\overline{DF}$, respectively. Let P be the intersection of YE and ZF and let G be the centroid of ABC. Show that the tangents at B and C to the circumcircle of ABC and the line PG are concurrent.

(Jakob Jurij Snoj)



First Solution. Due to the collinearity of A, X and D, there is a homothety at X sending the circumcircle of ABC to Ω. This homothety also sends the tangent at A to the circumcircle of ABC to the tangent at D of Ω.

1 point.

The homothety in G with ratio $-\frac{1}{2}$ sends the circumcircle of ABC to its nine-point circle and the tangent at A to the circumcircle to the tangent at D to the nine-point circle. These tangents are therefore parallel. It follows that Ω and the nine-point circle of ABC share a tangent at D.

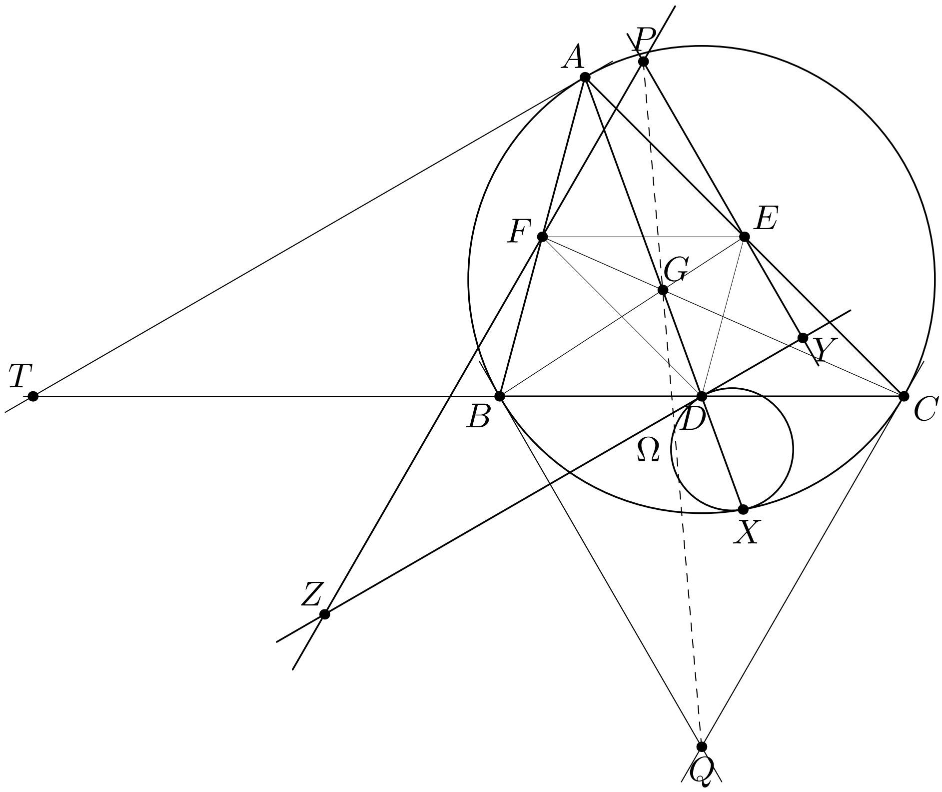
2 points.

As Y and Z lie on the perpendicular bisectors of DE and DF, respectively, it follows that $|YD| = |YE|$ and YE is also tangent to the nine-point circle of ABC - similarly, ZF is also tangent to this circle. We conclude that the nine-point circle of ABC is the incircle of PYZ.

3 points.

Finally, the homothety at G sending the circumcircle of ABC to its nine-point circle sends the tangents through A, B and C to the circumcircle of ABC respectively to the tangents at D, E and F to the nine-point circle of ABC. It, therefore, sends the intersection of tangents at B and C to the circumcircle of ABC to the point P, thus proving the desired collinearity.

4 points.



Second Solution. As in the previous solution, we prove that the tangent at D to Ω is parallel to the tangent at A to the circumcircle of ABC.

1 point.

Let $\sphericalangle BAC = \alpha$, $\sphericalangle ABC = \beta$, $\sphericalangle ACB = \gamma$. Let Q be the intersection of the tangents at B and C to the circumcircle of ABC. It now suffices to prove that G lies on PQ.

0 points.

Let T be the point on BC such that the line TA is tangent to the circumcircle of ABC at A. By the *tangent-chord theorem*, we have $\sphericalangle TAB = \gamma$, which implies $\sphericalangle ATB = \beta - \gamma$. Since $ZY \parallel TA$ and $\sphericalangle YDC = \sphericalangle ATB = \beta - \gamma$.

1 point.

By definion of Y, we have $|EY| = |DY|$. Since DE is a midline of ABC, we have $\sphericalangle EDC = \sphericalangle ABC = \beta$. Thus

$$\sphericalangle DEY = \sphericalangle EDY = \sphericalangle EDC - \sphericalangle YDC = \beta - (\beta - \gamma) = \gamma.$$

2 points.

Furthermore, note that $\sphericalangle BED + \sphericalangle DBE = \sphericalangle EDC = \beta$, and $\sphericalangle QBC = \alpha$, by the *tangent-chord theorem.* This implies

$$\sphericalangle QBE + \sphericalangle BEY = (\sphericalangle QBC + \sphericalangle DBE) + (\sphericalangle BED + \sphericalangle DEY) = \alpha + \beta + \gamma = 180^\circ$$

so $QB \parallel PE$. By analogous reasoning we conclude that $QC \parallel PF$.

2 points.

Since EF is a midline of ABC, we have that the sides EP, PF, EF of triangle PEF are parallel to the sides QB, QC, BC of triangle QBC, respectively. Those triangles are not congruent because $EF = \frac{BC}{2}$, so there exists a homothety which maps PEF to QBC. The centre of the homothety is the intersection of BE, CF, and PQ, which implies that G lies on PQ and we are done.

4 points.

Notes on marking:

- In the second solution, once we obtain $\sphericalangle DEY = \sphericalangle EDY = \gamma$, we can conclude that YE is tangent to the circumcircle of DEF and finish as in the first solution.
- The final **4 points** in either solution can only be awarded if the student correctly proves the other steps of the problem. Otherwise, a contestant can only obtain up to **2 points** for this part of the solution.
- No points are deducted if the student fails to argue that $\triangle PYZ$ and the triangle formed by the tangents through A, B and C to the circumcircle of ABC are not congruent.
- Analytic approaches are only awarded points if their results are correctly interpreted by geometric means.

Problem 3. Let $\mathbb{N}$ denote the set of all positive integers. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$x^2-y^2+2y(f(x)+f(y))$$

is a square of an integer for all positive integers x and y.

(Ivan Novak)

First Solution. Throughout the solution, let $P(a,b)$ denote the assertion "$a^2-b^2+2b(f(a)+f(b))$ is a perfect square".

Let p be a prime. Then $P(p,p)$ implies $4pf(p)$ is a perfect square, which implies $p\mid f(p)$.

1 point.

Let y be any positive integer, and let p be any prime. $P(p,y)$ implies $p^2+2yf(p)+2yf(y)-y^2$ is a perfect square. Taking the assertion modulo p, it follows that $2yf(y)-y^2$ is a quadratic residue modulo p. It is a well known fact that if a positive integer is a quadratic residue modulo all primes, it must be a perfect square. We conclude that $2yf(y)-y^2$ is a perfect square for all $y\in\mathbb{N}$.

3 points.

Define $g(y)$ to be $\sqrt{2yf(y)-y^2}$.
$P(1,y)$ implies $1-y^2+2yf(1)+2yf(y)=g(y)^2+2yf(1)+1$ is a perfect square, and since

$$g(y)^2+2yf(1)+1>g(y)^2,$$

we have a following chain of inequalities:

$$g(y)^2+2yf(1)+1\geqslant(g(y)+1)^2\implies 2yf(1)+1\geqslant 2g(y)+1\implies$$

$$yf(1)\geqslant g(y)\implies y^2f(1)^2\geqslant 2yf(y)-y^2\implies\frac{f(1)^2+1}{2}y\geqslant f(y).$$

1 point.

Since $p\mid f(p)$ and $\frac{f(p)}{p}\leqslant\frac{f(1)^2+1}{2}$ for any prime p, it follows from Pigeonhole principle that there exists a positive integer a such that $f(p)=ap$ for infinitely many primes p.

2 points.

Let p be any prime such that $f(p)=ap$ and let n be any positive integer. $P(p,n)$ implies

$$p^2-n^2+2nap+2nf(n)=(p+na)^2-n^2a^2-n^2+2nf(n)$$

is a perfect square.

However, this means that $2nf(n)-n^2-n^2a^2$ can be written as a difference of squares in infinitely many ways, which is only possible if it equals 0. Thus, $2nf(n)=n^2a^2+n^2$, or, equivalently, $f(n)=\frac{n(a^2+1)}{2}$ for all $n\in\mathbb{N}$. This also implies $\frac{a^2+1}{2}=a$, which gives us $a=1$.
Therefore, $f(n)=n$ for all $n\in\mathbb{N}$. It can be easily checked that the identity function is indeed a solution.

3 points.

Second Solution. Similarly as in the first solution, we conclude that $p\mid f(p)$. Also, from $P(4p,4p)$, we also conclude that $p\mid f(4p)$ for every odd prime p.

1 point.

Fix an odd prime number p, and let $A=\frac{f(p)}{p}$ and $B=\frac{f(4p)}{p}$.
Now, from $P(4p,p)$, we have that $15p^2+2p^2(A+B)$ is a square, which means $15+2(A+B)$ is also a square. Multiplying by 4 yields the fact that $60+8(A+B)$ is a square.
From $P(p,4p)$, we have that $-15p^2+8p^2(A+B)$ is a square, which means $-15+8(A+B)$ is a square.
We then have

$$75=60+8(A+B)-(-15+8(A+B)).$$

Since there are finitely many ways to write 75 as a difference of two squares, we conclude that $A+B$ can take only finitely many different values as p ranges over all primes. The same is true for A.

4 points.

Therefore, by Pigeonhole principle, there exists a positive integer a such that $f(p)=ap$ for infinitely many primes p.

2 points.

The problem can now be finished the same way as in the first solution.

3 points.

Notes on marking:

- Both solutions follow a similar structure. In the first part, a constant a is found such that $f(p) = ap$ for infinitely many primes p, and in the second part the problem is completed using the first fact. The points from different solutions are not additive.
- In the first solution, it is not expected from the students to prove the key lemma about quadratic residues that is used; it suffices to state it. On the other hand, merely stating the lemma is not worth any points on its own.
- If a student solves the problem under the assumption that $f(p) = ap$ for infinitely many p, but they don't prove this fact, they can get at most **1 point** out of the last **3 points**.
- In the second solution, if a student discusses $P(1, 4)$ and $P(4, 1)$ similarly to how $P(p, 4p)$ and $P(4p, p)$ are discussed, and concludes that $f(1)$ and $f(4)$ can take on finitely many different values, they should get **1 point** out of **4** for that part of the solution.

Problem 4. Find all positive integers d for which there exist polynomials $P(x)$ and $Q(x)$ with real coefficients such that degree of P equals d and

$$P(x)^2 + 1 = (x^2 + 1)Q(x)^2.$$

(Ivan Novak)

First Solution.

$$P(x)^2 + 1 = (x^2 + 1)Q(x)^2 \tag{1}$$

Let P and Q be polynomials satisfying the conditions. Note that the degree of the left hand side in (1) the equality is $2d$, and the degree of the right hand side is $2 + 2\deg Q$, which implies $\deg Q = d - 1$.
Suppose that r is a real root of Q. Then $P(r)^2 + 1 = 0$, which is clearly impossible. We conclude that Q has no real roots. Since Q has real coefficients, we conclude that Q has even degree since its roots must come in conjugate pairs. Thus, d must be odd.

1 point.

Now we prove that for any odd d, there exist polynomials satisfying the conditions. Let $\mathbb{R}[x, \sqrt{x^2+1}]$ be the set of all functions of the form $A + B\sqrt{x^2+1}$, where A and B are polynomials with real coefficients. Note that each element of $\mathbb{R}[x, x^2+1]$ can be uniquely associated with a pair of polynomials (A, B). Consider a function $n : \mathbb{R}[x, \sqrt{x^2+1}] \to \mathbb{R}$ defined by

$$n(A + B\sqrt{x^2+1}) = A^2 - (x^2+1)B^2$$

for all real polynomials A and B. Note that the equality (1) is equivalent to the equality

$$n(P + \sqrt{x^2+1}Q) = -1.$$

1 point.

Note that

$$n((A+\sqrt{x^2+1}B)(C+\sqrt{x^2+1}D)) = n(AC+(x^2+1)BD+\sqrt{x^2+1}(AD+BC)) = (AC+(x^2+1)BD)^2-(x^2+1)(AD+BC)^2.$$

On the other hand,

$$n(A + \sqrt{x^2+1}B)n(C + \sqrt{x^2+1}D) = (A^2 - (x^2+1)B^2)(C^2 - (x^2+1)D^2).$$

It can easily be checked that the two expressions are equal. Hence, the function n is multiplicative.

1 point.

Note that $n(x + \sqrt{x^2+1}) = -1$.

1 point.

Then, using the multiplicative property, $n((x + \sqrt{x^2+1})^d) = -1$ as well. Let $(x + \sqrt{x^2+1})^d = P + \sqrt{x^2+1}Q$ for some polynomials P and Q. By binomial theorem, we have

$$P + \sqrt{x^2+1}Q = (x + \sqrt{x^2+1})^d = \sum_{j \text{ odd}} \binom{d}{j} x^{d-j}(x^2+1)^{\frac{d-j}{2}} + \sum_{j \text{ even}} \binom{d}{j} x^j (x^2+1)^{\frac{d-1-j}{2}} \sqrt{x^2+1}.$$

It's now easy to see that P has degree d, since it is a sum of polynomials which have degree d and positive leading coefficients, and P and Q satisfy the starting equality. Thus, all odd positive integers are solutions.

6 points.

Second Solution.

$$P(x)^2 + 1 = (x^2 + 1)Q(x)^2 \tag{2}$$

Similarly as in the first solution, we conclude that d needs to be odd.

1 point.

Let us now prove that for every odd d such polynomials $P(x)$ and $Q(x)$ exist. Fix an odd positive integer d. Observing the roots of polynomials $P(x)$ and $Q(x)$, we can easily see from (2) that $P(x)$ and $Q(x)$ don't have a common root. Differentiating (2), we get :

$$P'(x)P(x) = x \cdot Q(x)^2 + (x^2+1) \cdot Q'(x)Q(x) = Q(x)\left(x \cdot Q(x) + (x^2+1) \cdot Q'(x)\right). \tag{3}$$

Since $P(x)$ and $Q(x)$ don't have common roots, from (3) we conclude that $Q(x)$ must divide $P'(x)$. Since they have the same degree, there must exist a real number u such that $uP'(x) = Q(x)$.

1 point.

Comparing coefficients in (2), we get that u must be $\frac{1}{d}$ or $-\frac{1}{d}$.

1 point.

We'll take $u = \frac{1}{d}$. Plugging in $Q(x) = P'(x)/d$ in (3), we get

$$P(x) = \frac{1}{d}\left(\frac{1}{d}x \cdot P'(x) + \frac{1}{d}(x^2+1)P''(x)\right). \tag{4}$$

We will now find all polynomials P of degree d which satisfy (4). Note that, by multiplying both sides with $P'(x)$ and integrating, each of these polynomials satisfies the equation $P(x)^2 + C = (x^2+1)\frac{(P'(x))^2}{d^2}$ for some $C \in \mathbb{R}$.

Denote $P(x) = \sum_{i=0}^{d} a_i x^i$. Then $P'(x) = \sum_{i=1}^{d} i a_i x^{i-1}$ and $P''(x) = \sum_{i=2}^{d} i(i-1)a_i x^{i-2}$. Writing out the coefficients in (4), we get

$$\sum_{i=0}^{d} a_i x^i = \frac{1}{d^2}\left(\sum_{i=1}^{d} i^2 a_i x^i + \sum_{i=2}^{d} i(i-1)a_i x^{i-2}\right)$$

Comparing the coefficients of x^k for all k on the left hand side and the right hand side of the above equation for $k \geqslant 0$, we get:

$$a_{d-1} = \frac{1}{d^2}\left((d-1)^2 a_{d-1}\right) \quad \text{which implies} \quad a_{d-1} = 0,$$

and also

$$a_k = \frac{1}{d^2}\left(k^2 a_k + (k+2)(k+1)a_{k+2}\right) \quad \text{which can be rewritten as}$$
$$a_{k+2} = \frac{d^2-k^2}{(k+2)(k+1)} \cdot a_k \quad \text{for all} \quad 0 \leqslant k \leqslant d-2.$$

1 point.

From here, we now have that $a_k = 0$ for all even $0 \leqslant k \leqslant d-2$ and that for all odd $0 \leqslant k \leqslant d-2$, we have $a_k = q_k a_1$ for some nonzero real coefficient q_k which is uniquely determined by the above recursion.

1 point.

It's easy to see that any such choice of coefficients $(a_k)_k$ with $a_1 \neq 0$ gives a solution to (4) which has degree d.

As we've already said, any solution to (4) is a solution to $P(x)^2 + C = (x^2+1)\frac{(P'(x))^2}{d^2}$ for some $C \in \mathbb{R}$. Considering the coefficient alongside x^0 in both sides and noting $a_0 = 0$, we get $C = a_1^2/d^2$. Thus, taking a solution with $a_1 = d$, we get the solution to $P(x)^2 + 1 = (x^2+1)\frac{(P'(x))^2}{d^2}$, which proves that every odd d is a solution to the problem.

5 points.

Third Solution.

$$P(x)^2 + 1 = (x^2+1)Q(x)^2 \tag{5}$$

Similarly as in the first solution, we conclude that d needs to be odd.

1 point.

Note that $d = 1$ is a solution, taking $P(x) = x$ and $Q(x) = 1$. Henceforth assume $d \geqslant 3$.

From $x^2+1 \mid P(x)^2+1$ we get $x^2+1 \mid P(x)^2 - x^2 \implies x^2+1 \mid (P(x)-x)(P(x)+x)$. It is not hard to see that the irreducible polynomial x^2+1 divides exactly one of the two factors. We can replace P by $-P$, so without loss of generality it is safe to assume that $P(x) = A(x)(x^2+1) + x$ for some real polynomial A. If we put this in (5), we obtain

$$(A(x) \cdot x + 1)^2 + A(x)^2 = Q(x)^2.$$

We will find real polynomials α and β such that $A(x) = 2\alpha(x)\beta(x)$, $Q(x) = \alpha(x)^2 + \beta(x)^2$, $xA(x)+1 = \alpha(x)^2 - \beta(x)^2$ and $\deg\alpha + \deg\beta = d-2$. Note that then $A(x)$ and $Q(x)$ satisfy the conditions due to the identities for Pythagorean triples.

2 points.

We thus need to find solutions to the equation

$$2x\alpha(x)\beta(x) + 1 = \alpha(x)^2 - \beta(x)^2 \tag{6}$$

where $\alpha, \beta \in \mathbb{R}[x]$ are polynomials with real coefficients. Notice that $(\alpha, \beta) = (1, -2x)$ is a pair of solutions.

1 point.

If we look at (6) as a quadratic equation in α we have

$$\alpha^2 - \alpha \cdot 2x\beta + 1 - \beta^2 = 0.$$

Roots α_1, α_2 must then satisfy $\alpha_1 + \alpha_2 = 2x\beta$. It is now easily verified that if (α, β) is a pair of polynomials which satisfy (6), then $(\beta, 2x\beta - \alpha)$ is another such pair.

1 point.

Thus starting with solution $(\alpha_0, \beta_0) = (1, -2x)$, we can recursively generate a sequence of solutions

$$(\alpha_{i+1}, \beta_{i+1}) = (\beta_i, 2x\beta_i - \alpha_i).$$

The degrees of $(\alpha_i, \beta_i)_{i \geqslant 0}$ now follow the pattern

$$(0,1), (1,2), (2,3), (3,4) \dots$$

More precisely, $\deg \alpha_i = i$, $\deg \beta_i = i + 1$ for all $i \geqslant 0$.
But then, if $d = 2i+1$ for some $i \geqslant 0$, the pair (α_i, β_i) gives a pair $(A, Q) = (2\alpha\beta, \alpha^2+\beta^2)$ such that $(xA(x)+1)^2+A(x)^2 = Q(x)^2$ and $\deg A = i + (i-1) = d-2$. Then, taking $P(x) = (x^2+1)A(x) + x$ yields a pair (P, Q) satisfying the original equation such that $\deg P = d$. We conclude that every odd d is a solution.

5 points.

Notes on marking:

- Points from different marking schemes are not additive.