Chapter 1

2009 Shortlist JBMO - Problems

1.1 Algebra

A1 Determine all integers a, b, c satisfying identities:

$$a+b+c=15$$

$$(a-3)^3+(b-5)^3+(c-7)^3=540.$$

A2 Find the maximum value of $z+x$, if (x,y,z,t) satisfies the conditions:

$$\begin{cases} x^2+y^2=4 \\ z^2+t^2=9 \\ xt+yz\geq 6 \end{cases}$$

A3 Find all values of the real parameter a, for which the system

$$(|x|+|y|-2)^2=1$$

$$y=ax+5$$

has exactly three solutions.

A4 Real numbers x,y,z satisfy

$$0<x,y,z<1$$

and

$$xyz=(1-x)(1-y)(1-z).$$

Show that

$$\frac{1}{4}\leq \max\{(1-x)y,(1-y)z,(1-z)x\}.$$

A5 Let x,y,z be positive real numbers. Prove that:

$$(x^2+y+1)(x^2+z+1)(y^2+z+1)(y^2+x+1)(z^2+x+1)(z^2+y+1)\geq (x+y+z)^6.$$

1.2 Combinatorics

C1 We have chosen 2009 distinct points in plane, and colored them in blue or red. On each blue - centered unit circle there are exactly two red points. Find the greatest possible number of blue points.

C2 Five players (A, B, C, D, E) take part in a bridge tournament. Every two players must play (as partners) against every other two players. Any two given players can be partners not more than once per a day. What is the least number of days needed for this tournament?

C3 a) In how many ways can we read the word SARAJEVO from the table below, if it is allowed to jump from cell to an adjacent cell (by vertex or a side) cell?
b) After the letter in one cell was deleted, only 525 ways to read the word SARAJEVO remained. Find all possible positions of that cell.

C4 Determine all pairs of (m, n) such that is possible to tile the table $m \times n$ with figure "corner" as in figure with condition that in that tilling does not exist rectangle (except $m \times n$) regularly covered with figures.

1.3 Geometry

G1 Parallelogram $ABCD$ is given with $AC > BD$, and O point of intersection of AC and BD. Circle with center at O and radius OA intersects extensions of AD and AB at points G and L, respectively. Let Z be intersection point of lines BD and GL. Prove that $\angle ZCA = 90^\circ$.

G2 In right trapezoid $ABCD$ ($AB \parallel CD$) the angle at vertex B measures 75°. Point H is the foot of the perpendicular from point A to the line BC. If $BH = DC$ and $AD + AH = 8$, find the area of $ABCD$.

G3 Parallelogram $ABCD$ with obtuse angle $\angle ABC$ is given. After rotation of the triangle ACD around the vertex C, we get a triangle $CD'A'$, such that points B, C and D' are collinear. Extensions of median of triangle $CD'A'$ that passes through D' intersects the straight line BD at point P. Prove that PC is the bisector of the angle $\angle BPD'$.

G4 Let $ABCDE$ be convex pentagon such that $AB + CD = BC + DE$ and k half circle with center on side AE that touches sides AB, BC, CD and DE of pentagon, respectively, at points P, Q, R and S (different from vertices of pentagon). Prove that

$PS \parallel AE$.

G5 Let A, B, C and O be four points in plane, such that $\angle ABC > 90^\circ$ and $OA = OB = OC$. Define the point $D \in AB$ and the line l such that $D \in l$, $AC \perp DC$ and $l \parallel AO$. Line l cuts AC at E and circumcircle of $\triangle ABC$ at F. Prove that the circumcircles of triangles BEF and CFD are tangent at F.

1.4 Number Theory

NT1 Determine all positive integer numbers k for which the numbers $k+9$ are perfect squares and the prime factors of k have to be only 2 and 3.

NT2 A group of $n > 1$ pirates of different age owned total of 2009 coins. Initially each pirate (except the youngest one) had one coin more than the next younger.

a) Find all possible values of n.
b) Every day a pirate was chosen. The chosen pirate gave a coin to each of the other pirates. If $n = 7$, find the largest possible number of coins a pirate can have after several days.

NT3 Find all pairs (x, y) of integers which satisfy the equation

$$(x+y)^2(x^2+y^2) = 2009^2.$$

NT4 Determine all prime numbers $p_1, p_2, \ldots, p_{12}, p_{13}$, $p_1 \leq p_2 \leq \ldots \leq p_{12} \leq p_{13}$, such that

$$p_1^2 + p_2^2 + \ldots + p_{12}^2 = p_{13}^2$$

and one of them is equal to $2p_1 + p_9$.

NT5 Show that there are infinitely many positive integers c, such that the following equations both have solutions in positive integers:

$$(x^2 - c)(y^2 - c) = z^2 - c$$

and

$$(x^2 + c)(y^2 - c) = z^2 - c.$$

Chapter 2

2009 Shortlist JBMO - Solutions

2.1 Algebra

A1 Determine all integers a, b, c satisfying identities:

$$a + b + c = 15$$

$$(a-3)^3 + (b-5)^3 + (c-7)^3 = 540.$$

Solution I: We will use the following fact:
Lemma: If x, y, z are integers such that

$$x + y + z = 0$$

than

$$x^3 + y^3 + z^3 = 3xyz.$$

Proof: Let

$$x + y + z = 0.$$

Than we have

$$x^3 + y^3 + z^3 = x^3 + y^3 + (-x-y)^3 = x^3 + y^3 - x^3 - y^3 - 3xy(x+y) = 3xyz.$$

$\blacksquare$

Now, from

$$a + b + c = 15$$

we obtain:

$$(a-3) + (b-5) + (c-7) = 0.$$

Using lemma and problem condition we get:

$$540 = (a-3)^3 + (b-5)^3 + (c-7)^3 = 3(a-3)(b-5)(c-7).$$

Now,

$$(a-3)(b-5)(c-7) = 180 = 2 \times 2 \times 3 \times 3 \times 5.$$

Since

$$(a-3) + (b-5) + (c-7) = 0$$

only possibility for the product $(a-3)(b-5)(c-7)$ is $4 \times 5 \times 9$.
Finally, we obtain following system of equations:

$$\begin{cases} a-3=-4 \\ b-5=-5, \\ c-7=9 \end{cases} \begin{cases} a-3=-5 \\ b-5=-4 \\ c-7=9 \end{cases} \begin{cases} a-3=-4 \\ b-5=-5, \\ c-7=9 \end{cases} \begin{cases} a-3=-5 \\ b-5=-4. \\ c-7=9 \end{cases}$$

From here we get:

$$(a,b,c) = \{(-1,0,16), (-2,1,16), (7,10,-2), (8,9,-2)\}.$$

Solution II: We use substitution $a-3=x$, $b-5=y$, $c-7=z$.
Now, equations are transformed to:

$$x+y+z=0$$

$$x^3+y^3+z^3=540.$$

Substituting $z=-x-y$ in second equation, we get:

$$-3xy^2-2x^2y=540$$

or

$$xy(x+y)=180$$

or

$$xyz=180.$$

Returning to starting problem we have:

$$(a-3)(b-5)(c-7)=180.$$

Solution proceeds as previous one.

A2 Find the maximum value of $z+x$, if (x,y,z,t) satisfies the conditions:

$$\begin{cases} x^2+y^2=4 \\ z^2+t^2=9 \\ xt+yz\geq 6 \end{cases}$$

Solution I: From the conditions we have

$$(x^2+y^2)(z^2+t^2)=36=(xt+yz)^2+(xz-yt)^2\geq 36+(xz-yt)^2$$

and this implies $xz-yt=0$.
Now it is clear that

$$x^2+z^2+y^2+t^2=(x+z)^2+(y-t)^2=13$$

and the maximum value of $z+x$ is $\sqrt{13}$. It is achieved for $x=\frac{4}{\sqrt{13}}$, $y=t=\frac{6}{\sqrt{13}}$ and $z=\frac{9}{\sqrt{13}}$.
Solution II: From inequality $xt+yz\geq 6$ and problem conditions we have:

$$(xt+yz)^2-36\geq 0\Leftrightarrow$$

$$(xt+yz)^2-(x^2+y^2)(z^2+t^2)\geq 0\Leftrightarrow$$

$$2xyzt-x^2y^2-y^2t^2\geq 0\Leftrightarrow$$

$$-(xz-yt)^2\geq 0.$$

From here we have $xz=yt$.
Furthermore,

$$x^2+y^2+z^2+t^2=(x+z)^2+(y-t)^2=13,$$

and it follows that

$$(x+z)^2\leq 13.$$

Thus,

$$x+z\leq\sqrt{13}.$$

Equality $x+z=\sqrt{13}$ eventually holds if we have $y=t$ and $z^2-x^2=5(z-x=\frac{5}{\sqrt{13}})$.
Therefore, $x=\frac{4}{\sqrt{13}}$, $y=t=\frac{6}{\sqrt{13}}$, $z=\frac{9}{\sqrt{13}}$.
A3 Find all values of the real parameter a, for which the system

$$(|x|+|y|-2)^2=1$$

$$y=ax+5$$

has exactly three solutions.
Solution: The first equation is equivalent to

$$|x| + |y| = 1$$

or

$$|x| + |y| = 3.$$

The graph of the first equation is symmetric with respect to both axes. In the first quadrant it is reduced to $x + y = 1$, whose graph is segment connecting points (1,0) and (0,1). Thus, the graph of

$$|x| + |y| = 1$$

is square with vertices (1,0), (0,1), (-1,0) and (0,-1). Similarly, the graph of

$$|x| + |y| = 3$$

is a square with vertices (3,0), (0,3), (-3,0) and (0,-3). The graph of the second equation of the system is a straight line with slope a passing through (0,5). This line intersects the graph of the first equation in three points exactly, when passing through one of the points (1,0) or (-1,0). This happens if and only if $a = 5$ or $a = -5$.
A4 Real numbers x, y, z satisfy

$$0 < x, y, z < 1$$

and

$$xyz = (1 - x)(1 - y)(1 - z).$$

Show that

$$\frac{1}{4} \leq \max\{(1 - x)y, (1 - y)z, (1 - z)x\}.$$

Solution: It is clear that $a(1-a) \leq \frac{1}{4}$ for any real numbers a (equivalent to $0 < (2a-1)^2$). Thus,

$$xyz = (1 - x)(1 - y)(1 - z)$$

$$(xyz)^2 = [x(1 - x)][y(1 - y)][z(1 - z)] \leq \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$xyz \leq \frac{1}{2^3}.$$

It implies that at least one of x, y, z is at less or equal to $\frac{1}{2}$. Let us say that $x \leq \frac{1}{2}$, and notice that $1 - x \geq \frac{1}{2}$.
Assume contrary to required result, that we have

$$\frac{1}{4} > \max\{(1 - x)y, (1 - y)x, (1 - z)x\}.$$

Now,
$$(1-x)y < \frac{1}{4}, \qquad (1-y)z < \frac{1}{4}, \qquad (1-z)x < \frac{1}{4}.$$
From here we deduce:
$$y < \frac{1}{4} \cdot \frac{1}{1-x} \leq \frac{1}{4} \cdot 2 = \frac{1}{2}.$$
Notice that $1 - y < \frac{1}{2}$.
Using same reasoning we conclude:
$$z < \frac{1}{2}, \qquad 1 - z > \frac{1}{2}.$$
Using these facts we derive:
$$\frac{1}{8} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} > xyz = (1-x)(1-y)(1-z) > \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$
Contradiction!
Remark: *The exercise along with its proof generalizes for any given (finite) number of numbers, and you can consider this new form in place of the proposed one:*
Exercise: *If for the real numbers $x_1, x_2, \ldots, x_n$ it is $0 < x_i < 1$, for all indices i, and*
$$x_1 x_2 \ldots x_n = (1-x_1)(1-x_2)\ldots(1-x_n),$$
show that
$$\frac{1}{4} \leq (1-x_i)x_{i+1} (\forall i)$$
(where $x_{n+1} = x_1$).
Or you can consider the following variation:
Exercise: *If for the real numbers $x_1, x_2, \ldots, x_{2009}$ it is $0 < x_i < 1$, for all indices i, and*
$$x_1 x_2 \ldots x_{2009} = (1-x_1)(1-x_2)\ldots(1-x_{2009}),$$
show that
$$\frac{1}{4} \leq (1-x_i)x_{i+1} (\forall i)$$
(where $x_{2010} = x_1$).
A5 Let x, y, z be positive real numbers. Prove that:
$$(x^2+y+1)(x^2+z+1)(y^2+z+1)(y^2+x+1)(z^2+x+1)(z^2+y+1) \geq (x+y+z)^6.$$

Solution I: Applying Cauchy-Schwarz's inequality:
$$(x^2+y+1)(z^2+y+1) = (x^2+y+1)(1+y+z^2) \geq (x+y+z)^2.$$

Using the same reasoning we deduce:

$$(x^2+z+1)(y^2+z+1) \geq (x+y+z)^2$$

and

$$(y^2+x+1)(z^2+x+1) \geq (x+y+z)^2.$$

Multiplying these three inequalities we get desired result.
Solution II: We have

$$(x^2+y+1)(z^2+y+1) \geq (x+y+z)^2 \Leftrightarrow$$

$$x^2z^2+x^2y+x^2+yz^2+y^2+y+z^2+y+1 \geq x^2+y^2+z^2+2xy+2yz+2zx \Leftrightarrow$$

$$(x^2z^2-2zx+1)+(x^2y-2xy+y)+(yz^2-2yz+y) \geq 0 \Leftrightarrow$$

$$(xz-1)^2+y(x-1)^2+y(z-1)^2 \geq 0$$

which is correct.
Using the same reasoning we get:

$$(x^2+z+1)(y^2+z+1) \geq (x+y+z)^2$$

$$(y^2+x+1)(z^2+x+1) \geq (x+y+z)^2.$$

Multiplying these three inequalities we get desired result. Equality is attained at $x = y = z = 1$.

2.2 Combinatorics

C1 We have chosen 2009 distinct points in plane, and colored them in blue or red. On each blue - centered unit circle there are exactly two red points. Find the greatest possible number of blue points.
Solution: Each pair of red points can belong to at most two unit blue - centered unit circles. As n red points form $\frac{n(n-1)}{2}$ pairs, we can have not more than twice that number of blue points, i.e. $n(n-1)$ blue points. Thus, the total number of points can not exceed

$$n+n(n-1)=n^2.$$

As $44^2 < 2009$, n must be at least 45. We can arrange 45 distinct red points on a segment of length 1, and color blue all but 16 ($= 45^2 - 2009$) points on intersections of the red - centered unit circles (all points of intersection are distinct, as no blue - centered unit circle can intersect the segment more than twice). Thus, the greatest possible number of blue points is 2009 - 45 = 1964.

C2 Five players (A, B, C, D, E) take part in a bridge tournament. Every two players must play (as partners) against every other two players. Any two given players can be partners not more than once per a day. What is the least number of days needed for this tournament?

Solution: A given pair must play with three other pairs and these plays must be in different days, so at three days are needed. Suppose that three days suffice. Let the pair AB play against CD on day x. Than $AB - DE$ and $CD - BE$ cannot play on day x. Then one of the other two plays of DE (with AC and BC) must be on day x. Similarly, one of the plays of BE with AC or AD must be on day x. Thus, two of the plays in the chain $BC - DE - AC - BE - AD$ are on day x (more than two among these cannot be on one day).

Consider the chain $AB - CD - EA - BD - CE - AB$. At least three days are needed for playing all the matches within it. For each of these days we conclude (as above) that there are exactly two of the plays in the chain $BC - DE - AC - BE - AD - BC$ on that day. This is impossible, as this chain consists of five plays.

It remains to show that four days will suffice:

Day 1: $AB - CD$, $AC - DE$, $AD - CE$, $AE - BC$

Day 2: $AB - DE$, $AC - BD$, $AD - BC$, $BE - CD$

Day 3: $AB - CE$, $AD - BE$, $AE - BD$, $BC - DE$

Day 4: $AC - BE$, $AE - CD$, $BD - CE$.

C3 a) In how many ways can we read the word SARAJEVO from the table below, if it is allowed to jump from cell to an adjacent cell (by vertex or a side) cell?

b) After the letter in one cell was deleted, only 525 ways to read the word SARAJEVO remained. Find all possible positions of that cell.

Solution: In the first of the tables below the number in each cell shows the number of ways to reach that cell from the start (which is the sum of the quantities in the cells, from which we can come), and in the second one are the number of ways to arrive from that cell to the end (which is the sum of the quantities in the cells, to which we can go).

a) The answer is 750, as seen from the second table.

b) If we delete the letter in a cell, the number of ways to read SARAJEVO will decrease by the product of the numbers in the corresponding cell in the two tables. As $750 - 525 = 225$, this product has to be 225. This happens only for two cells on the third row. Here is the table with the products:

C4 Determine all pairs of (m, n) such that is possible to tile the table $m \times n$ with figure

”corner” as in figure with condition that in that tilling does not exist rectangle (except $m \times n$) regularly covered with figures.

Solution: Every ”corner” covers exactly 3 squares, so necessary condition that tilling exists is $3|mn$.

First, we shall prove that for tilling with our condition it is necessary that both m, n for $m, n > 3$ must be even. Suppose contrary, that $m > 3$ is odd (without losing generality). Look at ”corners” that cover squares on side of length m of table $m \times n$. Because m is odd, there must be ”corner” which covers exactly one square on side. But any placement of that corner forces existence of 2×3 rectangle in tilling. Thus, m and n for $m, n > 3$ must be even and at least one among them is divisible by 3.

Notice that in corners of table $m \times n$ ”corner” must be placed as in figure.

If one of m and n is 2 then condition forces that only table is 2×3 or 3×2. If we try to find desired tilling when $m = 4$ then we are forced to stop at table 4×6 because of the conditions of problem.

We easily find example of desired tilling for table 6×6 and tilling for $6 \times 2k$.

Thus, it will be helpful to prove that desired tilling exists for tables $6k \times 4l$, for $k, l \geq 2$. Divide that table at rectangle 6×4 and tile that rectangle as we described. Now, change placement of problematic ”corners” as in figure.

Thus, we get desired tilling for this type of table.

Similarly, we prove existence in case $6k \times (4k + 2)$ where $m, l \geq 2$. But, we first divide table at two tables $6k \times 6$ and $6k \times 4(l-1)$. Divide them at rectangles 6×6 and 6×4. Tile them as we described earlier, and arrange problematic ”corners” as in previous case.

So, 2×3, 3×2, $6 \times 2k$, $2k \times 6k \geq 2$, and $6k \times 4l$ for $k, l \geq 2$ and $6k \times (4l + 2)$ for $k, l \geq 2$.

Remark: Problem is inspired by problem given at Romanian Selection Test 2000, but it is completely different.

Remark: Alternatively, problem can be relaxed with question: ”Does such tilling exist for concrete values of m and n”.

2.3 Geometry

G1 Parallelogram $ABCD$ is given with $AC > BD$, and O point of intersection of AC and BD. Circle with center at O and radius OA intersects extensions of AD and AB at points G and L, respectively. Let Z be intersection point of lines BD and GL. Prove that $\angle ZCA = 90°$.

Solution: From the point L we draw parallel line to BD that intersects lines AC and

AG at points N and R respectively. Since $DO = OB$, we have that $NR = NL$, and point N is the midpoint of segment LR.
Let K be the midpoint of GL. Now, $NK \parallel RG$, and

$$\angle AGL = \angle NKL = \angle ACL.$$

Therefore, from the cyclic quadrilateral $NKCL$ we deduce:

$$\angle KCN = \angle KLN.$$

Now, since $LR \parallel DZ$, we have

$$\angle KLN = \angle KZO.$$

It implies that quadrilateral $OKCZ$ is cyclic, and

$$\angle OKZ = \angle OCZ.$$

Since $OK \perp GL$, we derive that $\angle ZCA = 90^\circ$.
G2 In right trapezoid $ABCD$ ($AB \parallel CD$) the angle at vertex B measures 75°. Point H is the foot of the perpendicular from point A to the line BC. If $BH = DC$ and $AD + AH = 8$, find the area of $ABCD$.
Solution: Continue the legs of the trapezoid until they intersect at point E. The triangles ABH and ECD are congruent (ASA). The area of $ABCD$ is equal to area of triangle EAH of hypotenuse

$$AE = AD + DE = AD + AH = 8.$$

Let M be the midpoint of AE. Then

$$ME = MA = MH = 4$$

and $\angle AMH = 30^\circ$. Now, the altitude from H to AM equals one half of MH, namely 2. Finally, area is 8.
G3 Parallelogram $ABCD$ with obtuse angle $\angle ABC$ is given. After rotation of the triangle ACD around the vertex C, we get a triangle $CD'A'$, such that points B, C and D' are collinear. Extensions of median of triangle $CD'A'$ that passes through D' intersects the straight line BD at point P. Prove that PC is the bisector of the angle $\angle BPD'$.
Solution: Let $AC \cap BD = \{X\}$ and $PD' \cap CA' = \{Y\}$. Because $AX = CX$ and $CY = YA'$, we deduce:

$$\triangle ABC \cong \triangle CDA \cong \triangle CD'A' \Rightarrow \triangle ABX \cong \triangle CD'Y, \triangle BCX \cong \triangle D'A'Y.$$

It follows that

$$\angle ABX = \angle CD'Y.$$

Let M and N be orthogonal projections of the point C on the straight lines PD' and BP, respectively, and Q is the orthogonal projection of the point A on the straight line BP. Because $CD' = AB$, we have that $\triangle ABQ \cong \triangle CD'M$.
We conclude that $CM = AQ$. But, $AX = CX$ and $\triangle AQX \cong \triangle CNX$. So, $CM = CN$ and PC is the bisector of the angle $\angle BPD'$.
G4 Let $ABCDE$ be convex pentagon such that $AB + CD = BC + DE$ and k half circle with center on side AE that touches sides AB, BC, CD and DE of pentagon, respectively, at points P, Q, R and S (different from vertices of pentagon). Prove that $PS \parallel AE$.
Solution: Let O be center of k. We deduce $BP = BQ$, $CQ = CR$, $DR = DS$, since those are tangent lines of circle k. Using the condition $AB + CD = BC + DE$, we derive:

$$AP + BP + CR + DR = BQ + CQ + DS + ES.$$

From here we have $AP = ES$.
Thus,

$$\triangle APO \cong \triangle ESO (AP = ES, \angle APO = \angle ESO = 90^\circ, PO = SO).$$

It implies,

$$\angle OPS = \angle OEP.$$

Therefore,

$$\angle APS = \angle APO + \angle OPS = 90^\circ + \angle OPS = 90^\circ + \angle OSP = \angle PSE.$$

Now, from quadrilateral $APSE$ we deduce:

$$2\angle EAP + 2\angle APS = \angle EAP + \angle APS + \angle PSE + \angle SEA = 360^\circ.$$

So,

$$\angle EAP + \angle APS = 180^\circ$$

and $APSE$ is isosceles trapezoid. Therefore, $AB \parallel PS$.
G5 Let A, B, C and O be four points in plane, such that $\angle ABC > 90^\circ$ and $OA = OB = OC$. Define the point $D \in AB$ and the line l such that $D \in l$, $AC \perp DC$ and $l \parallel AO$. Line l cuts AC at E and circumcircle of $\triangle ABC$ at F. Prove that the circumcircles of triangles BEF and CFD are tangent at F.

Solution: Let $l \cap AC = \{K\}$ and define G to be the mirror image of the point A with respect to O. Then AG is a diameter of the circumcircle of the triangle ABC, therefore $AC \perp CG$. On the other hand we have $AC \perp DC$, and it implies that points D, C, G are collinear.
Moreover, as $AE \perp DG$ and $DE \perp AG$, we obtain that E is the orthocenter of triangle ADG and $GE \perp AD$. As AG is a diameter, we have $AB \perp BG$, and since $AD \perp GE$, the points E, G, and B are collinear.
Notice that

$$\angle CAG = 90^\circ - \angle AGC = \angle KDC$$

and

$$\angle CAG = \angle GFC,$$

since both subtend the same arc.
Hence,

$$\angle FDG = \angle GFC.$$

Therefore, GF is tangent to the circumcircle of the triangle CDF at point F.
We claim that line GF is also tangent to the circumcircle of triangle BEF at point F, which concludes the proof.
The claim is equivalent to $\angle GBF = \angle EFG$. Denote by F' the second intersection point - other than F - of line l with the circumcircle of triangle ABC. Observe that $\angle GBF = \angle GF'F$, because both angles subtend the same arc, and $\angle FF'G = \angle EFG$, since AG is the bisector line of the chord FF', and we are done.

2.4 Number Theory

NT1 Determine all positive integer numbers k for which the numbers $k+9$ are perfect squares and the prime factors of k have to be only 2 and 3.
Solution: We have an integer x such that

$$x^2 = k + 9$$

$k = 2^a 3^b, a, b \geq 0, a, b \in \mathbb{Z}$.
Therefore,

$$(x-3)(x+3) = k.$$

If $b = 0$ then we have $k = 16$.
If $b > 0$ then we have $3|k+9$. Hence, $3|x^2$ and $9|k$.

Therefore, we have $b \geq 2$. Let $x = 3y$.

$$(y-1)(y+1) = 2^a 3^{b-2}.$$

If $a = 0$ then $b = 3$ and we have $k = 27$.
If $a \geq 1$, than numbers $y-1$ and $y+1$ are even numbers. Therefore, we have $a \geq 2$, and

$$\frac{y-1}{2} \cdot \frac{y+1}{2} = 2^{a-2} 3^{b-2}.$$

Since the numbers $\frac{y-1}{2}$, $\frac{y+1}{2}$ are consecutive numbers, these numbers have to be powers of 2 and 3. Let $m = a-2$, $n = b-2$.
• If $2^m - 3^m = 1$ then we have $m \geq n$. For $n = 0$ we have $m = 1$, $a = 3$, $b = 2$ and $k = 72$. For $n > 0$ using $\mod 3$ we have that m is even number. Let $m = 2t$. Therefore,

$$(2^t - 1)(2^t + 1) = 3^n.$$

Hence, $t = 1$, $m = 2$, $n = 1$ and $a = 4$, $b = 3$, $k = 432$.
• If $3^n - 2^m = 1$, then $m > 0$. For $m = 1$ we have $n = 1$, $a = 3$, $b = 3$, $k = 216$. For $m > 1$ using $\mod 4$ we have that n is even number. Let $n = 2t$.

$$(3^t - 1)(3^t + 1) = 2^m.$$

Therefore, $t = 1$, $n = 2$, $m = 3$, $a = 5$, $b = 4$, $k = 2572$.
Set of solutions: $\{16, 27, 72, 216, 432, 2592\}$.
NT2 A group of $n > 1$ pirates of different age owned total of 2009 coins. Initially each pirate (except the youngest one) had one coin more than the next younger.
a) Find all possible values of n.
b) Every day a pirate was chosen. The chosen pirate gave a coin to each of the other pirates. If $n = 7$, find the largest possible number of coins a pirate can have after several days.
Solution:
a) If n is odd, then it is a divisor of $2009 = 7 \times 7 \times 41$. If $n > 49$, then n is at least 7×41, while the average pirate has 7 coins, so the initial division is impossible. So, we can have $n = 7$, $n = 41$ or $n = 49$. Each of these cases is possible (e.g. if $n = 49$, the average pirate has 41 coins, so the initial amounts are from $41 - 24 = 17$ to $41 + 24 = 65$).
If n is even, then 2009 is multiple of the sum S of the oldest and the youngest pirate. If $S < 7 \times 41$, then S is at most 39 and the pairs of pirates of sum S is at least 41, so we must have at least 82 pirates, a contradiction. So we can have just $S = 7 \times 41 = 287$ and

$S = 49 \times 41 = 2009$; respectively, $n = 2 \times 7 = 14$ or $n = 2 \times 1 = 2$. Each of these cases is possible (e.g. if $n = 14$, the initial amounts are from $144 - 7 = 137$ to $143 + 7 = 150$).
In total, n is one of the numbers 2, 7, 13, 41 and 49.
b) If $n = 7$, the average pirate has $7 \times 41 = 287$ coins, so the initial amounts are from 284 to 290; they have different residues modulo 7. The operation decreases on of the amounts by 6 and increases the other ones by 1, so the residues will be different at all times. The largest possible amount in one pirate will be achieved if all the others have as little as possible, namely 0,1,2,3,4 and 5 coins (the residues modulo 7 have to be different). If this happens, the wealthiest pirate will have $2009 - 14 = 1994$ coins. Indeed, this can be achieved e.g. if every day (until that moment) the coins are given by the second wealthiest: while he has more than 5 coins, he can provide the 6 coins needed, and when he has no more than five, the coins at the poorest six pirates have to be 0,1,2,3,4,5. Thus, $n = 1994$ can be achieved.
NT3 Find all pairs (x, y) of integers which satisfy the equation

$$(x + y)^2(x^2 + y^2) = 2009^2.$$

Solution: Let $x + y = s$, $x \times y = p$ with $s \in \mathbb{Z}^*$ and $p \in \mathbb{Z}$. The given equation can be written in the form

$$s^2(s^2 - 2p) = 2009^2$$

or

$$s^2 - 2p = \left(\frac{2009}{2}\right)^2.$$

So, s divide $2009 = 7^2 \times 41$ and it follows that $p \neq 0$.
If $p > 0$, then $2009^2 = s^2(s^2 - 2p) = s^4 - 2ps^2 < s^4$. We obtain that s divide 2009 and $|s| \geq 49$. Thus, $s \in \{\pm 49, \pm 287, \pm 2009\}$.
• For $s = \pm 49$, we have $p = 360$, and $(x, y) = \{(40, 9), (9, 40), (-40, -9), (-9, -40)\}$.
• For $s \in \{\pm 287, \pm 2009\}$ the equation has no integer solution.
If $p < 0$, then $2009^2 = s^4 - 2ps^2 > s^4$. We obtain that s divides 2009 and $|s| \leq 41$. Thus, $s \in \{\pm 1, \pm 7, \pm 41\}$. For these values of s the equation has no integer solution.
So, the given equation has only the solutions $(40, 9), (9, 40), (-40, -9), (-9, -40)$.
NT4 Determine all prime numbers $p_1, p_2, \ldots, p_{12}, p_{13}$, $p_1 \leq p_2 \leq \ldots \leq p_{12} \leq p_{13}$, such that

$$p_1^2 + p_2^2 + \ldots + p_{12}^2 = p_{13}^2$$

and one of them is equal to $2p_1 + p_9$.
Solution: Obviously, $p_{13} \neq 2$, because sum of squares of 12 prime numbers is greater or equal to $12 \times 2^2 = 48$. Thus, p_{13} is odd number and $p_{13} \geq 7$.

We have that $n^2 \equiv 1(\mod 8)$, when n is odd. Let k be the number of prime numbers equal to 2. Looking at equation modulo 8 we get:

$$4k + 12 - k \equiv 1(\mod 8).$$

So, $k \equiv 7(\mod 8)$ and because $k \leq 12$ we get $k = 7$. Therefore, $p_1 = p_2 = \ldots = p_7 = 2$. Furthermore, we are looking for solutions of equations:

$$28 + p_8^2 + p_9^2 + p_{10}^2 + p_{11}^2 + p_{12}^2 = p_{13}^2$$

where $p_8, p_9, \ldots, p_{13}$ are odd prime numbers and one of them is equal to $p_9 + 4$.
Now, we know that when n is not divisible by 3, $n^2 \equiv 1(\mod 3)$. Let s be number of prime numbers equal to 3. Looking at equation modulo 3 we get:

$$28 + 5 - s \equiv 1(\mod 3).$$

Thus, $s \equiv 2(\mod 3)$ and because $s \leq 5$, s is either 2 or 5. We will consider both cases.
i. When $s = 2$, we get $p_8 = p_9 = 3$. Thus, we are looking for prime numbers $p_{10} \leq p_{11} \leq p_{12} \leq p_{13}$ greater than 3 and at least one of them is 7 (certainly $p_{13} \neq 7$), that satisfy

$$46 + p_{10}^2 + p_{11}^2 + p_{12}^2 = p_{13}^2.$$

We know that $n^2 \equiv 1(\mod 5)$ or $n^2 = 4(\mod 5)$ when n is not divisible by 5. It is not possible that $p_{10} = p_{11} = 5$, because in that case p_{12} must be equal to 7 and left side in that case should be divisible by 5 - contradicts the fact that $p_{13} \geq 7$. So, we proved that $p_{10} = 5$ or $p_{10} = 7$.
If $p_{10} = 5$ than $p_{11} = 7$ because p_{11} is the least of remaining prime numbers. Thus, we are looking for solutions of equation

$$120 = p_{13}^2 - p_{12}^2$$

in prime numbers. Now, from

$$2^3 \cdot 3 \cdot 5 = (p_{12} - p_{12})(p_{13} + p_{12})$$

that desired solutions are $p_{12} = 7$, $p_{13} = 13$; $p_{12} = 13$, $p_{13} = 17$; $p_{12} = 29$, $p_{13} = 31$.
If $p_{10} = 7$ we are solving equation:

$$95 + p_{11}^2 + p_{12}^2 = p_{13}^2$$

in prime numbers greater than 5. But left side can give residues 0 or 3 modulo 5, while right side can give only 1 or 4 modulo 5. So, in this case we do not have solution.

ii. When $s=5$ we get equation:

$$28+45=73=p_{13}^2,$$

but 73 is not square or integer and we do not have solution in this case.
Finally, only solutions are:
$\{(2,2,2,2,2,2,2,3,3,5,7,7,13),(2,2,2,2,2,2,2,3,3,5,7,13,17),(2,2,2,2,2,2,2,3,3,5,7,29,31)\}$.
NT5 Show that there are infinitely many positive integers c, such that the following equations both have solutions in positive integers:

$$(x^2-c)(y^2-c)=z^2-c$$

and

$$(x^2+c)(y^2-c)=z^2-c.$$

Solution: The firs equation always has solutions, namely the triples $\{x,x+1,x(x+1)-c\}$ for all $x\in\mathbb{N}$. Indeed,

$$(x^2-c)((x+1)^2-c)=x^2(x+1)^2-2c(x^2+(x+1)^2)+c^2=(x(x+1)-c)^2-c.$$

For second equation, we try $z=|xy-c|$. We need

$$(x^2+c)(y^2-c)=(xy-c)^2$$

or

$$x^2y^2+c(y^2-x^2)-c^2=x^2y^2-2xyc+c^2.$$

Cancelling the common terms we get

$$c(x^2-y^2+2xy)=2c^2$$

or

$$c=\frac{x^2-y^2+2xy}{2}.$$

Therefore, all c of this form will work. This expression is a positive integer if x and y have the same parity, and it clearly takes infinitely many positive values. We only need to check $z\neq 0$, i.e. $c\neq xy$, which is true for $x\neq y$. For example, one can take

$$y=x-2$$

and

$$z=\frac{x^2-(x-2)^2+2x(x-2)}{2}=x^2-2.$$

Thus, $\{(x,x-2,2x-2)\}$ is a solution for $c=x^2-2$.