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SHORT LIST PROBLEM WITH SOLUTION

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PROBLEM SHORTLIST

(with solutions)

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ALGEBRA

A1

Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{a^5+b^5+c^2}+\frac{1}{b^5+c^5+a^2}+\frac{1}{c^5+a^5+b^2}\le 1.$$

Solution

First we remark that

$$a^5+b^5\ge ab(a^3+b^3).$$

Indeed

$$a^5+b^5\ge ab(a^3+b^3) \Leftrightarrow a^5-a^4b-ab^4+b^5\ge 0 \Leftrightarrow a^4(a-b)-b^4(a-b)\ge 0$$
$$\Leftrightarrow (a-b)(a^4-b^4)\ge 0 \Leftrightarrow (a-b)^2(a^2+b^2)(a+b)\ge 0.$$

We rewrite the inequality as

$$\frac{1}{a^5+b^5+abc^3}+\frac{1}{b^5+c^5+bca^3}+\frac{1}{c^5+a^5+cab^3}\le 1.$$

On the other hand the following inequality is true

$$a^5+b^5+abc^3\ge ab(a^3+b^3+c^3),$$

and similar for the other two.

Finally, using AM-GM we get:

$$\frac{1}{a^5+b^5+c^2}+\frac{1}{b^5+c^5+a^2}+\frac{1}{c^5+a^5+b^2}\le\frac{1}{a^3+b^3+c^3}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=\frac{a+b+c}{a^3+b^3+c^3}$$
$$\le\frac{a+b+c}{\frac{(a+b+c)^3}{9}}=\frac{9}{(a+b+c)^2}\le\frac{9}{(3\sqrt[3]{abc})^2}=1.$$

A2

Consider the sequence of rational numbers defned by $x_1 = \frac{4}{3}$ and $x_{n+1} = \frac{x_n^2}{x_n^2 - x_n + 1}$, $n \geq 1$.

Show that the numerator of the lowest term expression of each sum $\sum_{k=1}^{n} x_k$ is a perfect square.

Solution

It is easily seen that the x_n are all rational numbers greater than 1. Rewrite the recurrence formula in the form $x_n = \frac{1}{x_{n+1}-1} - \frac{1}{x_n - 1}$, $n \geq 1$, to get

$$\sum_{k=1}^{n} x_k = \frac{1}{x_{n+1}-1} - \frac{1}{x_1 - 1} = \frac{x_n^2 - x_n + 1}{x_n - 1} - 3 = \frac{(x_n - 2)^2}{x_n - 1}.$$

Finally, express the positive rational number $x_n - 1$ in lowest terms, $x_n - 1 = \frac{a}{b}$, to deduce that $\frac{(a-b)^2}{ab}$ expresses $\sum_{k=1}^{n} x_k$ the lowest terms.

Since $\gcd(a,b) = 1 \Rightarrow \gcd(a-b,b) = 1 \Rightarrow \gcd((a-b)^2,b) = 1$. Similarly we can prove that $\gcd((a-b)^2,a) = 1$, which implies that $\gcd((a-b)^2,ab) = 1$.

The conclusion follows.

A3

Find all the functions $f:\mathbb{N}\to\mathbb{N}$ such that:

$$n+f(m)\mid f(n)+nf(m) \qquad (1)$$

for any $m,n\in\mathbb{N}$

Solution

We will consider 2 cases, whether the range of the functions is infinite or finite or in other words the function take infinite or finite values.

Case 1. The Function has an infinite range. Let's fix a random natural number n and let m be any natural number. Then using (1) we have

$$n+f(m)\mid f(n)+nf(m)=f(n)-n^2+n(f(m)+n) \quad \Rightarrow \quad n+f(m)\mid f(n)-n^2.$$

Since n is a fixed natural number, then $f(n)-n^2$ is as well a fixed natural number, and since the above results is true for any m and the function f has an infinite range, we can choose m such that $n+f(m)>|f(n)-n^2|$. This implies that $f(n)=n^2$ for any natural number n. We now check that it is a solution. Since

$$n+f(m)=n+m^2$$

and

$$f(n)+nf(m)=n^2+nm^2=n(n+m^2)$$

it is straightforward that $n+f(m)\mid f(n)+nf(m)$

Case 2. The Function has a finite range. Since the function takes finite values, then it exists a natural number k such that $1\le f(n)\le k$ for any natural number n. It is clear that it exists at least one natural number s (where $1\le s\le k$) such that $f(n)=s$ for infinite natural numbers n. Let m,n be any natural numbers such that $f(m)=f(n)=s$. Using (1) we have

$$n+s\mid s+ns=s-s^2+s(n+s) \quad \Rightarrow \quad n+s\mid s^2-s.$$

Since this is true for any natural number n such that $f(n)=s$ and since exist infinite natural numbers n such that $f(n)=s$, we can choose the natural number n such that $n+s>s^2-s$, which implies that $s^2=s\Rightarrow s=1$, or in other words $f(n)=1$ for an infinite natural number n

Let's fix a random natural number m and let n be any natural number $f(n)=1$. Then using (1) we have

$$n+f(m)\mid 1+nf(m)=1-(f(m))^2+f(m)(n+f(m)) \quad \Rightarrow \quad n+f(m)\mid (f(m))^2-1$$

Since m is a fixed a random natural number, then $(f(m))^2-1$ is a fixed non-negative integer and since n is any natural nummber such that $f(n)=1$ and since exist infinite numbers n such that $f(n)=1$, we can choose the the natural number n such that $n+f(m)>(f(m))^2-1$. This implies $f(m)=1$ for any natural number m. We now check that it is a solution. Since

$$n+f(m)=n+1$$

and

$$f(n)+nf(m)=1+n$$

it is straightforward that $n+f(m)\mid f(n)+nf(m)$.

So, all the functions that satisfy the given condition are $f(n)=n^2$ for any $n\in\mathbb{N}$ or $f(n)=1$ for any $n\in\mathbb{N}$.

A4

Let $M=\{(a,b,c)\in\mathbb{R}^3 : 0<a,b,c<\frac{1}{2} \text{ with } a+b+c=1\}$ and $f:M\to\mathbb{R}$ given as

$$f(a,b,c)=4\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{1}{abc}$$

Find the best (real) bounds α and β such that

$$f(M)=\{f(a,b,c):(a,b,c)\in M\}\subseteq[\alpha,\beta]$$

and determine whether any of them is achievable.

Solution

Let $\forall(a,b,c)\in M$, $\alpha\le f(a,b,c)\le\beta$ and supose that there are no better bounds, i.e. α is the largest possible and β is the smallest possible. Now,

$$\begin{aligned}\alpha\le f(a,b,c)\le\beta &\Leftrightarrow \alpha abc\le 4(ab+bc+ca)-1\le\beta abc\\ &\Leftrightarrow (\alpha-8)abc\le 4(ab+bc+ca)-8abc-1\le(\beta-8)abc\\ &\Leftrightarrow (\alpha-8)abc\le 1-2(a+b+c)+4(ab+bc+ca)-8abc\le(\beta-8)abc\\ &\Leftrightarrow (\alpha-8)abc\le(1-2a)(1-2b)(1-2c)\le(\beta-8)abc\end{aligned}$$

For $\alpha<8$, we have

$$(1-2a)(1-2b)(1-2c)\ge 0>(\alpha-8)abc.$$

So $\alpha\ge 8$. But if we take $\varepsilon>0$ small and $a=b=\frac{1}{4}+\varepsilon$, $c=\frac{1}{2}-2\varepsilon$, we'll have:

$$(\alpha-8)\left(\tfrac{1}{4}+\varepsilon\right)\left(\tfrac{1}{4}+\varepsilon\right)\left(\tfrac{1}{2}-2\varepsilon\right)\le\left(\tfrac{1}{2}-2\varepsilon\right)\left(\tfrac{1}{2}-2\varepsilon\right)4\varepsilon$$

Taking $\varepsilon\to 0^+$, we get $\alpha-8\le 0$. So $\alpha=8$ and it can never be achieved. For the right side, note that there is a triangle whose side-lengths are a,b,c. For this triangle, denote $p=\frac{1}{2}$ the half-perimeter, S the area and r,R respectively the radius of incircle,outcircle. Using the relations $R=\frac{abc}{4S}$ and $S=pr$, we will have:

$$\begin{aligned}(1-2a)(1-2b)(1-2c)\le(\beta-8)abc &\Leftrightarrow (p-a)(p-b)(p-c)\le\frac{(\beta-8)abc}{8}\\ &\Leftrightarrow \frac{S^2}{p}\le\frac{(\beta-8)abc}{8}\\ &\Leftrightarrow 2\frac{S}{p}\le\frac{(\beta-8)abc}{4S}\\ &\Leftrightarrow \frac{R}{r}\ge 2(\beta-8)^{-1}\end{aligned}$$

Since the least value of $\frac{R}{r}$ is 2 (this is a well-known classic inequality), and it is achievable at $a=b=c=\frac{1}{3}$, we must have $\beta=9$.

Answer: $\alpha=8$ not achievable and $\beta=9$ achievable.

A5

Consider integers $m \geq 2$ and $n \geq 1$. Show that there is a polynomial $P(x)$ of degree equal to n with integer coefficients such that $P(0), P(1), \ldots, P(n)$ are all perfect powers of m.

Solution

Let $a_0, a_1, \ldots, a_n$ be integers to be chosen later, and consider the polynomial $P(x) = \frac{1}{n!}Q(x)$ where

$$Q(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k \prod_{\substack{0 \leq i \leq n \\ i \neq k}} (x-i).$$

Observe that for $l \in \{0, 1, \ldots, n\}$ we have

$$\begin{aligned} P(l) &= \frac{1}{n!}(-1)^{n-l}\binom{n}{l} a_l \prod_{\substack{0 \leq i \leq n \\ i \neq l}} (l-i) \\ &= \frac{1}{n!}(-1)^{n-l}\binom{n}{l} a_l l!(-1)^{n-l}(n-l)! \\ &= a_l \end{aligned}$$

So $P(x)$ is the unique polynomial of degree at most n such that $P(l) = a_l$. (Any two **polynomials of degree** at most n agreeing on $n+1$ distinct values are equal.) Note in **particular that**

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \prod_{\substack{0 \leq i \leq n \\ i \neq k}} (x-i) = n! \qquad (*)$$

If p is a prime dividing $n!$, we let r_p be maximal such that p^{r_p} divides $n!$. If p divides m, then there is an integer d_p such that $m^{d_p} \equiv 0 \pmod{p^{r_p}}$, for example $d_p = r_p$ will do. If p does not divide m, then there is an integer d_p such that $m^{d_p} \equiv 1 \pmod{p^{r_p}}$, for example, by Euler's theorem, $d_p = \varphi(p^{r_p})$ will do. Let $d = d_1 d_2 \ldots d_p$ and observe that for every positive integer t we have $m^{td} \equiv 0 \pmod{p^{r_p}}$ if $p \mid m$ and $m^{td} \equiv 1 \pmod{p^{r_p}}$ if $p \nmid m$.

Now let $t_0, \ldots, t_n$ be positive integers to be chosen later and define $a_k = m^{t_k d}$. We will show that the polynomial $P(x)$ has integer coefficients. We will also show that there is an appropriate choice of $t_0, \ldots, t_n$ such that $P(x)$ has degree exactly equal to n.

To show that $P(x)$ has integer coefficients it is enough to show that for every p dividing $n!$, all coefficients of $Q(x)$ are multiples of p^{r_p}. This is immediate if p divides m as all a_k's are multiples of p^{r_p}. If p does not divide m then we have $a_k \equiv 1 \pmod{p^{r_p}}$ for every $0 \leq k \leq n$ and so by (*)

$$Q(x) \equiv \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \prod_{\substack{0 \le i \le n \\ i \ne k}} (x-i) \equiv n! \equiv 0 \pmod{p^{r_p}}.$$

This shows that all coefficients of $Q(x)$ are indeed multiples of p^{r_p}. It remains to show that there is a choice of $t_0, \dots, t_n$ guaranteeing that the degree of $P(x)$ is exactly equal to n. One such choice is $t_0 = 2$ and $t_1 = \dots = t_n = 1$. This works because if $P(x)$ had degree less than n, then looking at the values $P(1), \dots, P(n)$ we would get that $P(x)$ is constant. But this is impossible as $P(0) \ne P(1)$.

Note. Looking at the coefficient of x^n in the definition of $P(x)$ it is not difficult to see that if we fix any $t_1, \dots, t_n$ and pick t_0 large enough we will get that this coefficient is non-zero. In particular, we can additionally guarantee that $P(0), P(1), \dots, P(n)$ are distinct perfect powers of m.

A6

Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$f(x+yf(x^2))=f(x)+xf(xy)$$

for all real numbers x and y.

Solution

Let $P(x,y)$ be the assertion $f(x+yf(x^2))=f(x)+xf(xy)$. $P(1,0)$ yields $f(0)=0$. If there exists $x_0\neq 0$ satisfying $f(x_0^2)=0$, then considering $P(x_0,y)$, we get $f(x_0y)=0$ for all $y\in\mathbb{R}$. In this case, since $x_0\neq 0$, we can write any real number c in the form x_0y for some y and hence we conclude that $f(c)=0$. It is clear that the zero function satisfies the given equation. Now assume that $f(x^2)\neq 0$ for all $x\neq 0$.

By $P(1,y)$ we have

$$f(1+yf(1))=f(1)+f(y).$$

If $f(1)\neq 1$, there exists a real number y satisfying $1+yf(1)=y$ which means that $f(1)=0$ which is a contradiction since $f(x^2)\neq 0$ for all $x\neq 0$. Therefore we get $f(1)=1$. Considering $P(x,-x/f(x^2))$ for $x\neq 0$, we obtain that

$$f(x)=-xf\left(-\frac{x^2}{f(x^2)}\right)\ \ \forall x\in\mathbb{R}\setminus\{0\}. \tag{1}$$

Replacing x by $-x$ in (1), we obtain $f(x)=-f(-x)$ for all $x\neq 0$. Since $f(0)=0$, we have $f(x)=-f(-x)$ for all $x\in\mathbb{R}$. Since f is odd, $P(x,-y)$ implies

$$f(x-yf(x^2))=f(x)-xf(xy)$$

and hence by adding $P(x,y)$ and $P(x,-y)$, we get

$$f(x+yf(x^2))+f(x-yf(x^2))=2f(x)\ \ \forall x,y\in\mathbb{R}$$

Putting $y=x/f(x^2)$ for $x\neq 0$ we get

$$f(2x)=2f(x)\ \ \forall x\in\mathbb{R}$$

and hence we have

$$f(x+yf(x^2))+f(x-yf(x^2))=f(2x)\ \ \forall x,y\in\mathbb{R}$$

It is clear that for any two real numbers u and v with $u\neq -v$, we can choose

$$x=\frac{u+v}{2}\ \text{ and }\ y=\frac{v-u}{2f\left(\left(\frac{u+v}{2}\right)^2\right)}$$

yielding

$$f(u)+f(v)=f(u+v). \tag{2}$$

Since f is odd, (2) is also true for $u=-v$ and hence we obtain that

$$f(x)+f(y)=f(x+y)\ \ \forall x,y\in\mathbb{R}.$$

Therefore, $P(x,y)$ implies

$$f(yf(x^2)) = xf(xy) \ \ \forall x, y \in \mathbb{R}.$$

Hence we have

$$f(f(x^2)) = xf(x) \ \ \forall x \in \mathbb{R} \tag{3}$$

and

$$f(xf(x^2)) = xf(x^2) \ \ \forall x \in \mathbb{R}. \tag{4}$$

Using (2) and (3), we get

$$\begin{aligned} xf(x) + yf(y) + xf(y) + yf(x) &= (x+y)(f(x)+f(y)) \\ &= f(f((x+y)^2)) \\ &= f(f(x^2+2xy+y^2)) \\ &= f\left(f(x^2)+f(y^2)+f(2xy)\right) \\ &= f(f(x^2)) + f(f(y^2)) + f(f(2xy)) \\ &= xf(x) + y(f(y)) + f(f(2xy)) \\ &= xf(x) + y(f(y)) + 2f(f(xy)) \end{aligned}$$

and hence

$$2f(f(xy)) = xf(y) + yf(x) \ \ \forall x, y \in \mathbb{R}. \tag{5}$$

Using (5), we have

$$2f(f(x)) = x + f(x) \ \ \forall x \in \mathbb{R}. \tag{6}$$

Using (3) and (6), we obtain that

$$2xf(x) = 2f(f(x^2)) = x^2 + f(x^2) \ \ \forall x \in \mathbb{R}. \tag{7}$$

Putting $y = f(x^2)$ in (5) yields

$$2f(f(xf(x^2))) = xf(f(x^2)) + f(x^2)f(x) \ \ \forall x \in \mathbb{R} \tag{8}$$

Using (4), we get $f(f(xf(x^2))) = xf(x^2)$ and by (3) and (8) we have

$$2xf(x^2) = 2f(f(xf(x^2))) = xf(f(x^2)) + f(x^2)f(x) = x^2 f(x) + f(x^2)f(x) \ \ \forall x \in \mathbb{R}. \tag{9}$$

Now using (7), write $f(x^2) = 2xf(x) - x^2$ in (9) to obtain that

$$2x(2xf(x) - x^2) = x^2 f(x) + (2xf(x) - x^2)f(x)$$

which is equivalent to

$$2x(x - f(x))^2 = 0 \ \ \forall x \in \mathbb{R}.$$

This shows that $f(x) = x \ \forall x \in \mathbb{R}$ which satisfies the original equation. Therefore all solutions are $f(x) = 0 \ \forall x \in \mathbb{R}$ and $f(x) = x \ \forall x \in \mathbb{R}$.

NUMBER THEORY

NT1

Find all pairs (x, y) of positive integers such that

$$x^3 + y^3 = x^2 + 42xy + y^2.$$

Solution

Let $d = (x, y)$ be the greatest common divisor of positive integers x and y.

So, $x = ad$, $y = bd$, where $d \in \mathbb{N}$, $(a,b) = 1$, $a,b \in \mathbb{N}$. We have

$$\begin{aligned} x^3 + y^3 = x^2 + 42xy + y^2 \quad &\Leftrightarrow \quad d^3(a^3 + b^3) = d^2(a^2 + 42ab + b^2) \\ &\Leftrightarrow \quad d(a+b)(a^2 - ab + b^2) = a^2 + 42ab + b^2 \\ &\Leftrightarrow \quad (da + db - 1)(a^2 - ab + b^2) = 43ab. \end{aligned}$$

If we denote $c = da + db - 1 \in \mathbb{N}$, then the equality $a^2c - abc + b^2c = 43ab$ implies the relations

$$\left.\begin{aligned} b|ca^2 \;\Rightarrow\; b|c \\ a|cb^2 \;\Rightarrow\; a|c \end{aligned}\right\} \;\Rightarrow\; (ab)|c$$

$$\begin{aligned} &\Leftrightarrow \quad c = mab, \; m \in \mathbb{N}^+ \\ &\Rightarrow \quad m(a^2 - ab + b^2) = 43 \\ &\Rightarrow \quad (a^2 - ab + b^2) | 43 \\ &\Leftrightarrow \quad a^2 - ab + b^2 = 1 \quad \text{or} \quad a^2 - ab + b^2 = 43. \end{aligned}$$

If $a^2 - ab + b^2 = 1$, then $(a-b)^2 = 1 - ab \geq 0 \Rightarrow a = b = 1$, $2d = 44$, $(x, y) = (22, 22)$.

If $a^2 - ab + b^2 = 43$, then, by virtue of simmetry, we suppose that $x \geq y \Rightarrow a \geq b$. We obtain that $43 = a^2 - ab + b^2 \geq ab \geq b^2 \Rightarrow b \in \{1,2,3,4,5,6\}$.

If $b = 1$, then $a = 7$, $d = 1$, $(x, y) = (7,1)$ or $(x, y) = (1,7)$.

If $b = 6$, then $a = 7$, $d = \frac{43}{13} \notin \mathbb{N}$.

For $b \in \{2,3,4,5\}$ there no positive integer solutions for a.

Finally, we have $(x, y) \in \{(1,7), (7,1), (22,22)\}$.

NT2

Find all functions $f:\mathbb{Z}_{>0}\to\mathbb{Z}_{>0}$ such that the number $xf(x)+f^2(y)+2xf(y)$ is a perfect square for all positive integers x, y.

Solution

Let p be a prime number. Then for $x=y=p$ the given condition gives us that the number $f^2(p)+3pf(p)$ is a perfect square. Then, $f^2(p)+3pf(p)=k^2$ for some positive integer k. Completing the square gives us that $(2f(p)+3p)^2-9p^2=4k^2$, or

$$(2f(p)+3p-2k)(2f(p)+3p+2k)=9p^2. \tag{1}$$

Since $2f(p)+3p+3k>3p$, we have the following 4 cases.

$$\begin{cases}2f(p)+3p+2k=9p\\2f(p)+3p-2k=p\end{cases} \text{ or } \begin{cases}2f(p)+3p+2k=p^2\\2f(p)+3p-2k=9\end{cases} \text{ or}$$

$$\begin{cases}2f(p)+3p+2k=3p^2\\2f(p)+3p-2k=3\end{cases} \text{ or } \begin{cases}2f(p)+3p+2k=9p^2\\2f(p)+3p-2k=1\end{cases}$$

Solving the systems, we have the following cases for $f(p)$.

$$f(p)=p \text{ or } f(p)=(\tfrac{p-3}{2})^2 \text{ or } f(p)=\frac{3p^2-6p-3}{4} \text{ or } f(p)=(\tfrac{3p-1}{2})^2.$$

In all cases, we see that $f(p)$ can be arbitrary large whenever p grows.

Now fix a positive integer x. From the given condition we have that

$$(f(y)+x)^2+xf(x)-x^2$$

is a perfect square. Since for y being a prime, let $y=q$, $f(q)$ can be arbitrary large and $xf(x)-x^2$ is fixed, it means that $xf(x)-x^2$ should be zero, since the difference of $(f(q)+x+1)^2$ and $(f(q)+x)^2$ can be arbitrary large.

After all, we conclude that $xf(x)=x^2$, so $f(x)=x$, which clearly satisfies the given condition.

NT3

Prove that for all positive integer n, there is a positive integer m, that $7^n \mid 3^m + 5^m - 1$.

Solution

We prove this by induction on n. The case $n=1$ is indeed trivial for $m=1$. Assume that the statement of the problem holds true for n, and we have $3^m + 5^m - 1 = 7^n l$ for some positive integer l which is not divisible by 7 (if not we are done). Since $3^6 \equiv 1 \pmod 7$ and $5^6 \equiv 1 \pmod 7$ we conclude that,

$$3^{6\cdot 7^{n-1}} \equiv 1 (\bmod 7^n), 5^{6\cdot 7^{n-1}} \equiv 1 (\bmod 7^n).$$

Since

$$v_7(3^{6.7^{n-1}} - 1) = v_7(3^6 - 1) + v_7(7^{n-1}) = n \text{ and } v_7(5^{6.7^{n-1}} - 1) = v_7(5^6 - 1) + v_7(7^{n-1}) = n.$$

Thus we can say that: $3^{6.7^{n-1}} = 1 + 7^n r, 5^{6.7^{n-1}} = 1 + 7^n s$ for some positive integers r, s. We find the reminder of r, s module 7. Note that:

$$\frac{y^{7^k} - 1}{7^{k+1}} = \frac{y-1}{7} \cdot \frac{1+y+..+y^6}{7} \cdot \ldots \cdot \frac{1+y^{7^{k-1}} + \ldots + y^{6.7^{k-1}}}{7} \qquad (*)$$

We use the above identity for $y = 3^6, 5^6$. Note that in both cases $y \equiv 1 (\bmod 7)$. Now we use the following lemma

Lemma. Let p be an odd prime such that $p \mid a-1$ then $\frac{a^p - 1}{a-1} \equiv p \ (\bmod p^2)$.

Proof. Take $a - 1 = b$, then $p \mid b$ now

$$\frac{a^p - 1}{a-1} = \frac{(b+1)^p - 1}{b} = b^{p-1} + \ldots + \binom{p}{2} b + p \equiv p \ (\bmod p^2)$$

since all the binomial coefficients is divisible by p. So our proof is complete. ■

Then by use of the lemma repeatedly we find that all the terms of the above identity (*) except the first term is congruent to 1 modulo 7. Thus we can find that :

$$\frac{y^{7^k} - 1}{7^{k+1}} \equiv \frac{y-1}{7} (\bmod 7)$$

Since

$$\frac{3^6 - 1}{7} = 104 \equiv -1, \frac{5^6 - 1}{7} \equiv 2232 \equiv -1 (\bmod 7)$$

we find that $r \equiv s \equiv -1 (\bmod 7)$, and by use of binomial theorem, we can easily find that

$$3^{6t.7^{n-1}} \equiv 1 + 7^n rt \ (\bmod 7^{n+1}), \quad 5^{6t.7^{n-1}} = 1 + 7^n st (\bmod 7^{n+1})$$

for all positive integers t.

Now take $m + 6t \cdot 7^{n-1}$ instead of m, (while we will specify the number t later) we can find that:

$$3^{m+6t\cdot 7^{n-1}} + 5^{m+6t\cdot 7^{n-1}} - 1 = 3^m \cdot 3^{6t\cdot 7^{n-1}} + 5^m \cdot 5^{6t\cdot 7^{n-1}} - 1$$

Taking modulo 7^{n+1} we can find that, the above expression is reduced to

$$3^m(1+7^n rt)+5^m(1+7^n st)-1 \equiv 3^m+5^m-1+5^m\cdot 7^n st+3^m\cdot 7^n rt$$

$$\equiv 7^n(l+(5^m s+3^m r)t) \pmod{7^{n+1}}$$

Now, the problem reduced to finding a positive integer t such that

$$l+(5^m s+3^m.r)t \equiv 0 \pmod 7$$

since

$$5^m s+3^m r \equiv -5^m-3^m \equiv 1 \pmod 7.$$

Since $3^m+5^m-1\equiv 0 \pmod 7$ whence, we find that $\gcd(5^m s+3^m r,7)=1$. Thus such integer t exists, so we are done!

NT4

Find all pairs of positive integers (x, y), such that x^2 is divisible by $2xy^2 - y^3 + 1$.

Solution

If $y = 1$, then $2x \mid x^2 \Leftrightarrow x = 2n,\ n \in \mathbb{N}$. So, the pairs $(x, y) = (2n, 1),\ n \in \mathbb{N}$ satisfy the required divizibility.

Let $y > 1$ such, that x^2 is divisible by $2xy^2 - y^3 + 1$. There exist $m \in \mathbb{N}$ such that

$$x^2 = m(2xy^2 - y^3 + 1)\text{, e.t. } x^2 - 2my^2x + (my^3 - m) = 0.$$

The discriminant of last quadratic equation is equal to $\Delta = 4m^2y^4 - 4my^3 + 4m$. Denote

$$A = 4m(y^2 - 1) + (y - 1)^2, \quad B = 4m(y^2 + 1) - (y + 1)^2.$$

For $y > 1,\ y \in \mathbb{N}$ and $m \in \mathbb{N}$ we have

$$A > 0,\ B = 4m(y^2 + 1) - (y + 1)^2 > 2(y^2 + 1) - (y + 1)^2 = (y - 1)^2 \geq 0 \Rightarrow B > 0.$$

We obtain the following estimations for the discriminant Δ:

$$\Delta + A = (2my^2 - y + 1)^2 \geq 0 \quad \Rightarrow \quad \Delta < (2my^2 - y + 1)^2;$$

$$\Delta - B = (2my^2 - y - 1)^2 \geq 0 \quad \Rightarrow \quad \Delta > (2my^2 - y - 1)^2.$$

Because the discriminant Δ must be a perfect square, we obtain the equalities:

$$\Delta = 4m^2y^4 - 4my^3 + 4m = (2my^2 - y)^2 \quad \Leftrightarrow \quad y^2 = 4m \quad \Rightarrow \quad y = 2k,\ k \in \mathbb{N},\ m = k^2,\ k \in \mathbb{N}.$$

The equation $x^2 - 8k^4x + k(8k^4 - k) = 0$ has the solutions $x = k$ and $x = 8k^4 - k$, where $k \in \mathbb{N}$.

Finaly, we obtain that all pairs of positive integers (x, y), such that x^2 is divisible by $2xy^2 - y^3 + 1$, are equal to $(x, y) \in \{(2k, 1), (k, 2k), (8k^4 - k, 2k) \mid k \in \mathbb{N}\}$.

NT5

Given a positive odd integer n, show that the arithmetic mean of fractional parts $\{\frac{k^{2n}}{p}\}$, $k=1,\ldots,\frac{p-1}{2}$, is the same for infinitely many primes p.

Solution

We show that the arithmetic mean in question is $\frac{1}{2}$ for infinitely many primes congruent to 1 modulo 4.

Notice that $\{\frac{k^{2n}}{p}\}=\frac{r_k}{p}$, where r_k is the remainder k^{2n} leaves upon division by p. Clearly, the r_k are quadratic residues modulo p.

If p is prime, and $p-1$ and n are relatively prime, then the r_k, $k=1,\ldots,\frac{p-1}{2}$, are pairwise distinct, since the k^{2n}, $k=1,\ldots,\frac{p-1}{2}$, are pairwise distinct modulo p, by Fermat's little theorem. In this case, the r_k, $k=1,\ldots,\frac{p-1}{2}$, form the set R of all $\frac{p-1}{2}$ quadratic residues modulo p in the range 1 through $p-1$.

If, in addition, p is congruent to 1 modulo 4, then -1 is a quadratic residue modulo p, and the assignment $r\mapsto p-r, r\in R$, defines a permutation of R. In this case,

$$\sum_{r\in R} r=\sum_{r\in R}(p-r)=\frac{p(p-1)}{2}-\sum_{r\in R} r,$$

so

$$\sum_{r\in R} r=\frac{p(p-1)}{4},$$

and the arithmetic mean in question is $\frac{1}{2}$.

Finally, since n is odd, infinitely many primes congruent to 1 modulo 4 are also congruent to 2 modulo n, by Dirichlet's theorem on arithmetic sequences of integers; for such a prime p, the numbers $p-1$ and n are clearly relatively prime. This completes the proof.

GEOMETRY

G1

Let ABC be an acute triangle. Variable points E and F are on sides AC and AB respectively such that $BC^2 = BA \cdot BF + CE \cdot CA$. As E and F vary prove that the circumcircle of AEF passes through a fixed point other than A.

Solution 1

Let H be the ortocenter of ABC and K,L,M be the feet of perpendiculars respectively from A,B,C to their opposite sides of ABC. Also let D be the intersection point of lines BE and CF. From power of point we have

$$BA \cdot BM = BC \cdot BK \tag{1}$$

and

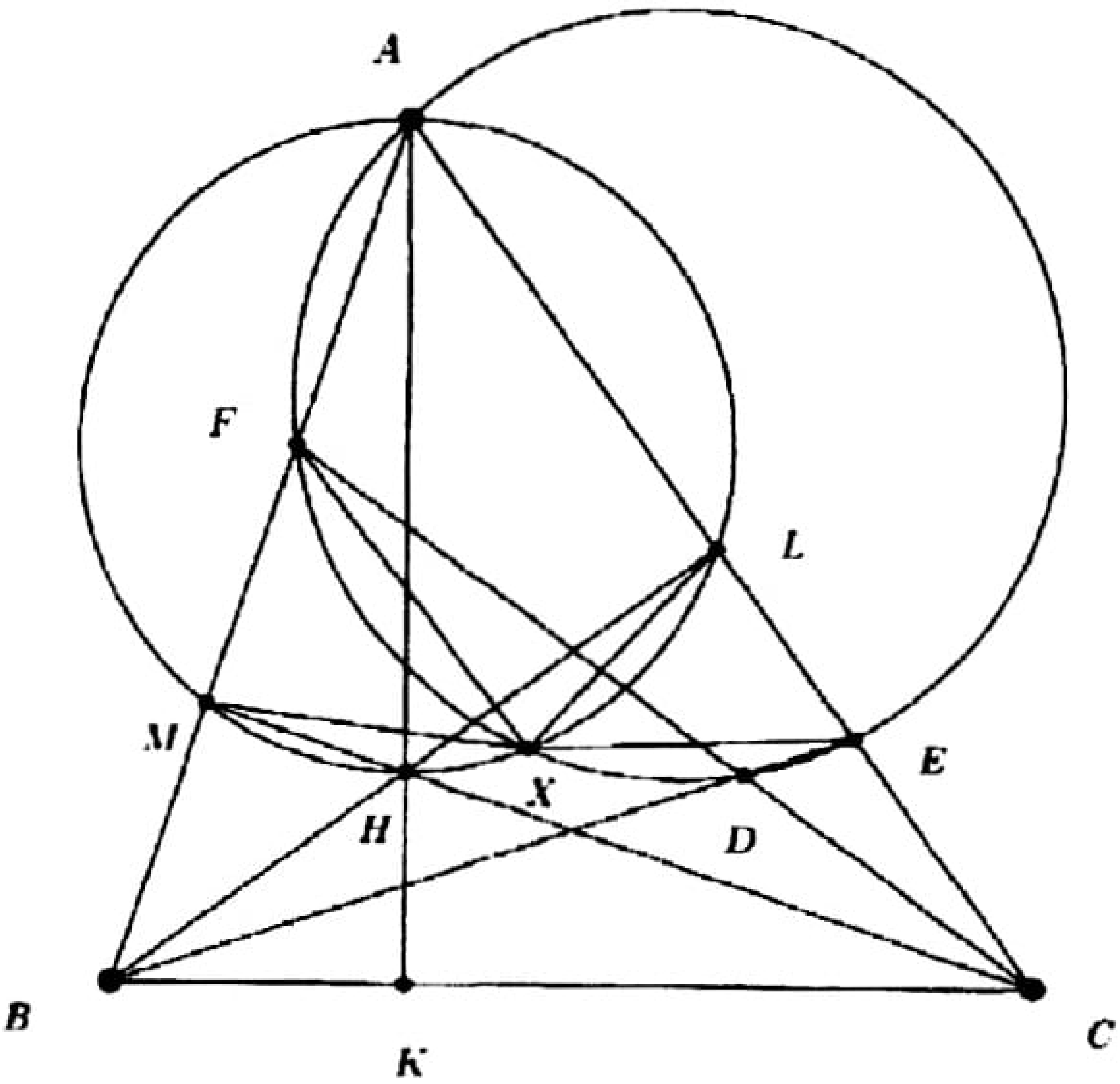
$$CA \cdot CL = CB \cdot CK \tag{2}$$

Adding (1) and (2) we have:

$$CA \cdot CL + BA \cdot BM = BC \cdot BK + CB \cdot CK = BC(BK + CK) = BC^2 \tag{3}$$

Combining (3) with the problem statement $BC^2 = BA \cdot BF + CE \cdot CA$ we have:

$$BA \cdot BF - BA \cdot BM = CA \cdot CL - CE \cdot CA$$
$$BA(BF - BM) = CA(CL - CE)$$
$$BA \cdot FM = CA \cdot LE$$
$$\frac{LE}{FM} = \frac{AB}{AC} = \frac{BL}{CM} \tag{4}$$



Where the last equality follows from $\triangle AMC \sim \triangle ALB$. Now since $\frac{LE}{FM} = \frac{BL}{CM}$ and $\measuredangle FMC = \measuredangle ELB = 90°$ we get that triangles $\triangle FMC \sim \triangle ELB$. From this similarity we get

$$\measuredangle AED = \measuredangle AEB = \measuredangle LEB = \measuredangle MFC = 180° - \measuredangle AFC = 180° - \measuredangle AFD,$$

meaning points A, D, E, F are concylic.

Since both pairs $\{E, F\}$ and $\{M, L\}$ satisfy the problem condition, we must have this fixed point we are looking for is the second intersection of the circumcircles around $AFDE$ and $AMHL$. Let this point be X. We now prove that X is fixed on the circumcircle of $AMHL$ (which would imply X is fixed).

From the concylicity we have

$$\measuredangle XLE = 180° - \measuredangle XLA = \measuredangle XMA = \measuredangle XMF \text{ and } \measuredangle XEL = \measuredangle XEA = 180° - \measuredangle XFA = \measuredangle XFM$$

and from here we get $\triangle XLE \sim \triangle XMF$. This similarity gives us

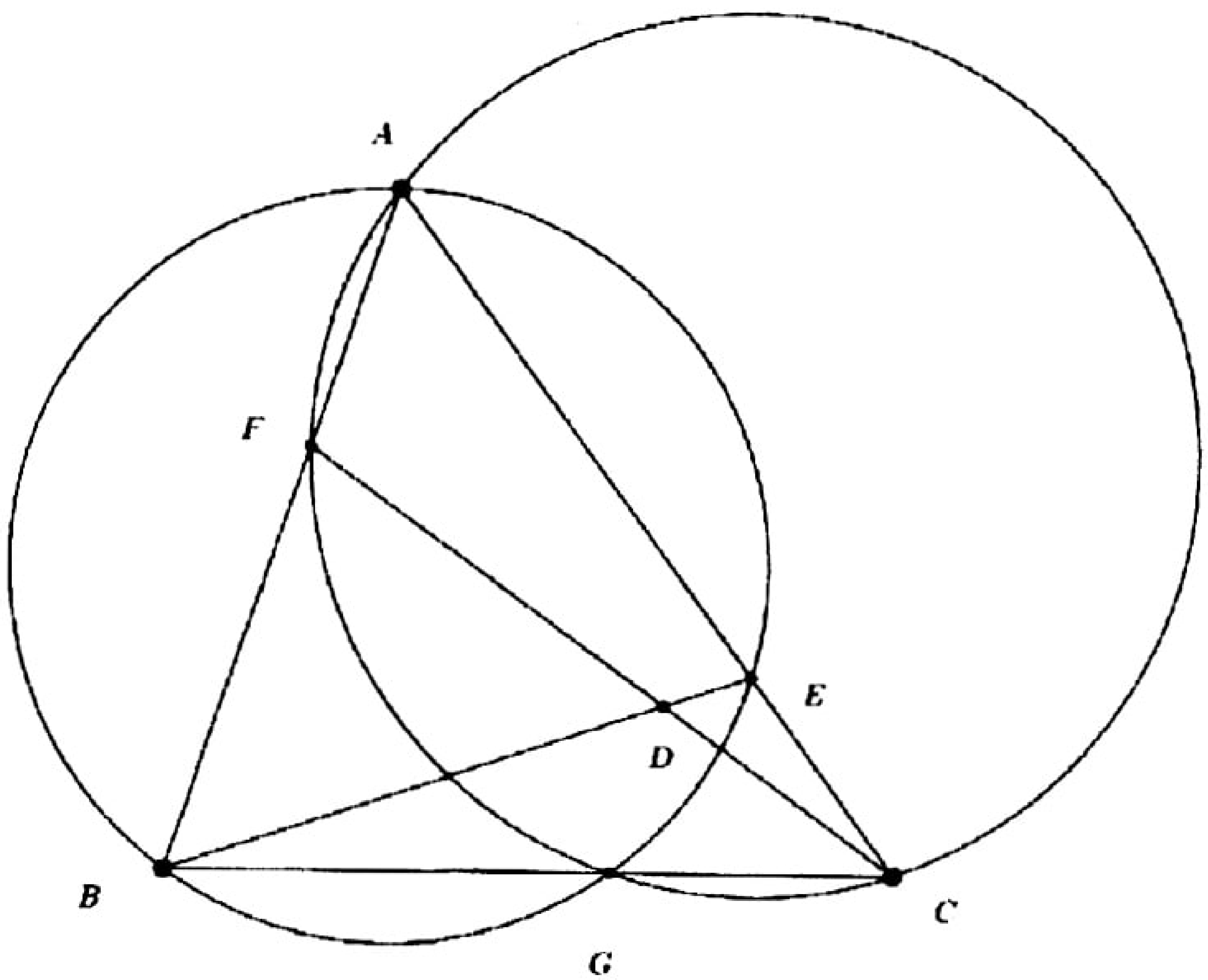
$$\frac{XL}{XM} = \frac{LE}{MF}. \tag{5}$$

Now combining (4) and (5) we get $\frac{XL}{XM} = \frac{AB}{AC}$ which is a fixed quantity. Since points M, L, the circumcircle of AML, and ratio $\frac{XL}{XM}$ are fixed, this implies that point X is fixed.

Solution 2

Let the D be the intersection of BE and CF and let circumcircle of triangle CFA intersect BC at point G. From power of point we have

$$BG \cdot BC = BF \cdot BA. \tag{6}$$



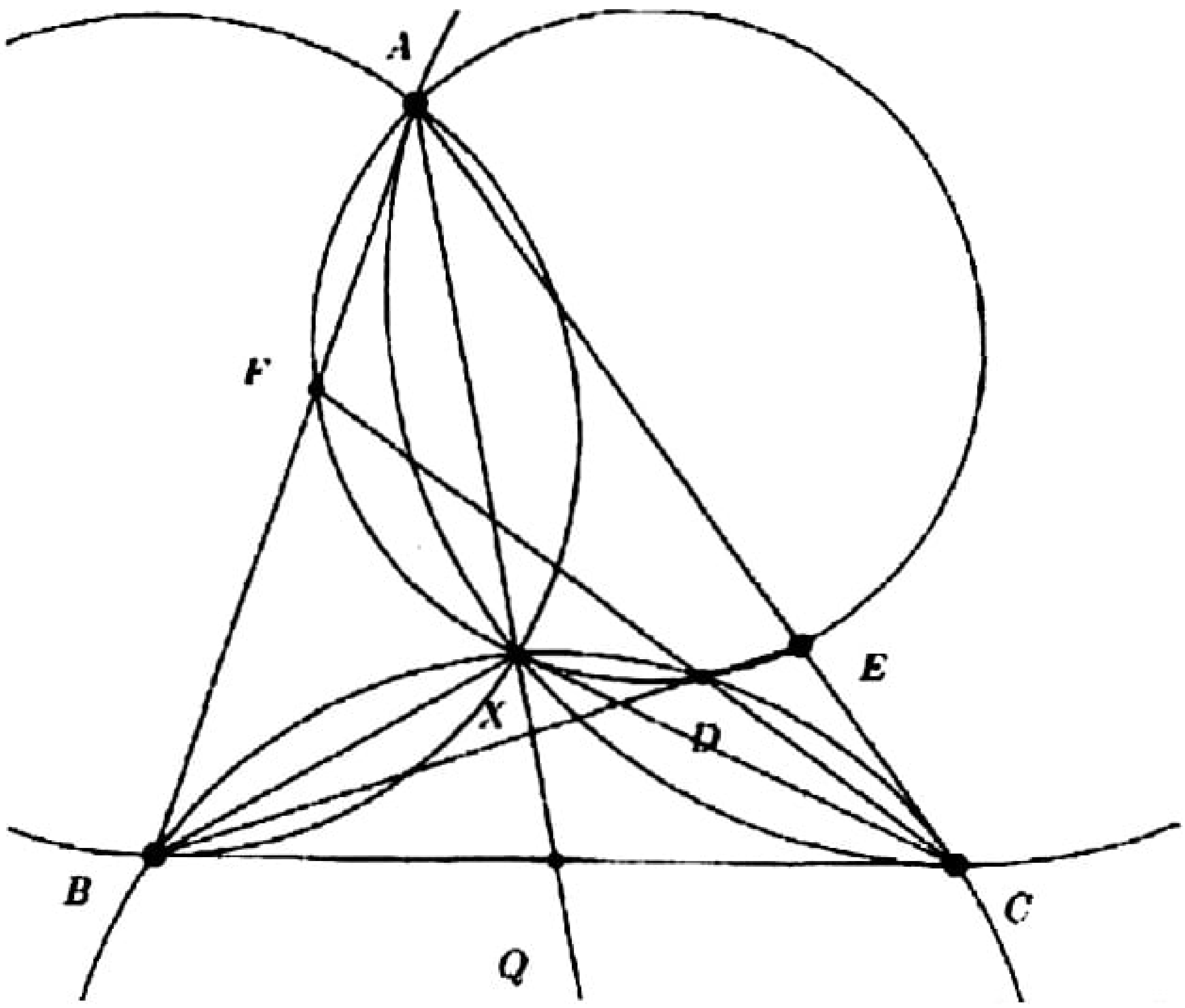
Combining (6) with the problem statement we get

$$BC^2 = BA \cdot BF + CE \cdot CA = BG \cdot BC + CE \cdot CA$$

and from here we get

$$CE \cdot CA = BC(BC - BG) = BC \cdot CG. \quad (7)$$

(7) implies that E, A, B, G are concyclic as well.



This gives us

$$\measuredangle GAC = \measuredangle GAE = \measuredangle GBE = \measuredangle CBD$$

and

$$\measuredangle BAC - \measuredangle GAC = \measuredangle GAB = \measuredangle GAF = \measuredangle GCF = \measuredangle BCD.$$

Adding these two equalities gives us

$$\measuredangle BAC = \measuredangle CBD + \measuredangle BCD = 180^\circ - \measuredangle BDC.$$

This implies that A, E, D, F are concyclic. Now let the second intersection of the circumcircles of BDC and $AFDE$ be X. We have

$$\measuredangle XAB = \measuredangle XAF = \measuredangle XDF = 180^\circ - \measuredangle XDC = \measuredangle XBC \quad (8)$$

and

$$\measuredangle XAC = \measuredangle XAE = 180^\circ - \measuredangle XDE = \measuredangle XDB = \measuredangle XCB \quad (9)$$

(8) and (9) imply that BC is tangent to the circumcircles of $\triangle XAB$ and $\triangle XAC$ respectively. Let AX, the radical axis of the two circumcircles, intersect BC at Q. Now we have by power of point

$$QB^2 = QX \cdot QA = QC^2$$

giving up that AX bisects BC. So X is the point on the median from A to side BC such that $\measuredangle BXC = 180^\circ - \measuredangle BAC$. This point is unique and we have proven that it is always on the circumcircle of $AEDF$.

G2

Let ABC be an acute triangle and D a variable point on side AC. Point E is on BD such that $BE=\frac{BC^2-CD\cdot CA}{BD}$. As D varies on side AC prove that the circumcircle of ADE passes through a fixed point other than A.

Solution

Let the circumcircle of triangle CED intersect BC at point G. From power of point we have

$$BG\cdot BC=BE\cdot BD. \qquad (1)$$

Combining (1) with the problem statement we get

$$\frac{BG\cdot BC}{BD}=BE=\frac{BC^2-CD\cdot CA}{BD}$$

and from here we get

$$CD\cdot CA=BC(BC-BG)=BC\cdot CG. \qquad (2)$$

(2) implies that D,A,B,G are concyclic as well. This gives us

$$\measuredangle BEC=\measuredangle BGD=180^\circ-\measuredangle BAD=180^\circ-\measuredangle CAB.$$

Now let the circumcircle of ADE and BEC intersect again at X. Since

$$\measuredangle XCB=\measuredangle XEB=180^\circ-\measuredangle XED=\measuredangle XAD=\measuredangle XAC$$

and

$$\measuredangle BXC=\measuredangle BEC=180^\circ-\measuredangle BAC$$

we have that X is on the unique circle through A and C tangent to side BC at point C and circumcircle of BHC where H is the ortocenter of triangle ABC. This intersection is unique and we are done.

G3

Let ABC be a triangle with $AB < AC$ inscribed into a circle c. The tangent of c at the point C meets the parallel from B to AC at the point D. The tangent of c at the point B meets the parallel from C to AB at the point E and the tangent of c at the point C at the point L. Suppose that the circumcircle c_1 of the triangle BDC meets AC at the point T and the circumcircle c_2 of the triangle BEC meets AB at the point S. Prove that the lines ST, BC, AL are concurrent.

Solution

We will prove first that the circle c_1 is tangent to AB at the point B. In order to prove this, we have to prove that $\measuredangle BDC = \measuredangle ABC$. Indeed, since $BD \parallel AC$, we have that $\measuredangle DBC = \measuredangle ACB$. Additionally, $\measuredangle BCD = \measuredangle BAC$ (by chord and tangent), which means that the triangles ABC, BDC have two equal angles and so the third ones are also equal. It follows that $\measuredangle BDC = \measuredangle ABC$, so c_1 is tangent to AB at the point B.

Similarly, the circle c_2 is tangent to AC at the point C.

As a consequence, $\measuredangle ABT = \measuredangle ACB$ (by chord and tangent) and also $\measuredangle BSC = \measuredangle ACB$.

By the above, we have that $\measuredangle ABT = \measuredangle BSC$, so the lines BT, SC are parallel.

Now, let ST intersect BC at the point K. It suffice to prove that K belongs to AL. From the trapezoid $BTCS$ we get that

$$\frac{BK}{KC} = \frac{BT}{SC} \qquad (1)$$

and from the similar triangles ABT, ASC, we have that

$$\frac{BT}{SC} = \frac{AB}{AS} \qquad (2).$$

By (1), (2) we get that

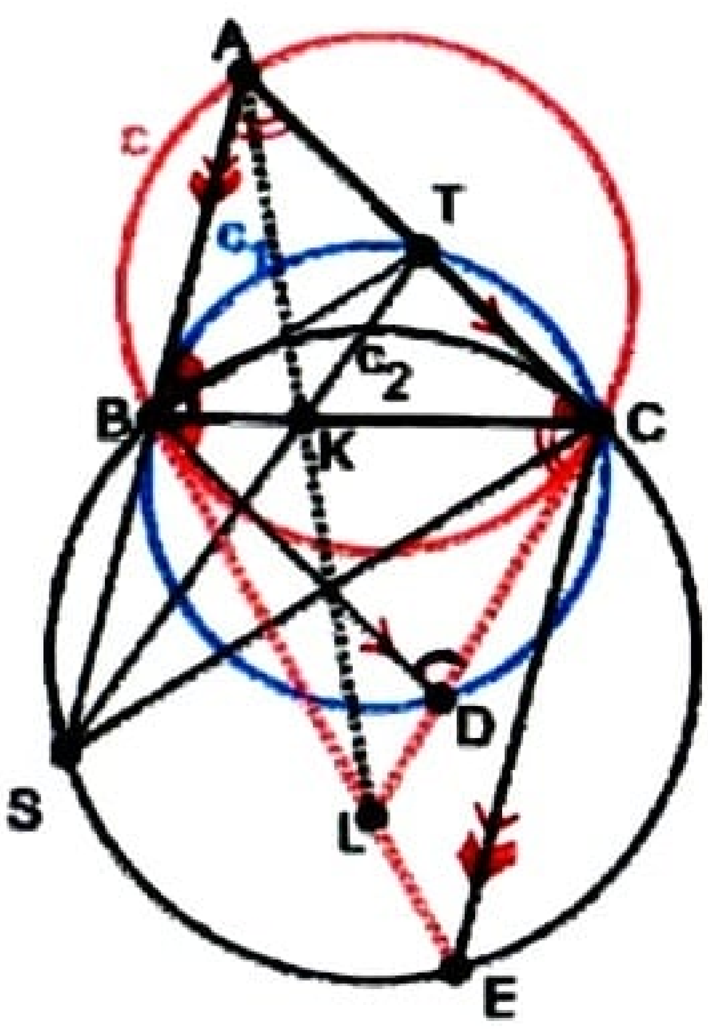
$$\frac{BK}{KC} = \frac{AB}{AS} \qquad (3).$$

From the power of point theorem, we have that

$$AC^2 = AB \cdot AS \Rightarrow AS = \frac{AC^2}{AB}.$$

Going back into (3), it gives that

$$\frac{BK}{KC} = \frac{AB^2}{AC^2}.$$



From the last one, it follows that K belongs to the symmedian of the triangle ABC.

Finally, recall that the well known fact that since LB and LC are tangents, it follows that AL is the symmedian of the triangle ABC, so K belongs to AL, as needed.

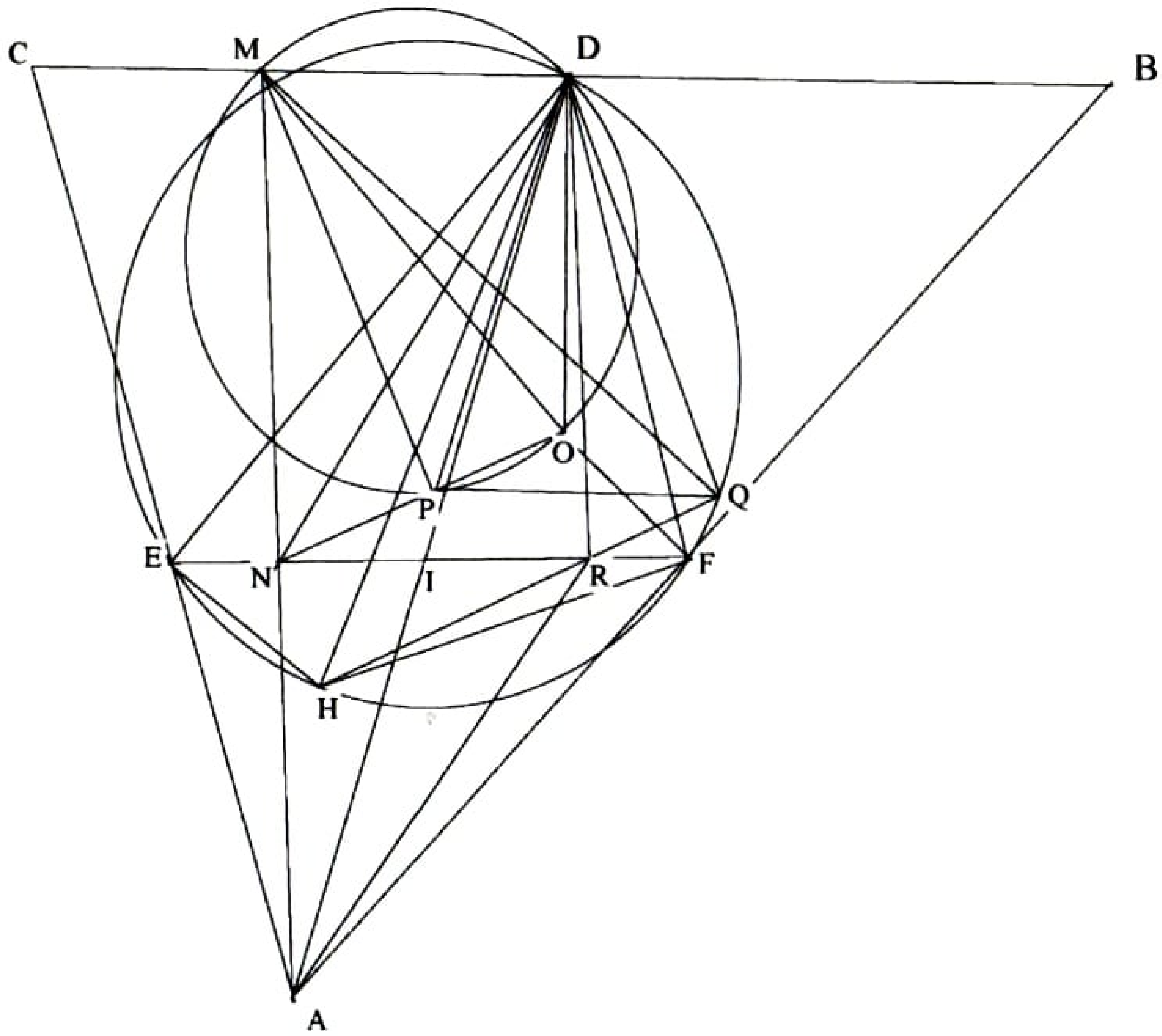
G4

The acuteangled triangle ABC with circumcenter O is given. The midpoints of the sides BC, CA and AB are D, E and F respectivelly. An arbitrary point M on the side BC, different of D, is choosen. The straight lines AM and EF intersects at the point N and the straight line ON cut again the circumscribed circle of the triangle ODM at the point P. Prove that the reflection of the point M with respect to the midpoint of the segment DP belongs on the nine points circle of the triangle ABC.

Solution.

The straight lines DO, EO and FO are the perpendicular bisectors of the sides BC, CA and AB respectively. It follows that $[OM]$ is the diameter of the circumscribed circle of the triangle ODM and $MP \perp ON$. The point O is the ortocenter of the triangle DEF (see the picture)

Let O_1 be the circumcenter of the triangle DEF and H be the diametrically opposite point of D. The circumscribed circle of the triangle DEF is the nine points circle of the triangle ABC. It follows that $EH \perp DE$, $FH \perp FD$ and $ED \parallel AF$, $DF \parallel AE$. So, the point H is the ortocenter of the triangle AEF.



Let $AD \cap EF = \{ I \}$ and R is the reflection of the point N with respect to the point I, i.e. $R \in (EF)$, $NI = RI$. The point I is the simmetry center of the parallelogram $AEDF$. It follows that the point I is the midpoint of the segment $[OH]$ and the quadrilaterals $AEDF$, $ANDR$, $HNOR$ are all parallelograms.

Let Q be the reflection of the point M with respect to the midpoint of the segment DP. It follows that the quadrilaterals $PQDM$ and $MNRD$ are the parallelograms, which imply that the quadrilateral $PQRN$ is a parallelogram. So, $NO \parallel HR$, $NP \parallel RQ$, which imply that the points H, R and Q are collinear. We obtain that $m\ (\angle DQH) = 90^\circ$, i.e. the point Q belongs on the nine points circle of the triangle ABC.

G5

Let ABC be an acute angled triangle with ortocenter H, centroid G and circumcircle ω. Let D and M respectively be the intersection of lines AH and AG with side BC. Rays MH and DG interect ω again at P and Q respectively. Prove that PD and QM intersect on ω.

Solution 1

Note that it is enough to prove that $\measuredangle DPA + \measuredangle MQA = 180°$.

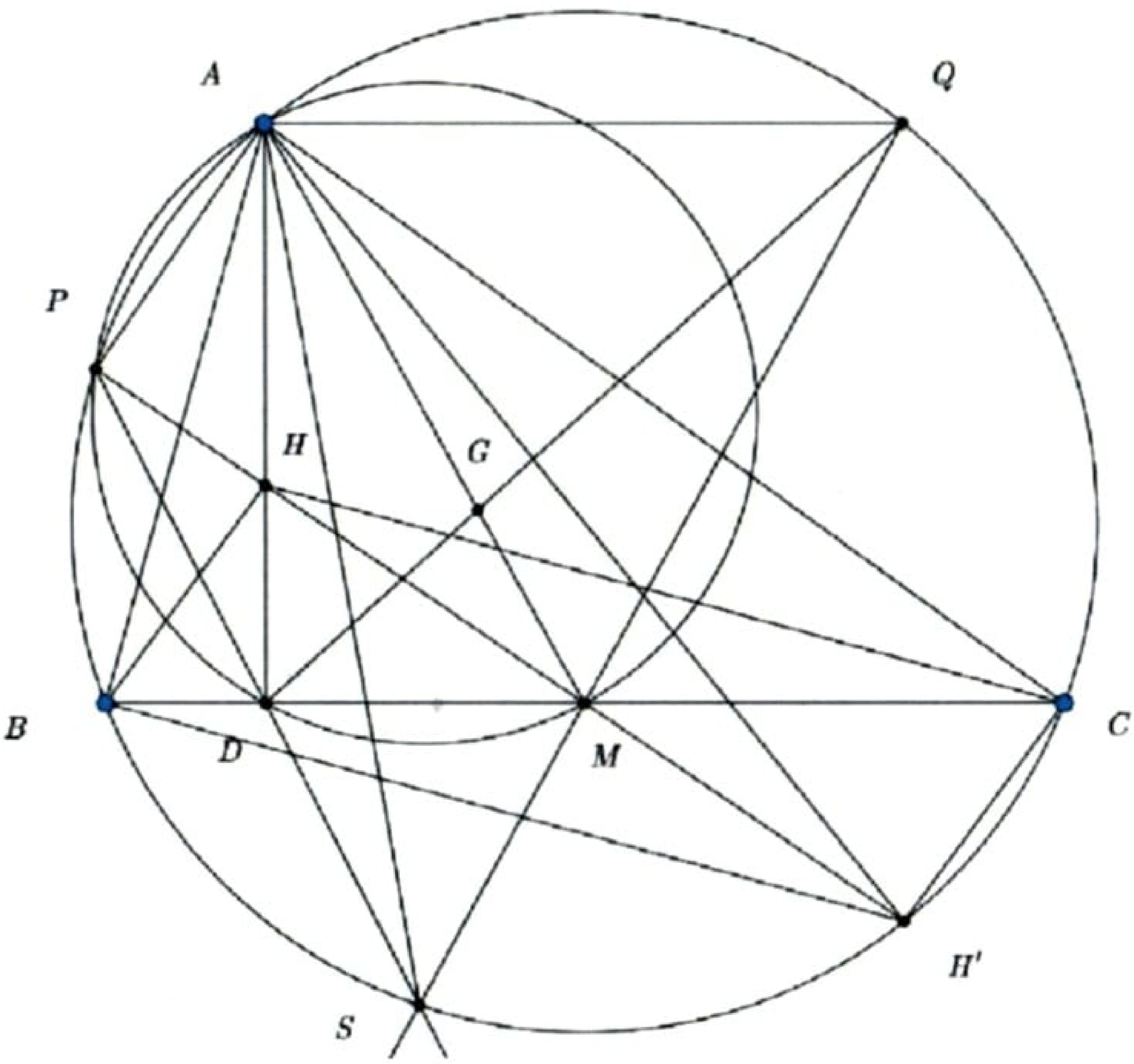
Without loss of generality assume that $AB < AC$. Let the reflection of H in point M be H'. Since $BHCH'$ is a paralelogram we get

$$\measuredangle BH'C = \measuredangle BHC = 180° - \measuredangle BAC$$

which means H' lies on ω. Also we get

$$\measuredangle ABH' = \measuredangle ABC + \measuredangle CBH' = \measuredangle ABC + \measuredangle BCH = 90°$$

since $CH \perp AB$. This mean AH' is the diameter of ω. This means $\measuredangle MPA = \measuredangle H'PA = 90°$. Since $\measuredangle MPA = \measuredangle MDA = 90°$ we get that M, D, P, A are concyclic.



This gives us $\measuredangle DPA + \measuredangle AMD = 180°$. So now it is enough to prove that $\measuredangle AMB = \measuredangle MQA$.

Taking the homothety with center G and factor -2 (the homothety taking the 9 point circle to the circumcircle of ABC), we get that Q and A are images of D and M respectively.

This means $AQ \parallel DM$ giving $AQ \parallel BC$. Since $AQ \parallel BC$ and A,B,C,Q are concyclic this means $ABCQ$ is an iscoseles trapezoid.

This means M lies on the perpendicular bisector of AQ. This gives us $MA = MQ$. Since $AQ \parallel BC$ this gives us $\measuredangle AQM = \measuredangle QAM = \measuredangle AMB$ which concludes our problem.

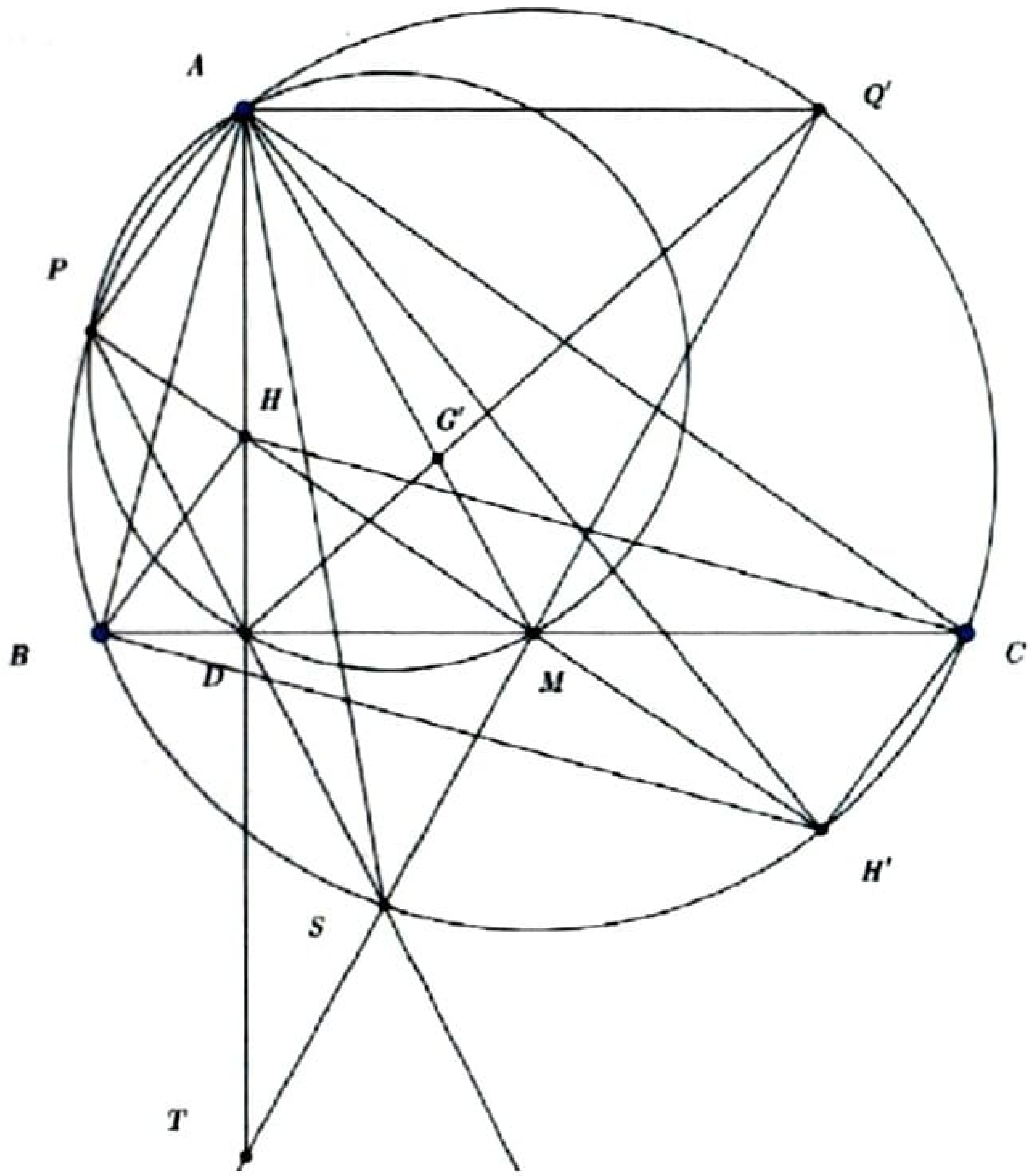
Solution 2

We prove that M, D, P, A are concyclic same as in solution 1. Let PD intersect ω again at S. We see that this gives us $\measuredangle SAH' = \measuredangle SPH' = \measuredangle DPM = \measuredangle DAM$. Combining this with $\measuredangle BAH' = \measuredangle CAD$ we get:

$$\measuredangle SAH' + \measuredangle SAB = \measuredangle H'AB = \measuredangle CAD = \measuredangle DAM + \measuredangle MAC$$

Giving us

$$\measuredangle SAB = \measuredangle MAC. \qquad (*)$$



Combining (*) with $\measuredangle ASB = \measuredangle ACB = \measuredangle ACM$ we get triangles ASB and ACM are similar. This gives us

$$\frac{AB}{AM} = \frac{SB}{CM}. \qquad (**)$$

Analogously we get triangles ASC and ABM are similar. This gives us

$$\frac{AC}{AM} = \frac{SC}{BM}. \qquad (***)$$

Combining (**) and (***) we get

$$\frac{AB}{AC} = \frac{SB}{SC} \qquad (1)$$

since $BM = CM$. Let SM intersect ω again at Q'. Let $Q'D$ intersect AM at G'. We wish to prove that $Q' \equiv Q$ and $G' \equiv G$. It is enough to prove that $AG' = 2G'M$.

Since triangles SMB and CMQ' are similar we get

$$\frac{SB}{SM} = \frac{CQ'}{CM}. \qquad (2)$$

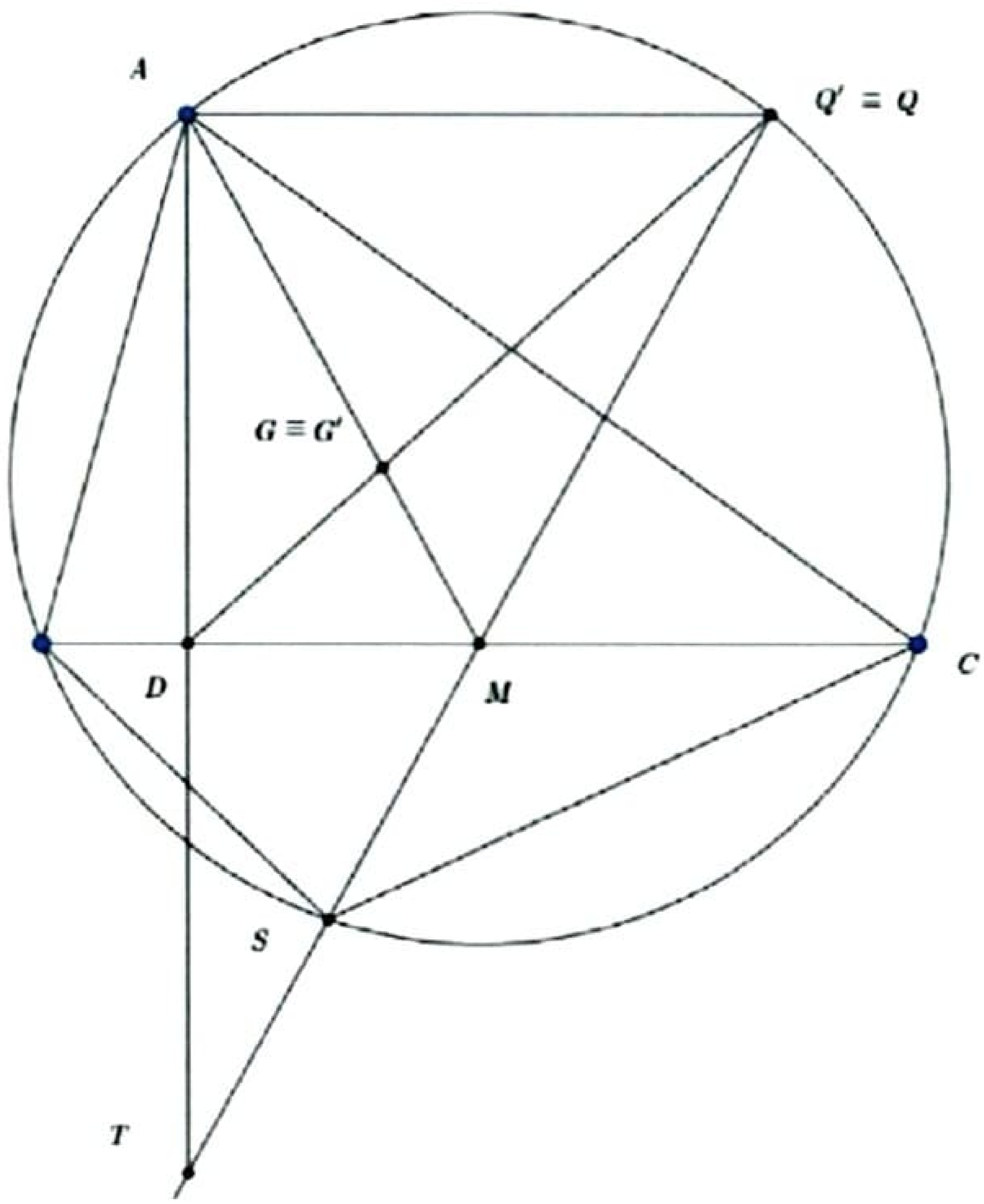
Analogously SMC and BMQ' are similar and we get

$$\frac{SM}{SC} = \frac{BM}{BQ'}. \qquad (3)$$

Multiplying (2) and (3) we get $\frac{SB}{SC} = \frac{CQ'}{BQ'}$. Combining that with (1) we get

$$\frac{AB}{AC} = \frac{CQ'}{BQ'}. \qquad (4)$$

Since Q' and A are on the same side of BC (4) gives us $AQ' \parallel BC$. This means that $ABCQ'$ is an isosceles trapezoid.



Let $Q'S$ intersect AD at point T. Since $MD \parallel AQ'$, $MA = MQ'$ and $\measuredangle TAQ' = 90^\circ$ we get that M is center of the circumcircle of the right triangle TAQ'.

Applying Menelaus' theorem on $D - G' - Q'$ and triangle AMT we get

$$\frac{AG'}{MG'} \cdot \frac{MQ'}{TQ'} \cdot \frac{TD}{AD} = 1 \quad (5)$$

Since $MA = MT$ and $MD \perp AT$ this means

$$DA = DT. \quad (6)$$

Also since M is the circumcenter of triangle TAQ' we get

$$2MQ' = TQ'. \quad (7)$$

Combining (5) with (6) and (7) we get $2MG' = AG'$. This gives us $G' \equiv G$ and $Q' \equiv Q$, thus proving the problem statement.

G6

Construct outside the acute-angled triangle ABC the isosceles triangles ABA_B, ABB_A, ACA_C, ACC_A, BCB_C and BCC_B, so that

$$AB = AB_A = BA_B,\ AC = AC_A = CA_C,\ BC = BC_B = CB_C$$

and

$$\measuredangle BAB_A = \measuredangle ABA_B = \measuredangle CAC_A = \measuredangle ACA_C = \measuredangle BCB_C = \measuredangle CBC_B = \alpha < 90^\circ.$$

Prove that the perpendiculars from A to B_AC_A, from B to A_BC_B and from C to A_CB_C are concurrent.

Solution.

Lemma. If BCD is the isoceles triangle which is outside the triangle ABC and has

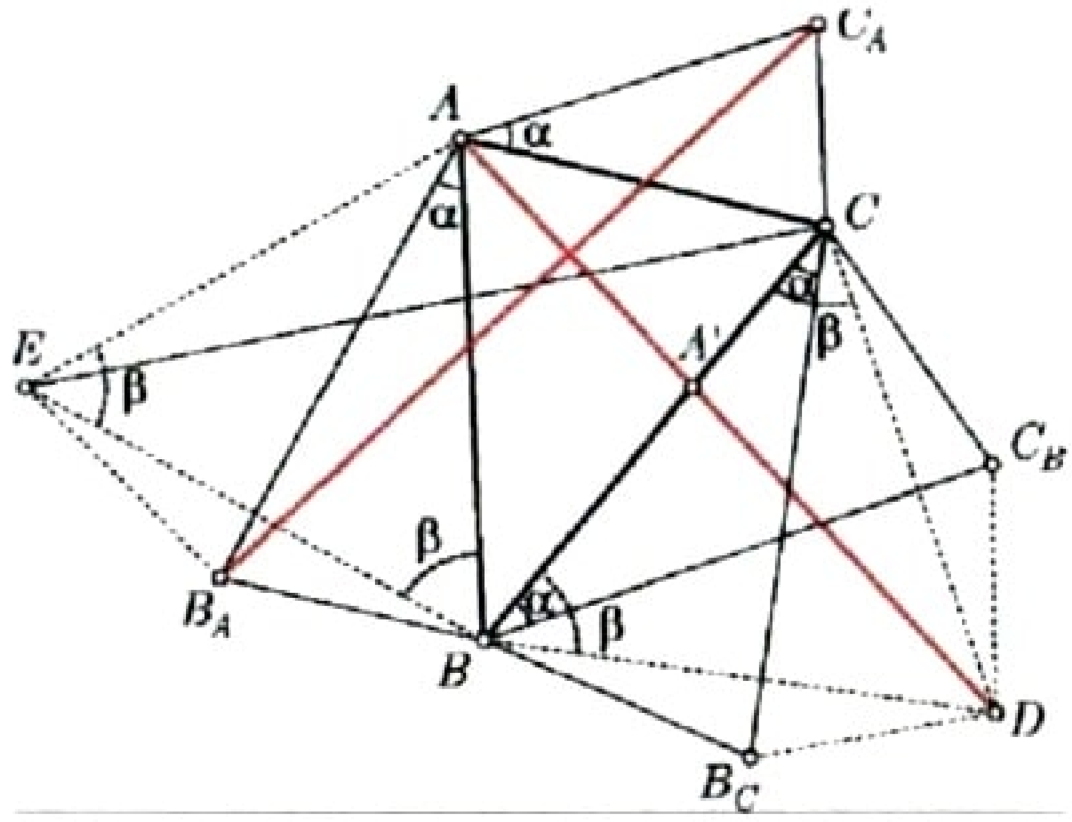
$$\measuredangle CBD = \measuredangle BCD = 90^\circ - \alpha \overset{\text{not}}{:=} \beta,$$

then $AD \perp B_AC_A$.

Proof of the lemma. Construct an isosceles triangle ABE outside the triangle ABC, so that $\measuredangle ABE = \measuredangle AEB = \beta$.

Then $AE = AB = AB_A$ and $\measuredangle EAB_A = \alpha$, so a rotation of center A and angle α sends C_A to C and B_A to E, hence $\measuredangle(\overrightarrow{B_AC_A}, \overrightarrow{EC}) = \alpha$ (the angle between vectors is considered oriented).

Also triangles EBA and BCD are similar, so a rotation of center B and angle β, followed by a dilation of ratio $\frac{EB}{AB} = \frac{BC}{BD}$ sends E to A and C to D, hence $\measuredangle(\overrightarrow{EC}, \overrightarrow{AD}) = \beta$ (also oriented angle).



This shows that

$$\measuredangle(\overrightarrow{B_AC_A}, \overrightarrow{AD}) = \measuredangle(\overrightarrow{B_AC_A}, \overrightarrow{EC}) + \measuredangle(\overrightarrow{EC}, \overrightarrow{AD}) = \alpha + \beta = 90^\circ. \blacksquare$$

Returning to the solution of the problem, denote A' the intersection of BC with the perpendicular from A to B_AC_A. Then A' belongs to the segment BC and

$$\frac{A'B}{A'C} = \frac{AB\sin(B+\beta)}{AC\sin(C+\beta)}.$$

Since similar relations are true for the intersections B', C' of the other two perpendiculars with the opposite sides, this yields

$$\frac{A'B}{A'C}\cdot\frac{B'C}{B'A}\cdot\frac{C'A}{C'B} = \frac{AB\sin(B+\beta)}{AC\sin(C+\beta)}\cdot\frac{BC\sin(C+\beta)}{BA\sin(A+\beta)}\cdot\frac{CA\sin(A+\beta)}{CB\sin(B+\beta)} = 1,$$

whence the conclusion.

Remark. The conditions 'acute-angled' and '$\alpha < 90^\circ$' are not essential, but without them there are cases when A' does not belong to the segment BC, or the perpendiculars become parallel.

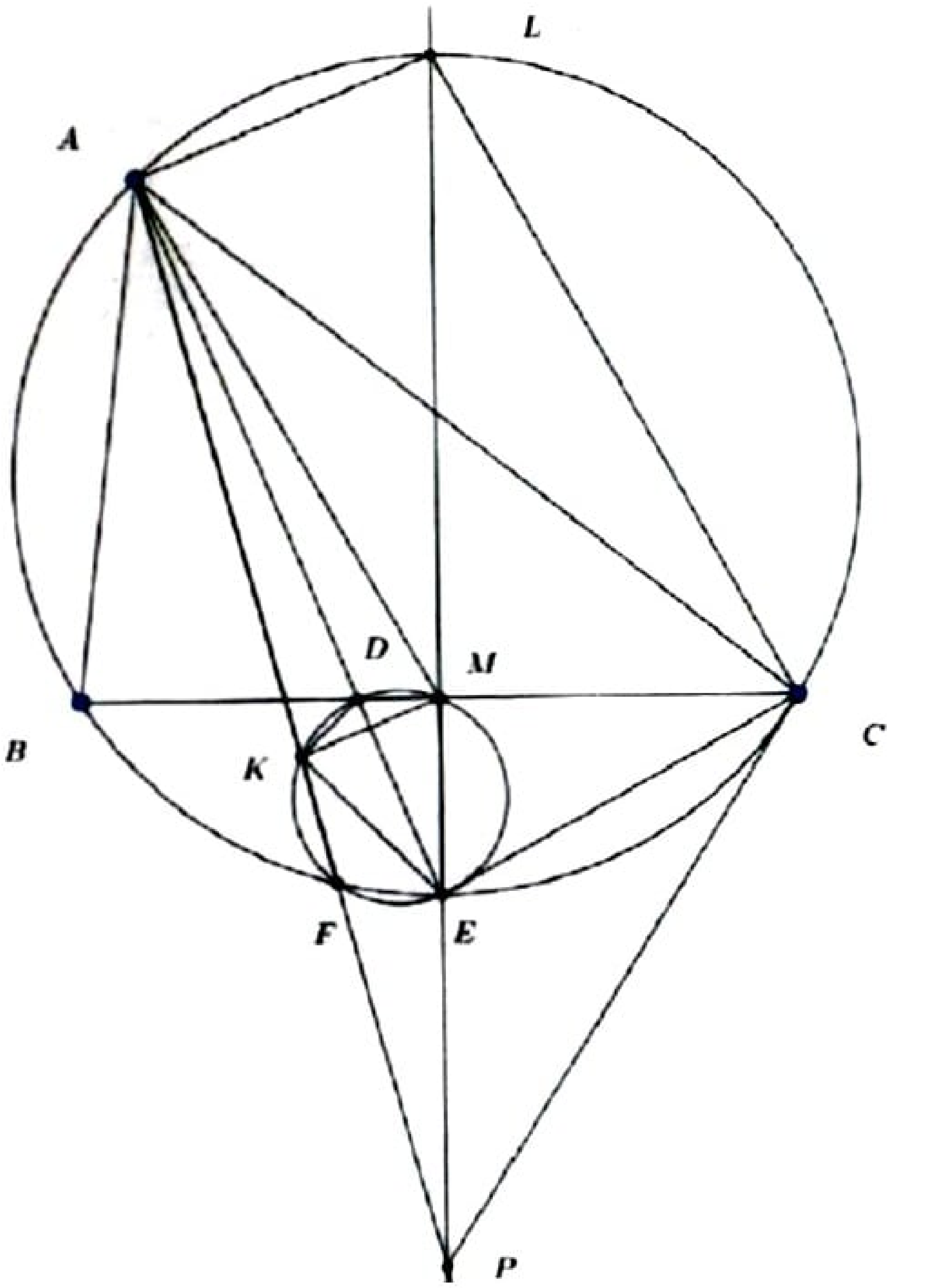
G7

Let ABC be an acute triangle with $AB \neq AC$ and circumcirle Γ. The angle bisector of BAC intersects BC and Γ at D and E respectively. Circle with diameter DE intersects Γ again at $F \neq E$. Point P is on AF such that $PB = PC$ and X and Y are feet of perpendiculars from P to AB and AC respectively. Let H and H' be the ortocenters of ABC and AXY respectively. AH meets Γ again at Q. If AH' and HH' intersect the circle with diameter AH again at points S and T, respectively, prove that the lines AT, HS and FQ are concurrent.

Solution

WLOG, assume $AB < AC$. Let M be the midpoint of side BC and let the circumcircle of DFE intersect AF again at K. Since

$$90^\circ + \measuredangle MED = 180^\circ - \measuredangle MDE = \measuredangle ABC + \frac{\measuredangle BAC}{2} = \measuredangle AFE = \measuredangle DFE + \measuredangle AFD = 90 + \measuredangle AFD$$



it follows that

$$\measuredangle AFD = \frac{\measuredangle ABC - \measuredangle ACB}{2} = \measuredangle MED$$

Because

$$\measuredangle DKE = \measuredangle DME = 90^\circ$$

and

$$\measuredangle KED = \measuredangle KFE = \measuredangle MED$$

we get $\triangle KDE \cong \triangle MDE$ from which it follows that DE is the perpendicular bisector of MK and here we get

$$\measuredangle FAD = \measuredangle KAD = \measuredangle MAD.$$

It is obvious that P is the intersection of ME and AF. Let ME intersect Γ again at L. From the angle bisector theorem in triangle $\triangle AMP$ we get

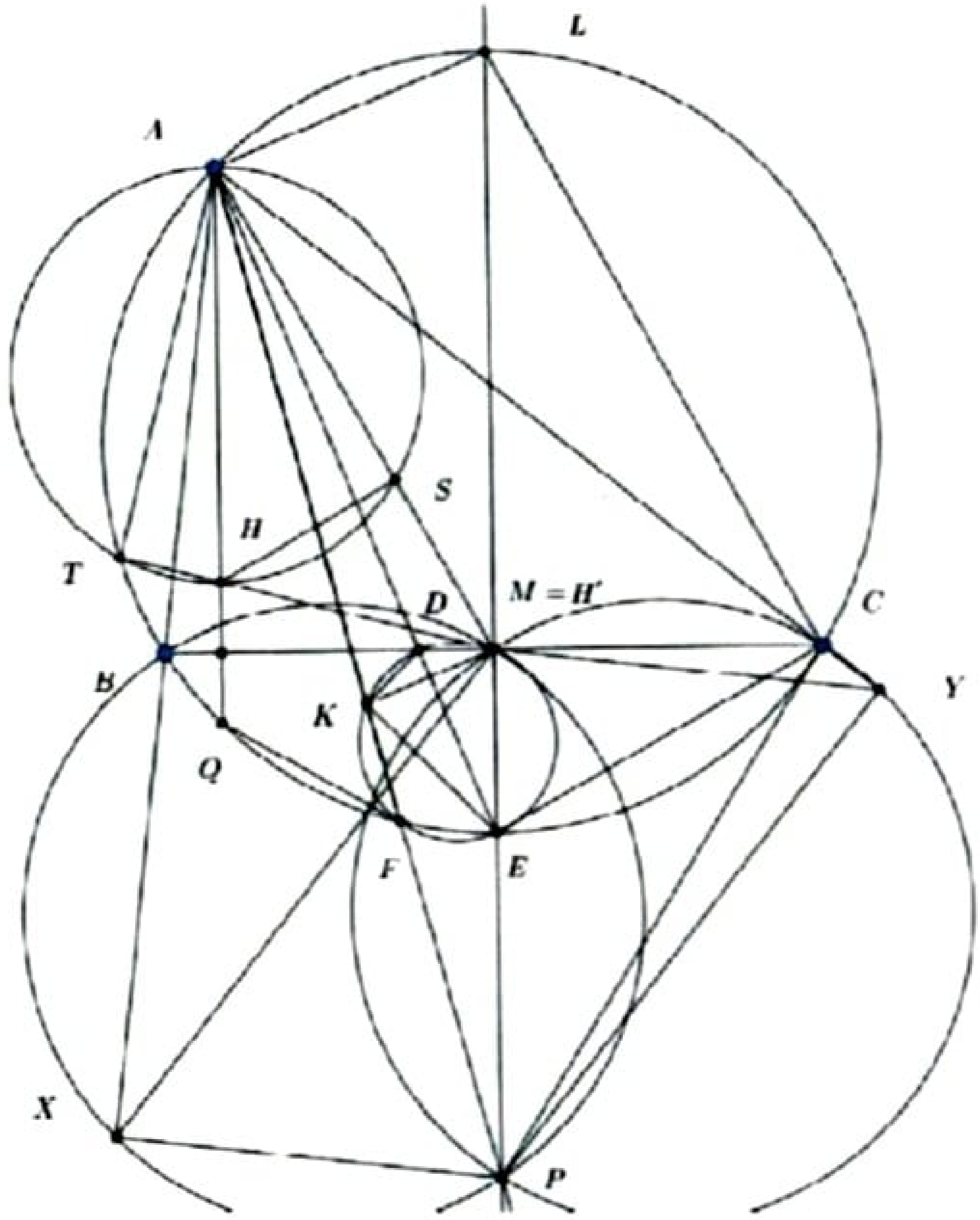
$$\frac{PE}{ME} = \frac{AP}{AM} = \frac{LP}{LM} \quad (1)$$

(LA is the external angle bisector of $\measuredangle MAP$ since LE is the diameter of Γ). Now we prove that CE and CL are angle bisectors of $\measuredangle MCP$. Let M' be the point on LE such that $\measuredangle M'CE = \measuredangle ECP$. From the angle bisector theorem we get

$$\frac{M'E}{PE} = \frac{CM'}{CP} = \frac{M'L}{PL} \quad (2)$$

Multiplying (1) and (2) we get $\frac{ME}{LM} = \frac{M'E}{M'L}$ adding 1 on both sides we get $LM = LM'$ from which it follows that $M \equiv M'$ and thus CE and CL are the bisectors of $\measuredangle MCP$.
Now we have

$$\measuredangle MPC = 90^\circ - \measuredangle MCP = 90^\circ - 2\measuredangle MCE = 90^\circ - 2\measuredangle EAC = 90^\circ - \measuredangle BAC.$$



Since X and Y are perpendicular to AB and AC we have $BXPM$ and $CYPM$ are concyclic. Here we get

$$\measuredangle MYC = \measuredangle MPC = 90^\circ - \measuredangle BAC$$

and it follows that $YM \perp AX$. Similarily we get $XM \perp AY$ and so M is the ortocentar of $\triangle AXY$ giving us $M \equiv H'$.
Since $ATHS$ and $ATQF$ are both concyclic it is enough to prove that $HSFQ$ is concyclic. Since

$$\measuredangle BQC = 180^\circ - \measuredangle BAC = \measuredangle BHC$$

and $HQ \perp BC$ it follows that BC is the perpendicular bisector of HQ. It is enough to prove that BC is the perpendicular bisector of SF. Let AM and TH meet Γ again at points A' and N respectively. Since HN passes through the midpoint of side BC and

$$\measuredangle BHC = 180 - \measuredangle BAC = \measuredangle BNC$$

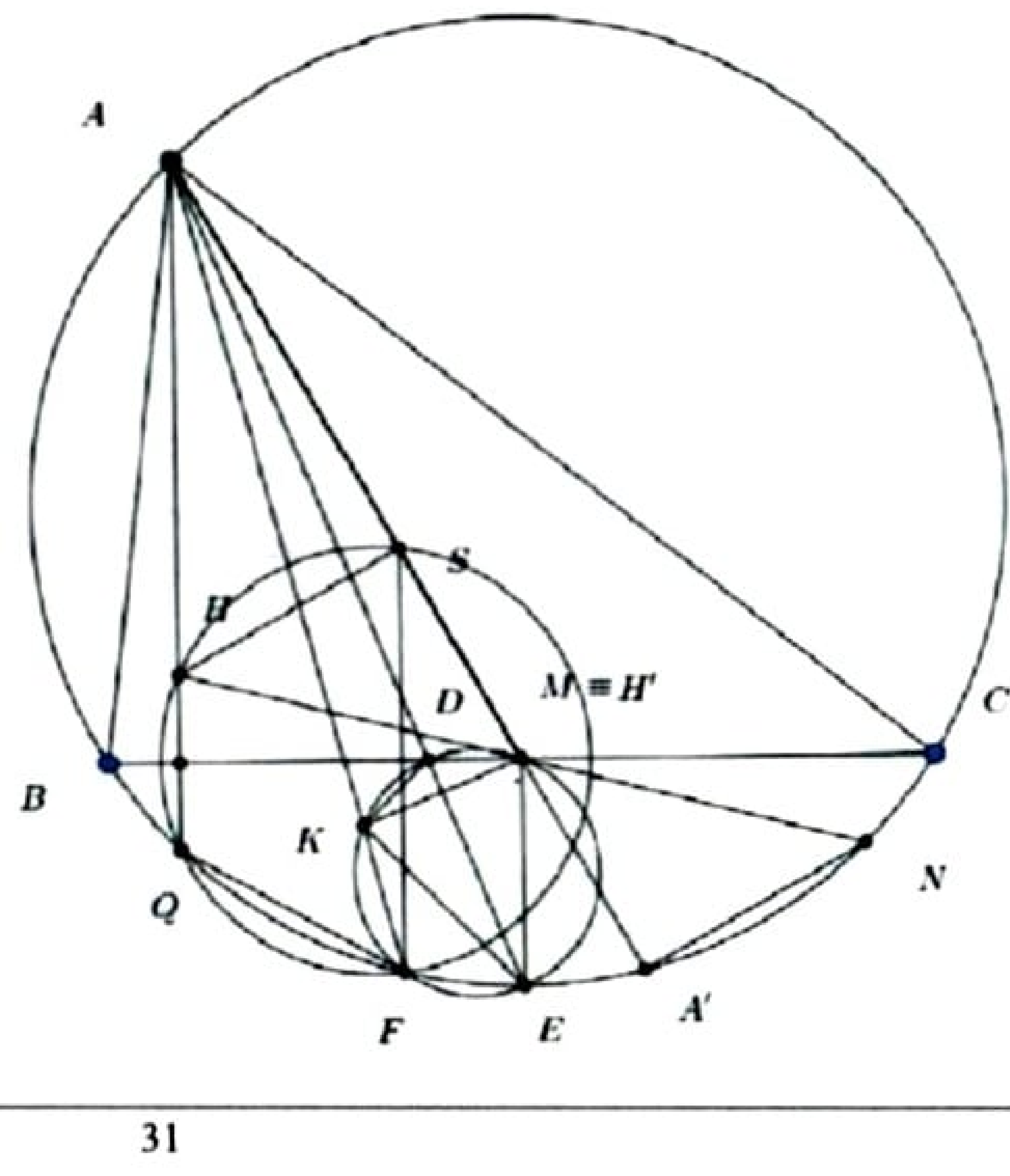
it follows that $BNCH$ is a paralelogram. From here we get that

$$\measuredangle NCB = \measuredangle HBC = 90^\circ - \measuredangle ACB$$

giving us $\measuredangle NCA = 90^\circ$ and similarily $\measuredangle NBA = 90^\circ$. This means AN is the diameter of Γ, so

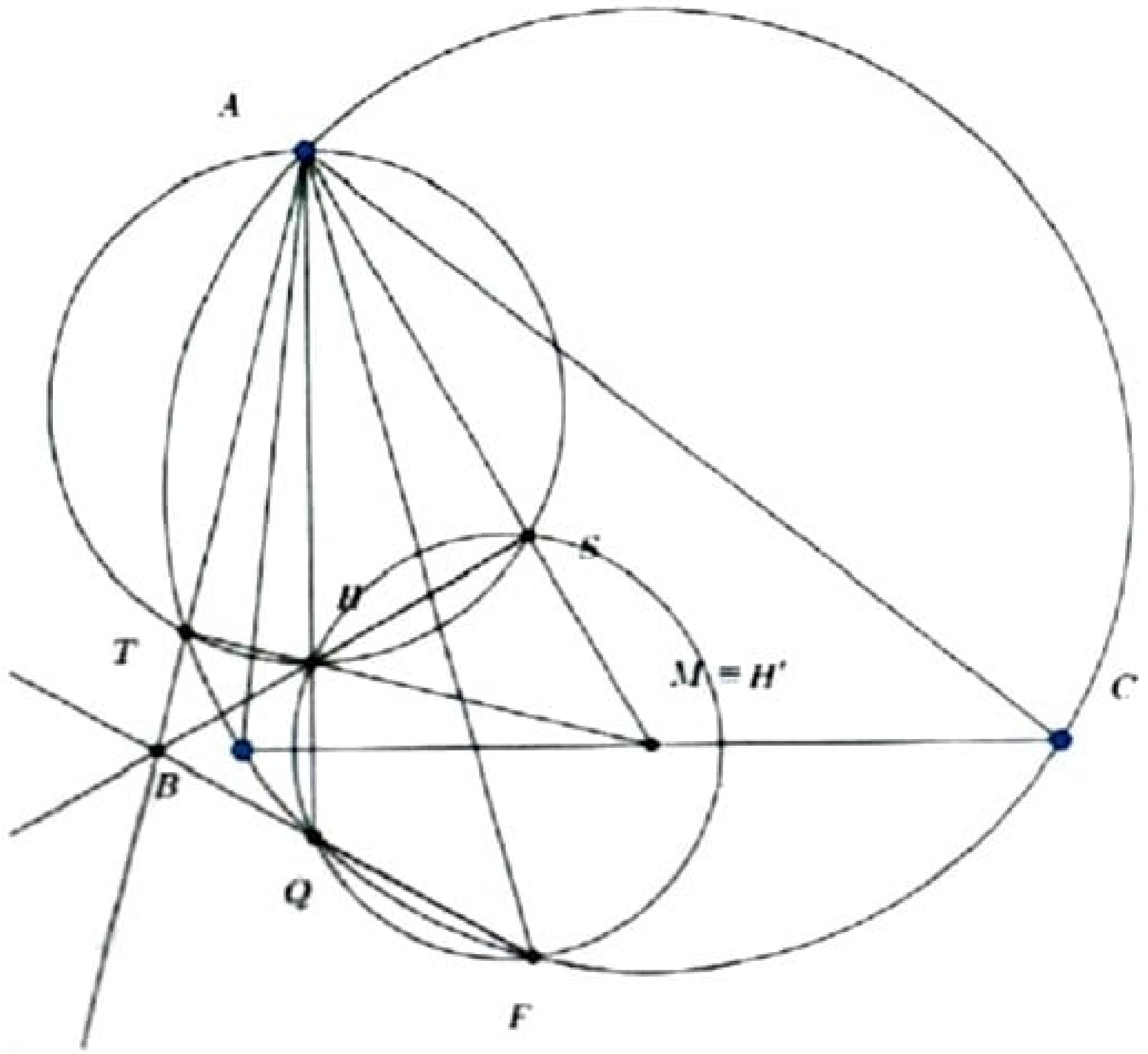
$$\measuredangle NA'S = \measuredangle NA'A = 90^\circ = \measuredangle HSA = \measuredangle HSA'$$

and from here we have $HS \parallel A'N$. Now since $HS \parallel A'N$ and M is the midpoint of HN (because $BHCN$ is a paralelogram) we get

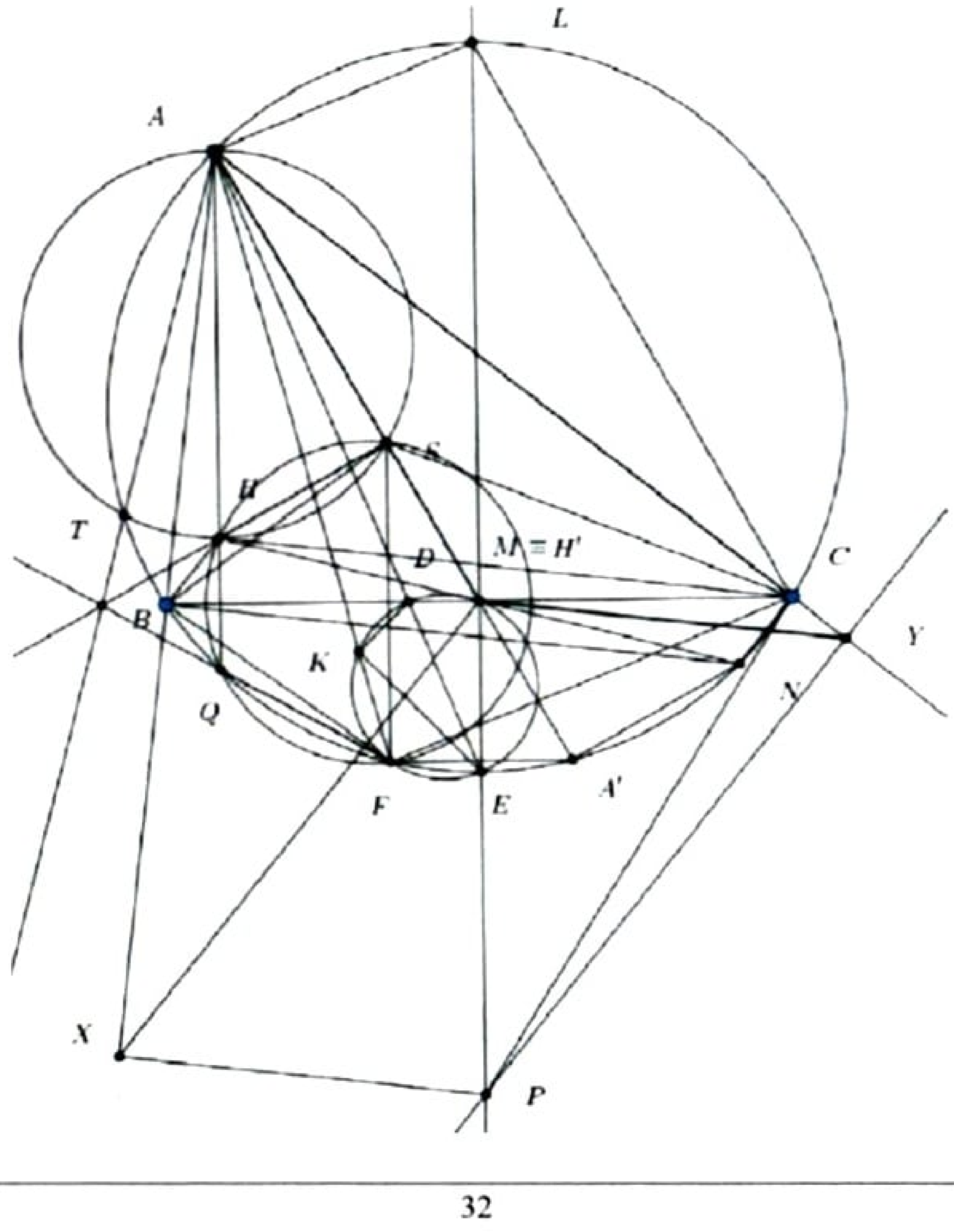


that $HSNA'$ is a paralelogram. Since

$$\measuredangle FAE = \measuredangle EAM = \measuredangle EAA'$$



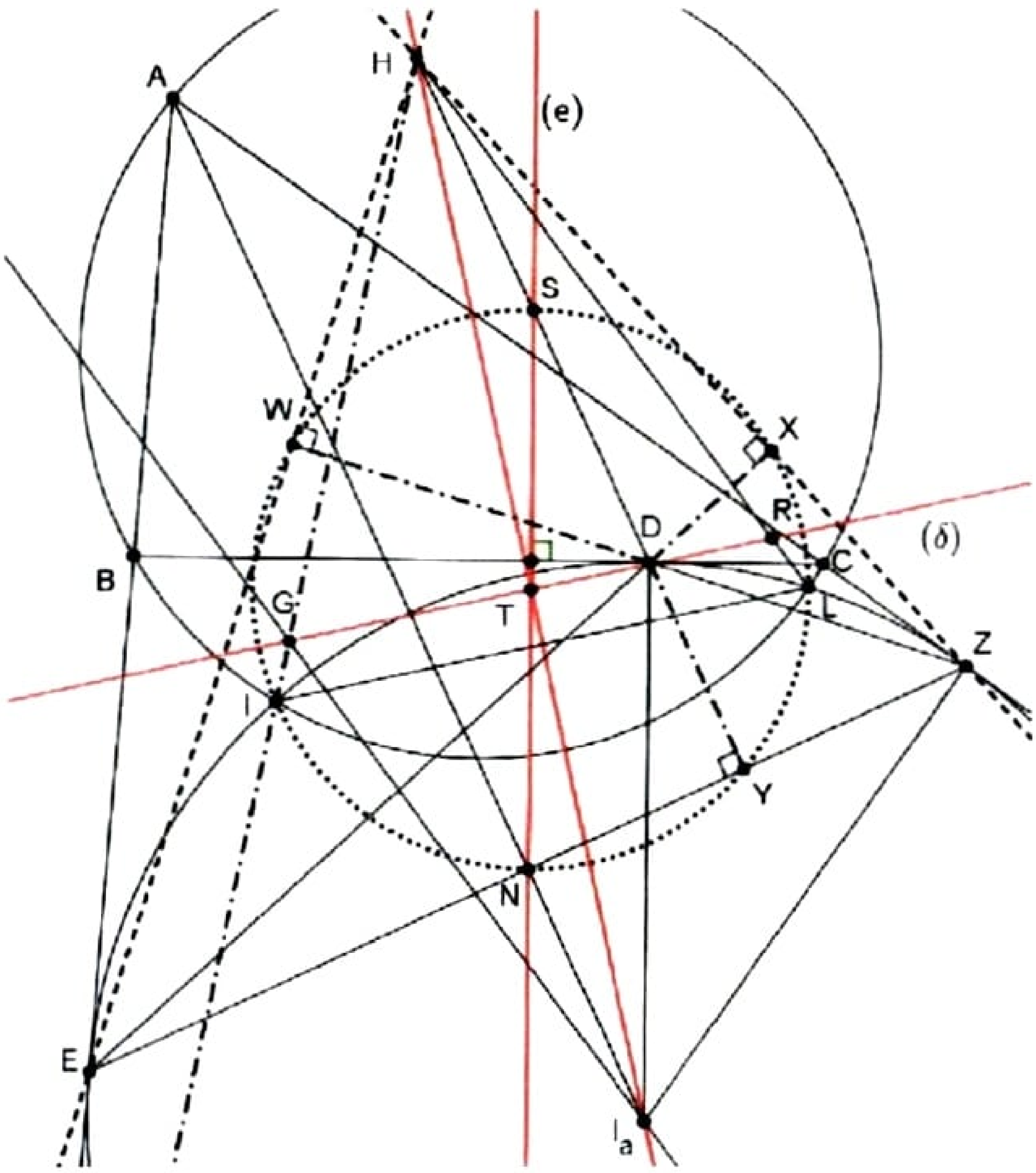
we get that $FA'BC$ is an isocelese trapezoid which means that ME is the perpendicular bisector of FA' (since it is the perpendicular bisector of BC).
This gives us $BF = CA' = BS$ and $CF = BA' = CS$ giving us that $SBFC$ is a deltoid, meaning that BC is the perpendicular bisector of FS. This means that $HSFQ$ is an isoceles trapezoid. Now from the radical axis theorem of the circumcircles of $HSFQ$, $HSAT$ and $ATQF$ we get that QF, HS, AT are concurrent.



G8

Given an acute triangle $\triangle ABC$ ($AC \neq AB$) and let (C) be its circumcircle. The excircle (C_1) corresponding to the vertex A, of center I_a, tangents to the side BC at the point D and to the extensions of the sides AB, AC at the points E, Z respectively. Let I and L are the intersection points of the circles (C) and (C_1), H the orthocenter of the triangle $\triangle EDZ$ and N the midpoint of segment EZ. The parallel line through the point I_a to the line HL meets the line HI at the point G. Prove that the perpendicular line (e) through the point N to the to the line BC and the parallel line (δ) through the point G to the line IL meet each other on the line HI_a.

Solution



We have $(e) \perp BC$ and $I_aD \perp BC$, so $(e) \| I_aD$. Let T, S be the midpoints of the segments HI_a, HD respectively and Y the point of intersection of the lines HD, EZ. Then, $TS \| I_aD$, $TS \perp BC$ and $SY \perp EZ$.

The Euler circle (ω) of the triangle EDZ passes through the points N, Y, S. Therefore, the segment SN is a diameter of the circle (ω). Thus, the center of (ω), let T', is the midpoint of the segment SN.

On the other hand, we know that the center of Euler circle (ω) is the midpoint T of HI_a. So $T \equiv T'$. Therefore, the line (e) passes through the points T, S.

Therefore, we get that the quadrilateral HSI_aN is parallelogram and its diagonals meet each other at the point T.

We consider the inversion $I(I_a, I_aZ^2)$. As $I_aZ^2 = I_aA \cdot I_aN$ we have $I(N) = A$. Similarly, if M_1, M_2 the midpoints of the segments DE, DZ respectively, we get, $I(M_1) = B$ and $I(M_2) = C$.

Therefore, the circumcircle (C) of the triangle ABC is the image of the circle (ω) under the inversion I and the points of the intersection of the circles and (ω) are invariant under this inversion. But it is well known that the circle of inversion passes through the points of the intersection of the circles (C) and (ω). Thus, the Euler circle (ω) passes through the points I, L.

Also, we consider the inversion $J(H, r^2)$ with

$$r^2 = HX \cdot HZ = HD \cdot HY = HW \cdot HE$$

where X, W, Y the traces of the altitudes of the triangle EDZ on its sides. Then, $J(Z) = X$, $J(D) = Y$ and $J(E) = W$. Therefore, the circumcircle (C_1) of the triangle ABC is the image of the circle (ω) under the inversion J. Thus, the circle of inversion J passes through the points I, L.

We conclude that $HI = HL$ and $HI_a \perp IL$ and since $(\delta) \parallel IL$, we have $HI_a \perp (\delta)$.

If, R is the point of intersection of the lines $(\delta), HL$, we get that quadrilateral HRI_aG is parallelogram and its diagonals meet each other at the point T. So, the perpendicular line (e) through the point N to the to the line BC and the parallel line (δ) through the point G to the line IL, meet each other on the line HI_a.

COMBINATORICS

C1

A grasshopper is sitting at an integer point in the Euclidean plane. Each second it jumps to another integer point in such a way that the jump vector is constant. A hunter that knows neither the starting point of the grasshopper nor the jump vector (but knows that the jump vector for each second is constant) wants to catch the grasshopper. Each second the hunter can choose one integer point in the plane and, if the grasshopper is there, he catches it. Can the hunter always catch the grasshopper in a finite amount of time?

Solution

The hunter can catch the grasshopper. Here is the strategy for him. Let f be any bijection between the set of positive integers and the set $\{((x,y),(u,v)):\ x,y,u,v\in\mathbb{Z}\}$, and denote

$$f(t)=((x_t,y_t),(u_t,v_t)).$$

In the second t, the hunter should hunt at the point (x_t+tu_t, y_t+tv_t). Let us show that this strategy indeed works.

Assume that the grasshopper starts at the point (x',y') and that the jump vector is (u',v'). Then in the second t the grasshopper is at the point $(x'+tu', y'+tv')$. Let

$$t' = f^{-1}((x',y'),(u',v')).$$

The hunter's strategy dictates that in the second t' he searches for the grasshopper at the point $(x_{t'}+t'u_{t'},\ y_{t'}+t'v_{t'})$, which is actually $(x'+t'u', y'+t'v')$, and this is precisely the point where the grasshopper is in the second t'. This completes the proof.

C2

Let n, a, b, c be natural numbers. Every point on the coordinate plane with integer coordinates is colored in one of n colors. Prove there exists c triangles whose vertices are colored in the same color, which are pairwise congruent, and which have a side whose lenght is divisible by a and a side whose lenght is divisible by b.

Solution

Let the colors be $d_1, d_2, d_3 \ldots, d_n$. Look at the coordinates

$$(k, 0+(n+1)abr), (k, ab+(n+1)abr), (k, 2ab+(n+1)abr), \ldots, (k, nab+(n+1)abr)$$

for integers k and r. By the pigeonhole principle there are two points of the same color. For every pair (k,r) we say the color d_i is (k,r)-good if at least two coordinates

$$(k, 0+(n+1)abr), (k, ab+(n+1)abr), (k, 2ab+(n+1)abr), \ldots, (k, nab+(n+1)abr)$$

are colored by color d_i. Fixing r and taking $k = 0, ab, 2ab, \ldots, n^2ab$ get that some color, say d_1, was (k,r)-good for at least $n+1$.

Among the $n+1$ pairs (x,y) there exists two which share the same x cordinate. We call such quadruple r-great. In every r-great quadruple there are two triangle whose vertecies are all the same color and whose two sides are divisible by ab. Taking

$$r = 0, 1, 2, \ldots, n(c(\binom{(n+1)(n^2+1)}{3}) + 1) + 1) + 1$$

we get that there is one color which is in a r-great quadruple for at least

$$c((\binom{(n+1)(n^2+1)}{3}) + 1) + 1$$

different values of r. Let this color be d_1. Since there are less than $\binom{(n+1)(n^2+1)}{3}$ possible triangles in any r-great quadruple (among $c((\binom{(n+1)(n^2+1)}{3}) + 1) + 1$ r-great quadruples with the color d_1) we get that there are $c+1$ triangles which are the same and the same color d_1 and with two sides divisible by ab. This concludes the problem.

C3

Let $n \geq 4$ points in the plane, no three of them are collinear. Prove that the number of parallelograms of area 1, formed by these points, is at most $\frac{n^2-3n}{4}$.

Solution

Fix a direction in the plane. We cannot have three points in the same line parallel to the direction so suppose that in that direction there are k pairs of points, each pair belonging to a parallel line to the fixed direction. Then there are at most $k-1$ parallelograms of area 1 formed by these k pairs of points.

Summing over all directions we get that the number of parallelograms of area 1 are at most $\binom{n}{2}-s$ where s is the number of different directions. But in that way we count every parallelogram two times, so the that the number of parallelograms of area 1 is at most $\frac{\binom{n}{2}-s}{2}$.

We will prove that $s \geq n$. Indeed, taking the convex hull of the n points, let x be a point on the boundary of the convex hull. Because the convex hull has at least three points on its boundary, we can take two points which are neighbors of x in the convex hull, say y, z these points. Then every segment starting from x has different direction from yz. So we have at least $n-1+1=n$ different directions. So the number of parallelograms is at most $\frac{\binom{n}{2}-n}{2}=\frac{n^2-3n}{4}$.

C4

For any set of points $A_1, A_2, \ldots, A_n$ *on* the plane, one defines $r(A_1, A_2, \ldots, A_n)$ as the radius of the smallest circle that contains all of these points. Prove that if $n \geq 3$, there are indices i, j, k such that

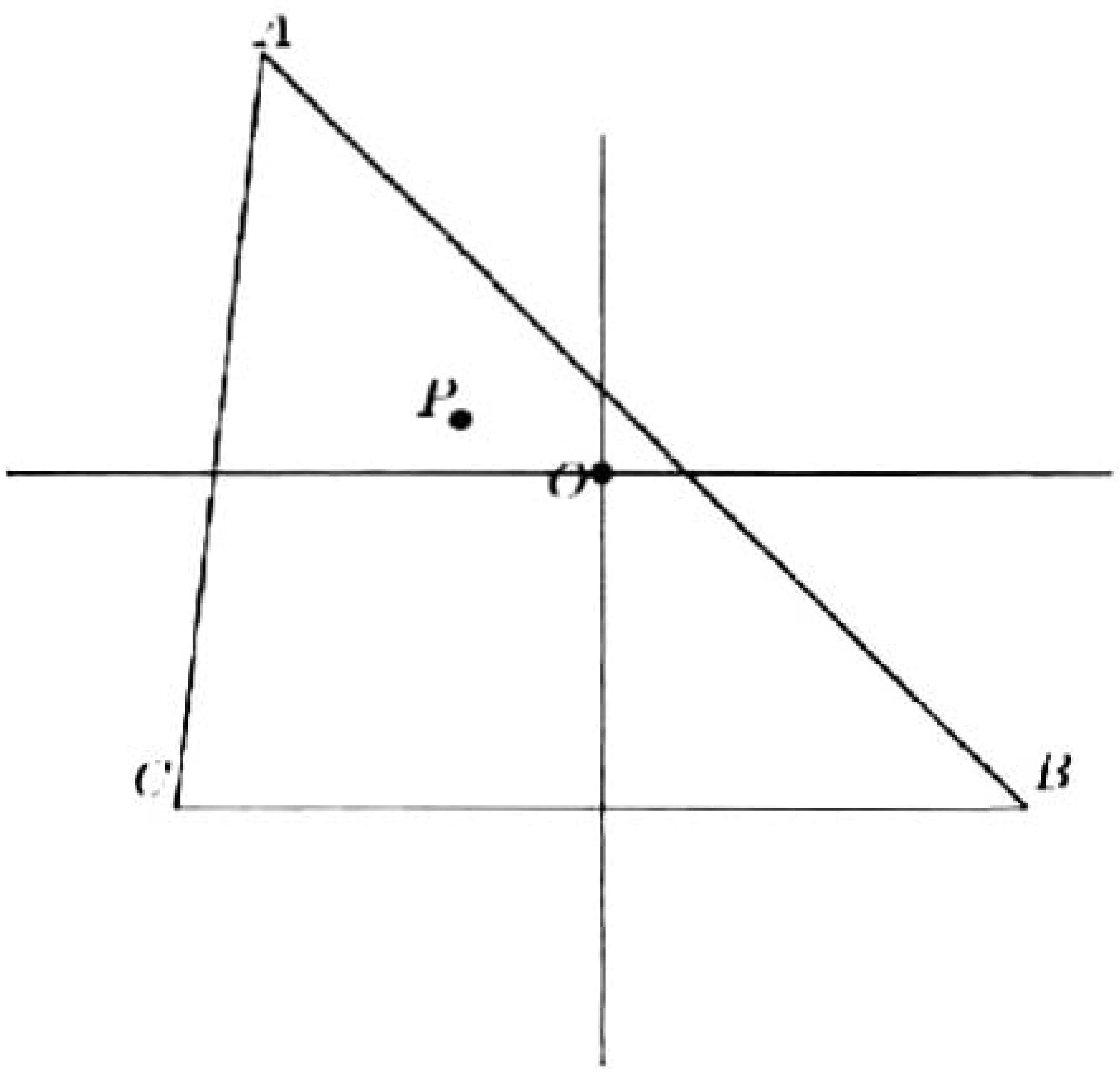
$$r(A_1, A_2, \ldots, A_n) = r(A_i, A_j, A_k).$$

Solution.

We start with a lemma.

Lemma. If the triangle ABC is acute, $r(A, B, C)$ is its circumradius and if it is obtuse, $r(A, B, C)$ is half the length of its longest side.

Proof.



Let us do the acute case first. The circumcircle contains the vertices, so $r(A, B, C)$ is not greater than the circumradius. Now, let us prove that no smaller circle contains all three vertices. If there is a smaller circle, let its center be P. Further, let the circumcenter be O. Since ABC is acute, O is in the interior. Consider the line that passes through O and is parallel to BC. Let us call it l_A and define l_B and l_C similarly. Now, consider the set of points that are on the opposite side of l_A with respect to A. Call this set S_A and define S_B and S_C similarly. It is easily seen (by geometry) that $S_A \cap S_B \cap S_C = \varnothing$. As such, assume $P \notin S_A$ without loss of generality. That is to say, P is on the same side of l_A as A. Now, consider the perpendicular bisector of BC and assume that P, w.l.o.g, is on the same side of this line as C. Under these circumstances, $|PB| \geq |OB|$. Thus, the smaller circle centered at P must exclude B.

In the obtuse case, let $\measuredangle BAC \geq 90°$. Then BC is the longest side. The circle with diameter BC contains all three vertices. Therefore, $r(A, B, C)$ is not greater than $\frac{1}{2}|BC|$. But any smaller circle will clearly exclude at least one of B and C. ∎

Now, let us return to the original problem. Note that there must be points A, B, C among $A_1, A_2, \ldots, A_n$ such that the circumcircle of ABC contains all n points. One can see this as follows: First start with a large circle that contains all n points. Then shrink it while keeping the center fixed, until one of the n points is on the circle and call this point A. Then shrink it keeping the point A in place and moving the center closer to A, until another point B is on the circle. Then keep the line AB fixed while moving the center toward it or away from it so that another C among the n points appears on the circle. It is easy to see that this procedure is doable.

Consider all such triples A, B, C such that the circumcircle of ABC contains all of $A_1, A_2, \dots, A_n$. Now choose the one among them with the smallest circumradius and let it be A_i, A_j, A_k. If $A_iA_jA_k$ is an acute triangle, any smaller circle will exclude one of A_i, A_j, A_k by the lemma above. Therefore,

$$r(A_1, A_2, \dots, A_n) = \textit{circumradius of } A_iA_jA_k = r(A_i, A_j, A_k).$$

If $A_iA_jA_k$ is an obtuse triangle, let A_i be its obtuse angle. We wish to prove that the circle with diameter A_jA_k contains all n points. This will mean that

$$r(A_1, A_2, \dots, A_n) = \frac{1}{2}|A_jA_k| = r(A_i, A_j, A_k)$$

and we will be done. If there are no points on the opposite side of A_jA_k w.r.t. A_i, then this assertion is clear. If there are some points on that side, choose the one X such that $\measuredangle A_jXA_k$ is smallest possible. Then the circumcircle of A_jXA_k contains all n points. However, by the choice of A_i, the circumradius of A_jXA_k cannot be less than that of $A_iA_jA_k$. Thus, $\measuredangle A_jXA_k \geq \measuredangle A_jA_iA_k \geq 90°$. As such, the circle with diameter A_jA_k contains all n points.

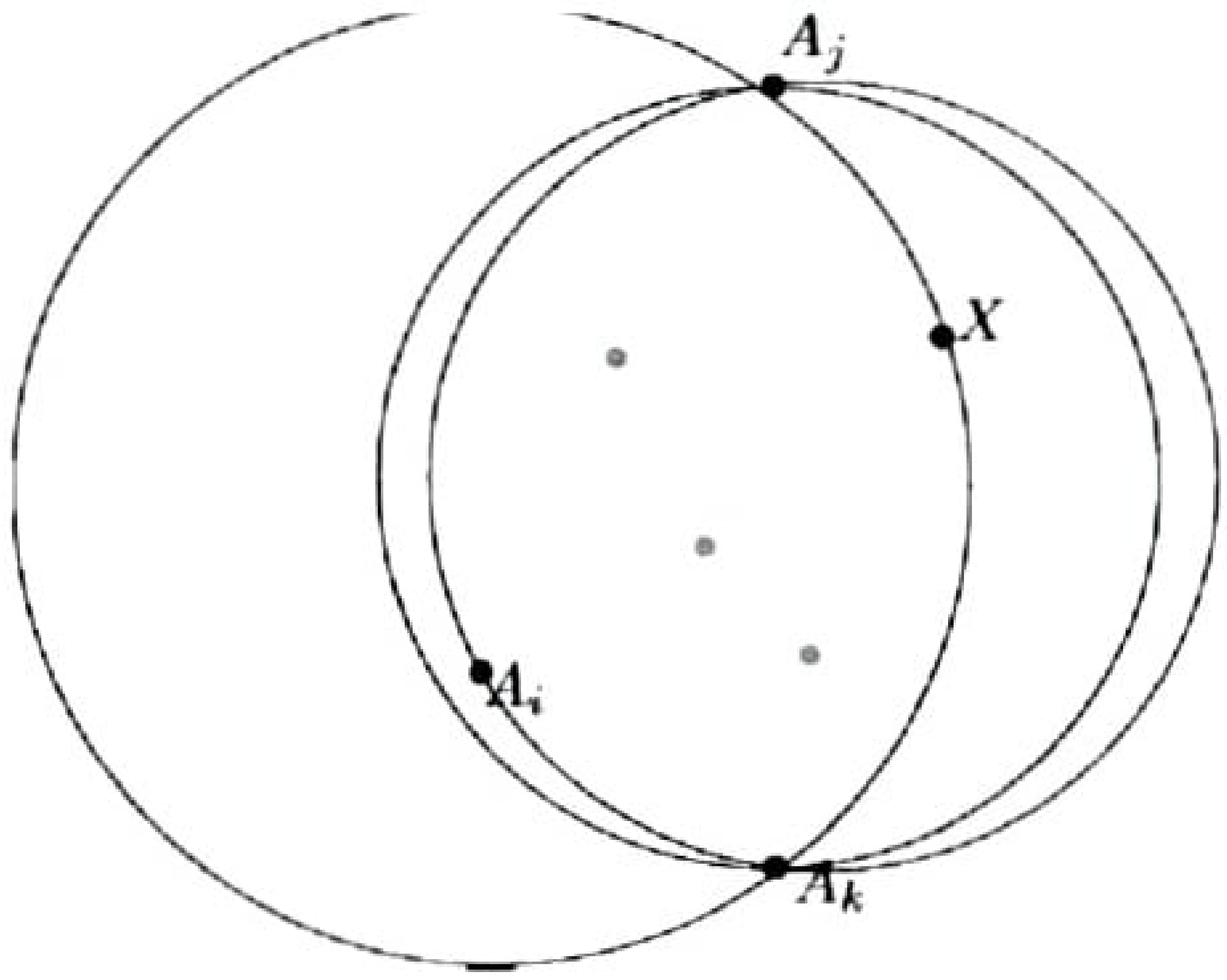


Figure 1: The circumcircles of $A_iA_jA_k$ and A_jXA_k as well as the circle with diameter A_jA_k are shown.

Remark. The problem selection committee recommended formulation of the task to improve.

C5

We have n students sitting at a round table. Initially each student is given one candy. At each step each student having candies either picks one of its candies and gives it to one of its neighbouring students, or distributes all of its candies to its neighbouring students in any way he wishes. A distribution of candies is called legal if it can be reached from the initial distribution via a sequence of steps.

Determine the number of legal distributions. (All the candies are udentical.)

Solution

The answer turns out to be $\binom{2n-1}{n}$ if n is odd and $\binom{2n-1}{n}-2\binom{\frac{3n}{2}-1}{n}$ if n is even.

Case 1. Suppose n is odd, say $n=2m+1$. In this case we will show that any distribution of candies is legal. Thus the number of legal distributions is indeed $\binom{2n-1}{n}$.

In this case we can achieve the above claim by letting each student to always distribute all of its candies to its two neighbouring students in some way. Thus at each step each candy will move either one position clockwise or one anticlockwise.

We now look at the initial distribution of candies and the required final distribution. We specify arbitrarily for each candy in the initial distribution, the position we wish this candy to end up in the required final distribution. Because n is odd, either the clockwise distance or the anticlockwise distance between the initial position of the candy and the required final position is even and at most m.

Thus after an even number of steps (at most m) we can move each candy to its required final position. (Note that if the candy reaches the required position earlier, we can move it back and forth until all candies reach their required position.) This completes the proof of our claim in this case.

Case 2. Suppose n is even, say $n=2m$. Let $x_1,\dots,x_{2m}$ be the students in this cyclic order. Observe that initially the students with even indices (even students) have at least one candy in total, and so do the students with odd indices (odd students). This property is preserved after each step.

We will show that every distribution in which the even students have at least one candy in total and the odd students also have at least one candy in total is legal.

Let us suppose that the required final distribution has a candies in odd positions and b candies in even positions. (Where $a,b\geq 1$.) It will be enough to reach any position with a candies in even positions and b candies in odd positions as then we can follow the same approach as in Case 1.

To achieve this we will first move all candies to students x_1 and x_2. This is easy by specifying that at each step x_1 moves all of its candies to x_2 while for $1\leq r\leq 2m-1$ student x_{r+1} moves all of its candies to x_r.

Suppose that we now have $a+k$ candies at x_1 and $b-k$ candies at x_2 where without loss of generality $k\geq 0$. If $k=0$ we have reached our target. If not, in the next step x_1 moves a candy to x_2 and x_2 moves a candy to x_3. In the next step x_1 (it still has $a+k-1\geq a>0$

candies) moves a candy to x_2, x_2 moves a candy to x_1 and x_3 moves a candy to x_2. We now have $a+k-1$ candies in x_1 and $b+1-k$ in x_2. Repeating this process another $k-1$ times we end up with a candies in x_1 and b candies in x_2 as required.

It remains to count the total number of legal configurations in this case. This is indeed equal to

$$\binom{2n-1}{n} - 2\binom{\frac{3n}{2}-1}{n}$$

as $\binom{2n-1}{n}$ counts the total number of configurations while $\binom{\frac{3n}{2}-1}{n}$ counts the number of illegal configurations where either all n candies belong to the $\frac{n}{2}$ odd positions or all n candies belong to the $\frac{n}{2}$ even positions.

C6

What is the least positive integer k such that, in every convex 101-gon, the sum of any k diagonals is greater than or equal to the sum of the remaining diagonals?

Solution

Let $PQ=1$. Consider a convex 101-gon such that one of its vertices is at P and the remaining 100 vertices are within ε of Q where ε is an arbitrarily small positive real. Let $k+l$ equal the total number $\frac{101\cdot 98}{2}=4949$ of diagonals. When $k\le 4851$, the sum of the k shortest diagonals is arbitrarily small. When $k\ge 4851$, the sum of the k shortest diagonals is arbitrarily close to $k-4851 = 98-l$ and the sum of the remaining diagonals is arbitrarily close to l. Therefore, we need to have $l\le 49$ and $k\ge 4900$.

We proceed to show that $k=4900$ works. To this end, colour all $l=49$ remaining diagonals green. To each green diagonal AB, apart from, possibly, the last one, we will assign two red diagonals AC and CB so that no green diagonal is ever coloured red and no diagonal is coloured red twice.

Suppose that we have already done this for $0\le i\le 48$ green diagonals (thus forming i red-red-green triangles) and let AB be up next. Let D be the set of all diagonals emanating from A or B and distinct from AB: we have $|D|=2\cdot 97=194$. Every red-red-green triangle formed thus far has at most two sides in D. Therefore, the subset E of all as-of-yet-uncoloured diagonals in D contains at least $194-2i$ elements.

When $i\le 47$, $194-2i\ge 100$. The total number of endpoints distinct from A and B of diagonals in D, however, is 99. Therefore, two diagonals in E have a common endpoint C and we can assign AC and CB to AB, as needed.

The case $i=48$ is slightly more tricky: this time, it is possible that no two diagonals in E have a common endpoint other than A and B, but, if so, then there are two diagonals in E that intersect in a point interior to both. Otherwise, at least one (say, a) of the two vertices adjacent to A is cut off from B by the diagonals emanating from A and at least one (say, b) of the two vertices adjacent to B is cut off from A by the diagonals emanating from B (and $a\ne b$). This leaves us with at most 97 suitable endpoints and at least 98 diagonals in E, a contradiction.

By the triangle inequality, this completes the solution.

Union of Mathematicians
of Macedonia