# 2010 Balkan MO Shortlist

– Algebra

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**A1** Let $a, b$ and $c$ be positive real numbers. Prove that

$$\frac{a^2b(b-c)}{a+b}+\frac{b^2c(c-a)}{b+c}+\frac{c^2a(a-b)}{c+a}\geq 0.$$

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**A2** Let the sequence $(a_n)_{n\in\mathbb{N}}$, where $\mathbb{N}$ denote the set of natural numbers, is given with $a_1=2$ and $a_{n+1}=a_n^2-a_n+1$. Find the minimum real number $L$, such that for every $k\in\mathbb{N}$

$$\sum_{i=1}^{k}\frac{1}{a_i}<L$$

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**A3** Let $a, b, c, d$ be positive real numbers. Prove that

$$(\frac{a}{a+b})^5+(\frac{b}{b+c})^5+(\frac{c}{c+d})^5+(\frac{d}{d+a})^5\geq\frac{1}{8}$$

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**A4** Let $n>2$ be a positive integer. Consider all numbers $S$ of the form

$$S=a_1a_2+a_2a_3+\ldots+a_{k-1}a_k$$

with $k>1$ and $a_i$ begin positive integers such that $a_1+a_2+\ldots+a_k=n$. Determine all the numbers that can be represented in the given form.

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**C1** In a soccer tournament each team plays exactly one game with all others. The winner gets $3$ points, the loser gets $0$ and each team gets $1$ point in case of a draw. It is known that $n$ teams ($n \geq 3$) participated in the tournament and the final classification is given by the arithmetical progression of the points, the last team having only 1 point.

- Prove that this configuration is unattainable when $n = 12$
- Find all values of $n$ and all configurations when this is possible

**C2** A grasshopper jumps on the plane from an integer point (point with integer coordinates) to another integer point according to the following rules: His first jump is of length $\sqrt{98}$, his second jump is of length $\sqrt{149}$, his next jump is of length $\sqrt{98}$, and so on, alternatively. What is the least possible odd number of moves in which the grasshopper could return to his starting point?

**C3** A strip of width $w$ is the set of all points which lie on, or between, two parallel lines distance $w$ apart. Let $S$ be a set of $n$ ($n \geq 3$) points on the plane such that any three different points of $S$ can be covered by a strip of width $1$.
Prove that $S$ can be covered by a strip of width $2$.

**C4** Integers are written in the cells of a table $2010 \times 2010$. Adding $1$ to all the numbers in a row or in a column is called a *move*. We say that a table is *equilibrium* if one can obtain after finitely many moves a table in which all the numbers are equal.

-Find the largest positive integer $n$, for which there exists an *equilibrium* table containing the numbers $2^0, 2^1, \ldots, 2^n$.
- For this $n$, find the maximal number that may be contained in such a table.

**C5** A train consist of $2010$ wagons containing gold coins, all of the same shape. Any two coins have equal weight provided that they are in the same wagon, and differ in weight if they are in different ones. The weight of a coin is one of the positive reals

$$m_1 < m_2 < \ldots < m_{2010}$$

Each wagon is marked by a label with one of the numbers $m_1, m_2, \ldots, m_{2010}$ (the numbers on different labels are different).

A controller has a pair of scales (allowing only to compare masses) at his disposal. During each measurement he can use an arbitrary number of coins from any of the wagons. The controller has the task to establish: if all labels show rightly the common weight of the coins in a wagon or if there exists at least one wrong label. What is the least number of measurement that the controller has to perform to accomplish his task?

**G1** Let $ABCDE$ be a pentagon with $\hat{A} = \hat{B} = \hat{C} = \hat{D} = 120°$. Prove that $4 \cdot AC \cdot BD \geq 3 \cdot AE \cdot ED$.

**G2** Consider a cyclic quadrilateral such that the midpoints of its sides form another cyclic quadrilateral. Prove that the area of the smaller circle is less than or equal to half the area of the bigger circle

**G3** The incircle of a triangle $A_0B_0C_0$ touches the sides $B_0C_0, C_0A_0, A_0B_0$ at the points $A, B, C$ respectively, and the incircle of the triangle $ABC$ with incenter $I$ touches the sides $BC, CA, AB$ at the points $A_1, B_1, C_1$, respectively. Let $\sigma(ABC)$ and $\sigma(A_1B_1C)$ be the areas of the triangles $ABC$ and $A_1B_1C$ respectively. Show that if $\sigma(ABC) = 2\sigma(A_1B_1C)$, then the lines $AA_0, BB_0, IC_1$ pass through a common point .

**G4** Let $ABC$ be a given triangle and $\ell$ be a line that meets the lines $BC, CA$ and $AB$ in $A_1, B_1$ and $C_1$ respectively. Let $A'$ be the midpoint, of the segment connecting the projections of $A_1$ onto the lines $AB$ and $AC$. Construct, analogously the points $B'$ and $C'$.
(a) Show that the points $A', B'$ and $C'$ are collinear on some line $\ell'$.
(b) Show that if $\ell$ contains the circumcenter of the triangle $ABC$, then $\ell'$ contains the center of it's Euler circle.

**G5** Let $ABC$ be an acute triangle with orthocentre $H$, and let $M$ be the midpoint of $AC$. The point $C_1$ on $AB$ is such that $CC_1$ is an altitude of the triangle $ABC$. Let $H_1$ be the reflection of $H$ in $AB$. The orthogonal projections of $C_1$ onto the lines $AH_1$, $AC$ and $BC$ are $P$, $Q$ and $R$, respectively. Let $M_1$ be the point such that the circumcentre of triangle $PQR$ is the midpoint of the segment $MM_1$.
Prove that $M_1$ lies on the segment $BH_1$.

**G6** In a triangle $ABC$ the excircle at the side $BC$ touches $BC$ in point $D$ and the lines $AB$ and $AC$ in points $E$ and $F$ respectively. Let $P$ be the projection of $D$ on $EF$. Prove that the circumcircle $k$ of the triangle $ABC$ passes through $P$ if and only if $k$ passes through the midpoint $M$ of the segment $EF$.

**G7** A triangle $ABC$ is given. Let $M$ be the midpoint of the side $AC$ of the triangle and $Z$ the image of point $B$ along the line $BM$. The circle with center $M$ and radius $MB$ intersects the lines $BA$ and $BC$ at the points $E$ and $G$ respectively. Let $H$ be the point of intersection of $EG$ with the line $AC$, and $K$ the point of intersection of $HZ$ with the line $EB$. The perpendicular from point $K$ to the line $BH$ intersects the lines $BZ$ and $BH$ at the points $L$ and $N$, respectively.
If $P$ is the second point of intersection of the circumscribed circles of the triangles $KZL$ and $BLN$, prove that, the lines $BZ, KN$ and $HP$ intersect at a common point.

**G8** Let $c(0, R)$ be a circle with diameter $AB$ and $C$ a point, on it different than $A$ and $B$ such that $\angle AOC > 90^o$. On the radius $OC$ we consider the point $K$ and the circle $(c_1)$ with center $K$ and radius $KC = R_1$. We draw the tangents $AD$ and $AE$ from $A$ to the circle $(c_1)$. Prove that the straight lines $AC, BK$ and $DE$ are concurrent

| – | Number Theory |
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| **N1** | Determine whether it is possible to partition $\mathbb{Z}$ into triples $(a, b, c)$ such that, for every triple, $\lvert a^3b + b^3c + c^3a \rvert$ is perfect square. |
| **N2** | Solve the following equation in positive integers: $x^3 = 2y^2 + 1$ |
| **N3** | For each integer $n$ ($n \geq 2$), let $f(n)$ denote the sum of all positive integers that are at most $n$ and not relatively prime to $n$.<br>Prove that $f(n+p) \neq f(n)$ for each such $n$ and every prime $p$. |