Chapter 1

2007 Shortlist JBMO - Problems

1.1 Algebra

A1 Let a be a real positive number such that $a^3 = 6(a+1)$. Prove that the equation $x^2 + ax + a^2 - 6 = 0$ has no solution in the set of the real number.

A2 Prove that $\frac{a^2 - bc}{2a^2 + bc} + \frac{b^2 - ca}{2b^2 + ca} + \frac{c^2 - ab}{2c^2 + ab} \leq 0$ for any real positive numbers a, b, c.

A3 Let A be **a** set of positive integers containing the number 1 and at least one more element. Given that for any two different elements m, n of A the number $\frac{m+1}{(m+1,n+1)}$ is also an element of A, prove that A coincides with the set of positive integers.

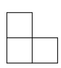
A4 Let a and b be positive integers bigger than 2. Prove that there exists a positive integer k and a sequence $n_1, n_2, \ldots, n_k$ consisting of positive integers, such that $n_1 = a$, $n_k = b$, and $(n_i + n_{i+1}) \mid n_i n_{i+1}$ for all $i = 1, 2, \ldots, k-1$.

A5 The real numbers x, y, z, m, n are positive, such that $m + n \geq 2$. Prove that

$$x\sqrt{yz(x+my)(x+nz)} + y\sqrt{xz(y+mx)(y+nz)} + z\sqrt{xy(z+mx)(x+ny)} \leq \frac{3(m+n)}{8}(x+y)(y+z)(z+x).$$

1.2 Combinatorics

C1 We call a tiling of an $m \times n$ rectangle with corners (see figure below) "regular" if there is no sub-rectangle which is tiled with corners. Prove that if for some m and n there exists a "regular" tiling of the $m \times n$ rectangular then there exists a "regular" tiling also for the $2m \times 2n$ rectangle.



C2 Consider 50 points in the plane, no three of them belonging to the same line. The points have been colored into four colors. Prove that there are at least 130 scalene triangles whose vertices are colored in the same color.

C3 The nonnegative integer n and $(2n+1)\times(2n+1)$ chessboard with squares colored alternatively black and white are given. For every natural number m with $1 < m < 2n+1$, an $m \times m$ square of the given chessboard that has more than half of its area colored in black, is called a B-square. If the given chessboard is a B-square, find in terms of n the total number of B-squares of this chessboard.

1.3 Geometry

G1 Let M be an interior point of the triangle ABC with angles $\sphericalangle BAC = 70^\circ$ and $\sphericalangle ABC = 80^\circ$. If $\sphericalangle ACM = 10^\circ$ and $\sphericalangle CBM = 20^\circ$, prove that $AB = MC$.

G2 Let $ABCD$ be a convex quadrilateral with $\sphericalangle DAC = \sphericalangle BDC = 36^\circ$, $\sphericalangle CBD = 18^\circ$ and $\sphericalangle BAC = 72^\circ$. If P is the point of intersection of the diagonals AC and BD, find the measure of $\sphericalangle APD$.

G3 Let the inscribed circle of the triangle $\triangle ABC$ touch side BC at M, side CA at N and side AB at P. Let D be a point from $[NP]$ such that $\frac{DP}{DN} = \frac{BD}{CD}$. Show that $DM \perp PN$.

G4 Let S be a point inside $\sphericalangle pOq$, and let k be a circle which contains S and touches the legs Op and Oq in points P and Q respectively. Straight line s parallel to Op from S intersects Oq in a point R. Let T be the point of intersection of the ray PS and circumscribed circle of $\triangle SQR$ and $T \neq S$. Prove that $OT \parallel SQ$ and OT is a tangent of the circumscribed circle of $\triangle SQR$.

1.4 Number Theory

NT1 Find all the pairs positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{[x,y]} + \frac{1}{(x,y)} = \frac{1}{2},$$

where (x, y) is the greatest common divisor of x, y and $[x, y]$ is the least common multiple of x, y.

NT2 Prove that the equation $x^{2006} - 4y^{2006} - 2006 = 4y^{2007} + 2007y$ has no solution in the set of the positive integers.

NT3 Let $n > 1$ be a positive integer and p a prime number such that $n \mid (p-1)$ and $p \mid (n^6 - 1)$. Prove that at least one of the numbers $p - n$ and $p + n$ is a perfect square.

NT4 Let a, b be two co-prime positive integers. A number is called **good** if it can be written in the form $ax + by$ for non-negative integers x, y. Define the function $f : \mathbb{Z} \to \mathbb{Z}$

as $f(n) = n - n_a - n_b$, where s_t represents the remainder of s upon division by t. Show that an integer n is **good** if and only if the infinite sequence $n, f(n), f(f(n)), \ldots$ contains only non-negative integers.

NT5 Let p be a prime number. Show that $7p + 3^p - 4$ is not a perfect square.

Chapter 2

2007 Shortlist JBMO - Solutions

2.1 Algebra

A1 Let a be a real positive number such that $a^3 = 6(a+1)$. Prove that the equation $x^2 + ax + a^2 - 6 = 0$ has no solution in the set of the real number.
Solution
The discriminant of the equation is $\Delta = 3(8 - a^2)$. If we accept that $\Delta \geq 0$, then $a \leq 2\sqrt{2}$ and $\frac{1}{a} \geq \frac{\sqrt{2}}{4}$, from where $a^2 \geq 6 + 6 \cdot \frac{\sqrt{2}}{4} = 6 + \frac{6}{a} \geq 6 + \frac{3\sqrt{2}}{2} > 8$ (contradiction).

A2 Prove that $\frac{a^2 - bc}{2a^2 + bc} + \frac{b^2 - ca}{2b^2 + ca} + \frac{c^2 - ab}{2c^2 + ab} \leq 0$ for any real positive numbers a, b, c.
Solution
The inequality rewrites as $\sum \frac{2a^2 + bc - 3bc}{2a^2 + bc} \leq 0$, or $3 - 3\sum \frac{bc}{2a^2 + bc} \leq 0$ in other words $\sum \frac{bc}{2a^2 + bc} \geq 1$.
Using Cauchy-Schwarz inequality we have

$$\sum \frac{bc}{2a^2 + bc} = \sum \frac{b^2c^2}{2a^2bc + b^2c^2} \geq \frac{\left(\sum bc\right)^2}{2abc\,(a+b+c) + \sum b^2c^2} = 1,$$

as claimed.

A3 Let A be **a** set of positive integers containing the number 1 and at least one more element. Given that for any two different elements m, n of A the number $\frac{m+1}{(m+1, n+1)}$ is also an element of A, prove that A coincides with the set of positive integers.
Solution
Let $a > 1$ be lowest number in $A \setminus \{1\}$. For $m = a$, $n = 1$ one gets $y = \frac{a+1}{(2, a+1)} \in A$.
Since $(2, a+1)$ is either 1 or 2, then $y = a + 1$ or $y = \frac{a+1}{2}$.

But $1 < \dfrac{a+1}{2} < a$, hence $y = a+1$. Applying the given property for $m = a+1$, $n = a$ one has $\dfrac{a+2}{(a+2, a+1)} = a+2 \in A$, and inductively $t \in A$ for all integers $t \geq a$.
Furthermore, take $m = 2a-1$, $n = 3a-1$ (now in A!); as $(m+1, n+1) = (2a, 3a) = a$ one obtains $\dfrac{2a}{a} = 2 \in A$, so $a = 2$, by the definition of a.
The conclusion follows immediately.

A4 Let a and b be positive integers bigger than 2. Prove that there exists a positive integer k and a sequence $n_1, n_2, \ldots, n_k$ consisting of positive integers, such that $n_1 = a$, $n_k = b$, and $(n_i + n_{i+1}) \mid n_i n_{i+1}$ for all $i = 1, 2, \ldots, k-1$.
Solution
We write $a \Leftrightarrow b$ if the required sequence exists. It is clear that $\Leftrightarrow$ is equivalence relation, i.e. $a \Leftrightarrow a$, ($a \Leftrightarrow b$ implies $b \Rightarrow a$) and ($a \Leftrightarrow b$, $b \Leftrightarrow c$ imply $a \Leftrightarrow c$).
We shall prove that for every $a \geq 3$, (a - an integer), $a \Leftrightarrow 3$.
If $a = 2^s t$, where $t > 1$ is an odd number, we take the sequence

$$2^s t,\ 2^s(t^2 - t),\ 2^s(t^2 + t),\ 2^s(t+1) = 2^{s+1} \cdot \frac{t+1}{2}.$$

Since $\dfrac{t+1}{2} < t$ after a finite number of steps we shall get a power of 2. On the other side, if $s > 1$ we have 2^s, $3 \cdot 2^s$, $3 \cdot 2^{s-1}$, $3 \cdot 2^{s-2}$, $\ldots, 3$.

A5 The real numbers x, y, z, m, n are positive, such that $m + n \geq 2$. Prove that

$$x\sqrt{yz(x+my)(x+nz)} + y\sqrt{xz(y+mx)(y+nz)} + z\sqrt{xy(z+mx)(x+ny)} \ \leq$$

$$\frac{3(m+n)}{8}(x+y)(y+z)(z+x).$$

Solution
Using the AM-GM inequality we have

$$\sqrt{yz(x+my)(x+nz)} = \sqrt{(xz+myz)(xy+nyz)} \leq \frac{xy+xz+(m+n)yz}{2},$$

$$\sqrt{xz(y+mx)(y+nz)} = \sqrt{(yz+mxz)(xy+nxz)} \leq \frac{xy+yz+(m+n)xz}{2},$$

$$\sqrt{xy(z+mx)(z+ny)} = \sqrt{(yz+mxy)(xz+nxy)} \leq \frac{xz+yz+(m+n)xy}{2}.$$

Thus it is enough to prove that

$$x\left[xy+xz+(m+n)yz\right] + y\left[xy+yz+(m+n)xz\right] + z\left[xy+yz+(m+n)xz\right] \leq$$

$$\leq \frac{3(m+n)}{4}(x+y)(y+z)(z+x),$$

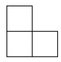
or

$$4\left[A+3(m+n)B\right]\leq 3(m+n)(A+2B)\Leftrightarrow 6(m+n)B\leq\left[3(m+n)-4\right]A,$$

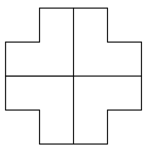
where $A=x^2y+x^2z+xy^2+y^2z+xz^2+yz^2$, $B=xyz$.
Because $m+n\geq 2$ we obtain the inequality $m+n\leq 3(m+n)-4$. From AM-GM inequality it follows that $6B\leq A$. From the last two inequalities we deduce that $6(m+n)B\leq[3(m+n)-4]A$. The inequality is proved.
Equality holds when $m=n=1$ and $x=y=z$.

2.2 Combinatorics

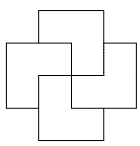
C1 We call a tiling of an $m\times n$ rectangle with corners (see figure below) "regular" if there is no sub-rectangle which is tiled with corners. Prove that if for some m and n there exists a "regular" tiling of the $m\times n$ rectangular then there exists a "regular" tiling also for the $2m\times 2n$ rectangle.



Solution
A corner-shaped tile consists of 3 squares. Let us call "center of the tile" the square that has two neighboring squares. Notice that in a "regular" tiling, the squares situated in the corners of the rectangle have to be covered by the "center" of a tile, otherwise a 2×3 (or 3×2) rectangle tiled with two tiles would form.
Consider a $2m\times 2n$ rectangle, divide it into four $m\times n$ rectangles by drawing its midlines, then do a "regular" tiling for each of these rectangles. In the center of the $2m\times 2n$ rectangle we will necessarily obtain the following configuration:



Now simply change the position of these four tiles into:



It is easy to see that this tiling is "regular".

C2 Consider 50 points in the plane, no three of them belonging to the same line. The points have been colored into four colors. Prove that there are at least 130 scalene triangles whose vertices are colored in the same color.

Solution

Since $50 = 4 \cdot 12 + 2$, according to the pigeonhole principle we will have at least 13 points colored in the same color. We start with the:

Lemma. Given $n > 8$ points in the plane, no three of them collinear, then there are at least $\dfrac{n(n-1)(n-8)}{6}$ scalene triangles with vertices among the given points.

Proof. There are $\dfrac{n(n-1)}{2}$ segments and $\dfrac{n(n-1)(n-2)}{6}$ triangles with vertices among the given points. We shall prove that there are at most $n(n-1)$ isosceles triangles. Indeed, for every segment AB we can construct at most two isosceles triangles (if we have three ABC, ABD and ABE, than C, D, E will be collinear). Hence we have at least

$$\frac{n(n-1)(n-2)}{6} - n(n-1) = \frac{n(n-1)(n-8)}{6} \text{ scalene triangles.}$$

For $n = 13$ we have $\dfrac{13 \cdot 12 \cdot 5}{6} = 130$, QED.

C3 The nonnegative integer n and $(2n+1) \times (2n+1)$ chessboard with squares colored alternatively black and white are given. For every natural number m with $1 < m < 2n+1$, an $m \times m$ square of the given chessboard that has more than half of its area colored in black, is called a B-square. If the given chessboard is a B-square, find in terms of n the total number of B-squares of this chessboard.

Solution

Every square with even side length will have an equal number of black and white 1×1 squares, so it isn't a B-square. In a square with odd side length, there is one more 1×1 black square than white squares, if it has black corner squares. So, a square with odd side length is a B-square either if it is a 1×1 black square or it has black corners.

Let the given $(2n+1) \times (2n+1)$ chessboard be a B-square and denote by b_i ($i = 1, 2, \ldots, n+1$) the lines of the chessboard, which have $n+1$ black 1×1 squares, by w_i ($i = 1, 2, \ldots, n$) the lines of the chessboard, which have n black 1×1 squares and by T_m ($m = 1, 3, 5, \ldots, 2n-1, 2n+1$) the total number of B-squares of dimension $m \times m$ of the given chessboard.

For T_1 we obtain $T_1 = (n+1)(n+1) + n \cdot n = (n+1)^2 + n^2$.

For computing T_3 we observe that there are n 3×3 B-squares, which have the black corners on each pair of lines (b_i, b_{i+1}) for $i = 1, 2, \ldots, n$ and there are $n-1$ 3×3 B-squares, which have the black corners on each pair of lines (w_i, w_{i+1}) for $i = 1, 2, \ldots, n-1$. So, we have

$$T_3 = n \cdot n + (n-1)(n-1) = n^2 + (n-1)^2.$$

By using similar arguments for each pair of lines (b_i, b_{i+2}) for $i = 1, 2, \ldots, n-1$ and for each pair of lines (w_i, w_{i+2}) for $i = 1, 2, \ldots, n-2$ we compute

$$T_5 = (n-1)(n-1) + (n-2)(n-2) = (n-1)^2 + (n-2)^2.$$

Step by step, we obtain

$$T_7 = (n-2)(n-2) + (n-3)(n-3) = (n-2)^2 + (n-3)^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$T_{2n-1} = 2 \cdot 2 + 1 \cdot 1 = 2^2 + 1^2,$$

$$T_{2n+1} = 1 \cdot 1 = 1^2.$$

The total number of B-squares of the given chessboard equals to

$$T_1 + T_3 + T_5 + \ldots + T_{2n+1} = 2(1^2 + 2^2 + \ldots + n^2) + (n+1)^2 =$$

$$\frac{n(n+1)(2n+1)}{3} + (n+1)^2 = \frac{(n+1)(2n^2+4n+3)}{3}.$$

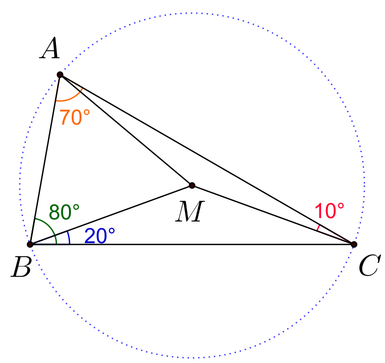
The problem is solved.

2.3 Geometry

G1 Let M be an interior point of the triangle ABC with angles $\sphericalangle BAC = 70^\circ$ and $\sphericalangle ABC = 80^\circ$. If $\sphericalangle ACM = 10^\circ$ and $\sphericalangle CBM = 20^\circ$, prove that $AB = MC$.

Solution

Let O be the circumcenter of the triangle ABC. Because the triangle ABC is acute, O is in the interior of $\Delta\, ABC$. Now we have that $\sphericalangle AOC = 2\sphericalangle ABC = 160^\circ$, so $\sphericalangle ACO = 10^\circ$ and $\sphericalangle BOC = 2\sphericalangle BAC = 140^\circ$, so $\sphericalangle CBO = 20^\circ$. Therefore $O \equiv M$, thus $MA = MB = MC$. Because $\sphericalangle ABO = 80^\circ - 20^\circ = 60^\circ$, the triangle ABM is equilateral and so $AB = MB = MC$.

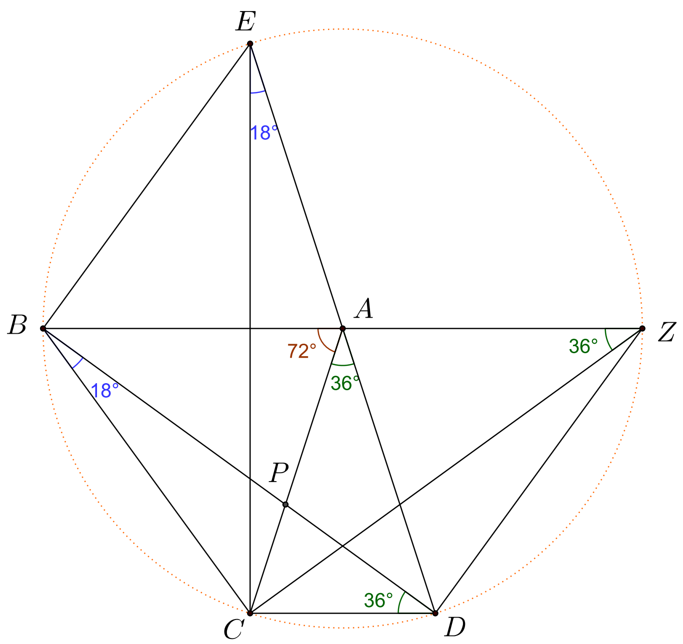


G2 Let $ABCD$ be a convex quadrilateral with $\sphericalangle DAC = \sphericalangle BDC = 36^\circ$, $\sphericalangle CBD = 18^\circ$ and $\sphericalangle BAC = 72^\circ$. If P is the point of intersection of the diagonals AC and BD, find the measure of $\sphericalangle APD$.

Solution

On the rays $(DA$ and $(BA$ we take points E and Z, respectively, such that $AC = AE = AZ$. Since $\sphericalangle DEC = \dfrac{\sphericalangle DAC}{2} = 18^\circ = \sphericalangle CBD$, the quadrilateral $DEBC$ is cyclic.

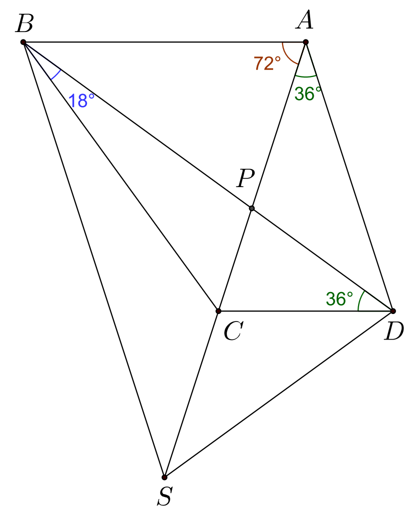
Similarly, the quadrilateral $CBZD$ is cyclic, because $\sphericalangle AZC = \dfrac{\sphericalangle BAC}{2} = 36^\circ = \sphericalangle BDC$. Therefore the pentagon $BCDZE$ is inscribed in the circle $k(A, AC)$. It gives $AC = AD$ and $\sphericalangle ACD = \sphericalangle ADC = \dfrac{180^\circ - 36^\circ}{2} = 72^\circ$, which gives $\sphericalangle ADP = 36^\circ$ and $\sphericalangle APD = 108^\circ$.



Alternative solution. Let X be the intersection point of the angle bisector of $\sphericalangle CAD$ and PD. As $\sphericalangle CAX = \sphericalangle CBX = 18^\circ$, $ABCX$ is cyclic, hence $\sphericalangle BXC = 72^\circ$. It follows that CXD is isosceles. From the SSA criterion for triangles AXC and AXD, it follows that either $\sphericalangle ACX = \sphericalangle ADX$, or $\sphericalangle ACX + \sphericalangle ADX = 180^\circ$. The latter being excluded, it follows that triangles AXC and AXD are congruent. Immediate angle chasing leads to the conclusion.

Alternative solution. Let S be the reflection of D in the line BC. Triangle BDS is isosceles, with $\sphericalangle DBS = 36^\circ$, hence $\sphericalangle SDB = \sphericalangle BSD = 72^\circ$. It follows that $ABSD$ is cyclic ($\sphericalangle BSD + \sphericalangle BAD = 180^\circ$), hence $\sphericalangle BAS = \sphericalangle BDS = 72^\circ$ which means that A, C, S

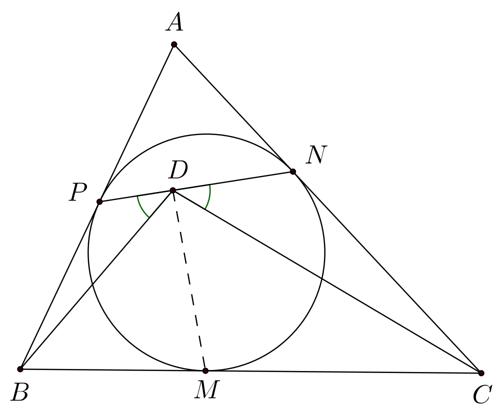
are collinear. C is the incenter of ΔBSD, therefore $\sphericalangle PSB = \sphericalangle PBS = 36°$, which leads to $\sphericalangle DPA = 108°$.



G3 Let the inscribed circle of the triangle $\triangle ABC$ touch side BC at M, side CA at N and side AB at P. Let D be a point from $[NP]$ such that $\frac{DP}{DN} = \frac{BD}{CD}$. Show that $DM \perp PN$.

Solution

From $AP = AN$ it follows that $\sphericalangle ANP = \sphericalangle APN$ or $\sphericalangle NPB = \sphericalangle PNC$ (both obtuse). Hence the triangles BDP and CND are similar (SSA) and $\sphericalangle CDN = \sphericalangle BDP$ and $\dfrac{CD}{BD} = \dfrac{CN}{BP} = \dfrac{CM}{BM}$. So DM is a bisector of the angle BDC, from where $NP \perp MD$.



G4 Let S be a point inside $\sphericalangle pOq$, and let k be a circle which contains S and touches the legs Op and Oq in points P and Q respectively. Straight line s parallel to Op from S intersects Oq in a point R. Let T be the point of intersection of the ray $(PS$ and circumscribed circle of $\triangle SQR$ and $T \neq S$. Prove that $OT \parallel SQ$ and OT is a tangent of the circumscribed circle of $\triangle SQR$.

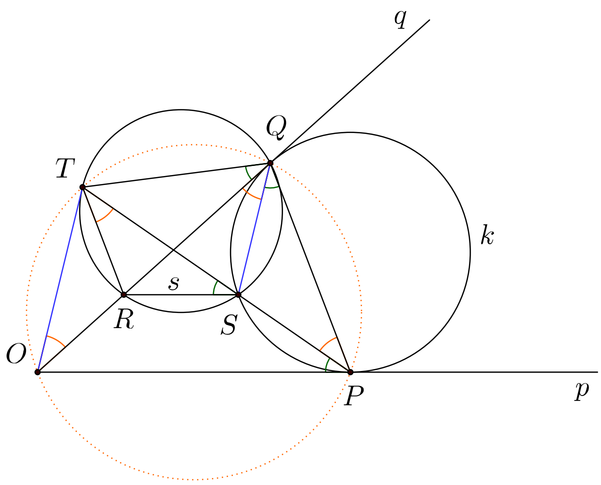
Solution

Let $\sphericalangle OPS = \varphi_1$ and $\sphericalangle OQS = \varphi_2$. We have that $\sphericalangle OPS = \sphericalangle PQS = \varphi_1$ and $\sphericalangle OQS = \sphericalangle QPS = \varphi_2$ (tangents to circle k).

Because $RS \parallel OP$ we have $\sphericalangle OPS = \sphericalangle RST = \varphi_1$ and $\sphericalangle RQT = \sphericalangle RST = \varphi_1$ (cyclic quadrilateral $RSQT$). So, we have as follows $\sphericalangle OPT = \varphi_1 = \sphericalangle RQT = \sphericalangle OQT$, which implies that the quadrilateral $OPQT$ is cyclic. From that we directly obtain $\sphericalangle QOT = \sphericalangle QPT = \varphi_2 = \sphericalangle OQS$, so $OT \parallel SQ$. From the cyclic quadrilateral $OPQT$ by easy calculation we get

$$\sphericalangle OTR = \sphericalangle OTP - \sphericalangle RTS = \sphericalangle OQP - \sphericalangle RQS = (\varphi_1 + \varphi_2) - \varphi_2 = \varphi_1 = \sphericalangle RQT.$$

Thus, OT is a tangent to the circumscribed circle of $\triangle SQR$.



2.4 Number Theory

NT1 Find all the pairs positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{[x, y]} + \frac{1}{(x, y)} = \frac{1}{2},$$

where (x, y) is the greatest common divisor of x, y and $[x, y]$ is the least common multiple of x, y.

Solution
We put $x = du$ and $y = dv$ where $d = (x, y)$. So we have $(u, v) = 1$. From the conclusion we obtain $2(u + 1)(v + 1) = duv$. Because $(v, v + 1) = 1$, v divides $2(u + 1)$.
Case 1. $u = v$. Hence $x = y = [x, y] = (x, y)$, which leads to the solution $x = 8$ and $y = 8$.
Case 2. $u < v$. Then $u + 1 \leq v \Leftrightarrow 2(u + 1) \leq 2v \Leftrightarrow \dfrac{2(u + 1)}{v} \leq 2$, so $\dfrac{2(u + 1)}{v} \in \{1, 2\}$.
But $\dfrac{2(u + 1)}{v} = \dfrac{du}{v + 1}$.
If $\dfrac{2(u + 1)}{v} = 1$ then we have $(d - 2)u = 3$. Therefore $(d, u) = (3, 3)$ or $(d, u) = (5, 1)$ so $(d, u, v) = (3, 3, 8)$ or $(d, u, v) = (5, 1, 4)$.
Thus we get $(x, y) = (9, 24)$ or $(x, y) = (5, 20)$.
If $\dfrac{2(u + 1)}{v} = 2$ we similarly get $(d - 2)u = 4$ from where $(d, u) = (3, 4)$, or $(d, u) = (4, 2)$, or $(d, u) = (6, 1)$. This leads $(x, y) = (12, 15)$ or $(x, y) = (8, 12)$ or $(x, y) = (6, 12)$.
Case 3. $u > v$. Because of the symmetry of u, v and x, y respectively we get exactly the symmetrical solutions of case 2.
Finally the pairs of (x, y) which are solutions of the problem are:
$(8, 8)$, $(9, 24)$, $(24, 9)$, $(5, 20)$, $(20, 5)$, $(12, 15)$, $(15, 12)$, $(8, 12)$, $(12, 8)$, $(6, 12)$, $(12, 6)$.

NT2 Prove that the equation $x^{2006} - 4y^{2006} - 2006 = 4y^{2007} + 2007y$ has no solution in the set of the positive integers.
Solution
We assume the contrary is true. So there are x and y that satisfy the equation. Hence we have

$$x^{2006} = 4y^{2007} + 4y^{2006} + 2007y + 2006$$
$$x^{2006} + 1 = 4y^{2006}(y + 1) + 2007(y + 1)$$
$$x^{2006} + 1 = (4y^{2006} + 2007)(y + 1).$$

But $4y^{2006} + 2007 \equiv 3 \pmod 4$, so $x^{2006} + 1$ will have at least one prime divisor of the type $4k + 3$. It is known (and easily obtainable by using Fermat's Little Theorem) that this is impossible.

NT3 Let $n > 1$ be a positive integer and p a prime number such that $n \mid (p - 1)$ and $p \mid (n^6 - 1)$. Prove that at least one of the numbers $p - n$ and $p + n$ is a perfect square.
Solution
Since $n \mid p - 1$, then $p = 1 + na$, where $a \geq 1$ is an integer. From the condition $p \mid n^6 - 1$, it follows that $p \mid n - 1$, $p \mid n + 1$, $p \mid n^2 + n + 1$ or $p \mid n^2 - n + 1$.
- Let $p \mid n - 1$. Then $n \geq p + 1 > n$ which is impossible.
- Let $p \mid n + 1$. Then $n + 1 \geq p = 1 + na$ which is possible only when $a = 1$ and $p = n + 1$, i.e. $p - n = 1 = 1^2$.
- Let $p \mid n^2 + n + 1$, i.e. $n^2 + n + 1 = pb$, where $b \geq 1$ is an integer.

The equality $p = 1 + na$ implies $n \mid b - 1$, from where $b = 1 + nc$, $c \geq 0$ is an integer. We have

$$n^2 + n + 1 = pb = (1 + na)(1 + nc) = 1 + (a + c)n + acn^2 \text{ or } n + 1 = acn + a + c.$$

If $ac \geq 1$ then $a + c \geq 2$, which is impossible. If $ac = 0$ then $c = 0$ and $a = n + 1$. Thus we obtain $p = n^2 + n + 1$ from where $p + n = n^2 + 2n + 1 = (n + 1)^2$.

• Let $p \mid n^2 - n + 1$, i.e. $n^2 - n + 1 = pb$ and analogously $b = 1 + nc$. So

$$n^2 - n + 1 = pb = (1 + na)(1 + nc) = 1 + (a + c)n + acn^2 \text{ or } n - 1 = acn + a + c.$$

Similarly, we have $c = 0$, $a = n - 1$ and $p = n^2 - n + 1$ from where $p - n = n^2 - 2n + 1 = (n - 1)^2$.

NT4 Let a, b be two co-prime positive integers. A number is called **good** if it can be written in the form $ax + by$ for non-negative integers x, y. Define the function $f : \mathbb{Z} \to \mathbb{Z}$ as $f(n) = n - n_a - n_b$, where s_t represents the remainder of s upon division by t. Show that an integer n is **good** if and only if the infinite sequence $n, f(n), f(f(n)), \ldots$ contains only non-negative integers.

Solution

If n is good then $n = ax + by$ also $n_a = (by)_a$ and $n_b = (ax)_b$ so

$$f(n) = ax - (ax)_b + by - (by)_a = by' + ax'$$

is also good, thus the sequence contains only good numbers which are non-negative.

Now we have to prove that if the sequence contains only non-negative integers then n is good. Because the sequence is non-increasing then the sequence will become constant from some point onwards. But $f(k) = k$ implies that k is a multiple of ab thus some term of the sequence is good. We are done if we prove the following:

Lemma: $f(n)$ is good implies n is good.

Proof of Lemma: $n = 2n - n_a - n_b - f(n) = ax' + by' - ax - by = a(x' - x) + b(y' - y)$ and $x' \geq x$ because $n \geq f(n) \Rightarrow n - n_a \geq f(n) - f(n)_a \Rightarrow ax' \geq ax + by - (by)_a \geq ax$. Similarly $y' \geq y$.

NT5 Let p be a prime number. Show that $7p + 3^p - 4$ is not a perfect square.

Solution

Assume that for a prime number p greater than 3, $m = 7p + 3^p - 4$ is a perfect square. Let $m = n^2$ for some $n \in \mathbb{Z}$. By Fermat's Little Theorem,

$$m = 7p + 3^p - 4 \equiv 3 - 4 \equiv -1 \pmod{p}.$$

If $p = 4k + 3$, $k \in \mathbb{Z}$, then again by Fermat's Little Theorem

$$-1 \equiv m^{2k+1} \equiv n^{4k+2} \equiv n^{p-1} \equiv 1 \pmod{p}, \text{ but } p > 3,$$

a contradiction. So $p \equiv 1 \pmod{4}$.

Therefore $m = 7p + 3^p - 4 \equiv 3 - 1 \equiv 2 \pmod{4}$. But this is a contradiction since 2 is not perfect square in $\pmod{4}$. For $p = 2$ we have $m = 19$ and for $p = 3$ we have $m = 44$.