1

Problems

1.1 The Forty-Fifth IMO
Athens, Greece, July 7–19, 2004

1.1.1 Contest Problems

First Day (July 12)

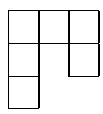
1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the line segment BC.

2. Find all polynomials $P(x)$ with real coefficients that satisfy the equality

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c)$$

for all triples a, b, c of real numbers such that $ab + bc + ca = 0$.

3. Determine all $m \times n$ rectangles that can be covered with *hooks* made up of 6 unit squares, as in the figure:



Rotations and reflections of hooks are allowed. The rectangle must be covered without gaps and overlaps. No part of a hook may cover area outside the rectangle.

Second Day (July 13)

4. Let $n \geq 3$ be an integer and $t_1, t_2, \dots, t_n$ positive real numbers such that
$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$
Show that t_i, t_j, t_k are the side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

5. In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies
$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$
Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

6. We call a positive integer *alternate* if its decimal digits are alternately odd and even. Find all positive integers n such that n has an alternate multiple.

1.1.2 Shortlisted Problems

1. **A1 (KOR)**[IMO4] Let $n \geq 3$ be an integer and $t_1, t_2, \dots, t_n$ positive real numbers such that
$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$
Show that t_i, t_j, t_k are the side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

2. **A2 (ROM)** An infinite sequence $a_0, a_1, a_2, \dots$ of real numbers satisfies the condition
$$a_n = |a_{n+1} - a_{n+2}| \text{ for every } n \geq 0$$
with a_0 and a_1 positive and distinct. Can this sequence be bounded?

3. **A3 (CAN)** Does there exist a function $s\colon \mathbb{Q} \to \{-1, 1\}$ such that if x and y are distinct rational numbers satisfying $xy = 1$ or $x + y \in \{0, 1\}$, then $s(x)s(y) = -1$? Justify your answer.

4. **A4 (KOR)**[IMO2] Find all polynomials $P(x)$ with real coefficients that satisfy the equality
$$P(a - b) + P(b - c) + P(c - a) = 2P(a + b + c)$$
for all triples a, b, c of real numbers such that $ab + bc + ca = 0$.

5. **A5 (THA)** Let $a, b, c > 0$ and $ab + bc + ca = 1$. Prove the inequality
$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

6. **A6 (RUS)** Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\left(x^2 + y^2 + 2f(xy)\right) = (f(x+y))^2 \quad \text{for all } \ x, y \in \mathbb{R}.$$

7. **A7 (IRE)** Let $a_1, a_2, \ldots, a_n$ be positive real numbers, $n > 1$. Denote by g_n their geometric mean, and by $A_1, A_2, \ldots, A_n$ the sequence of arithmetic means defined by $A_k = \frac{a_1+a_2+\cdots+a_k}{k}$, $k = 1, 2, \ldots, n$. Let G_n be the geometric mean of $A_1, A_2, \ldots, A_n$. Prove the inequality

$$n \sqrt[n]{\frac{G_n}{A_n}} + \frac{g_n}{G_n} \le n + 1$$

and establish the cases of equality.

8. **C1 (PUR)** There are 10001 students at a university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:
   (i) Each pair of students are in exactly one club.
   (ii) For each student and each society, the student is in exactly one club of the society.
   (iii) Each club has an odd number of students. In addition, a club with $2m + 1$ students (m is a positive integer) is in exactly m societies.
   Find all possible values of k.

9. **C2 (GER)** Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are distinct. Each intersection point must be colored with one of n distinct colors so that each color is used at least once, and exactly k distinct colors occur on each circle. Find all values of $n \ge 2$ and k for which such a coloring is possible.

10. **C3 (AUS)** The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \ge 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices (where each pair of vertices are joined by an edge).

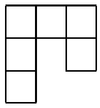
11. **C4 (POL)** Consider a matrix of size $n \times n$ whose entries are real numbers of absolute value not exceeding 1, and the sum of all entries is 0. Let n be an even positive integer. Determine the least number C such that every such matrix necessarily has a row or a column with the sum of its entries not exceeding C in absolute value.

12. **C5 (NZL)** Let N be a positive integer. Two players A and B, taking turns, write numbers from the set $\{1, \ldots, N\}$ on a blackboard. A begins the game by writing 1 on his first move. Then, if a player has written n on

a certain move, his adversary is allowed to write $n+1$ or $2n$ (provided the number he writes does not exceed N). The player who writes N wins. We say that N is of type A or of type B according as A or B has a winning strategy.
(a) Determine whether $N = 2004$ is of type A or of type B.
(b) Find the least $N > 2004$ whose type is different from that of 2004.

13. **C6 (IRN)** For an $n \times n$ matrix A, let X_i be the set of entries in row i, and Y_j the set of entries in column j, $1 \le i, j \le n$. We say that A is *golden* if $X_1, \dots, X_n, Y_1, \dots, Y_n$ are distinct sets. Find the least integer n such that there exists a 2004×2004 golden matrix with entries in the set $\{1, 2, \dots, n\}$.

14. **C7 (EST)**^IMO3 Determine all $m \times n$ rectangles that can be covered with *hooks* made up of 6 unit squares, as in the figure:



Rotations and reflections of hooks are allowed. The rectangle must be covered without gaps and overlaps. No part of a hook may cover area outside the rectangle.

15. **C8 (POL)** For a finite graph G, let $f(G)$ be the number of triangles and $g(G)$ the number of tetrahedra formed by edges of G. Find the least constant c such that
$$g(G)^3 \le c \cdot f(G)^4 \text{ for every graph } G.$$

16. **G1 (ROM)**^IMO1 Let ABC be an acute-angled triangle with $AB \ne AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the line segment BC.

17. **G2 (KAZ)** The circle Γ and the line ℓ do not intersect. Let AB be the diameter of Γ perpendicular to ℓ, with B closer to ℓ than A. An arbitrary point $C \ne A, B$ is chosen on Γ. The line AC intersects ℓ at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect ℓ at F, and let AF intersect Γ at $G \ne A$. Prove that the reflection of G in AB lies on the line CF.

18. **G3 (KOR)** Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcenters of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extension points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60°$.

19. **G4 (POL)**[IMO5] In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

20. **G5 (Dušan Djukić, SMN)** Let $A_1A_2 \dots A_n$ be a regular n-gon. The points $B_1, \dots, B_{n-1}$ are defined as follows:
(i) If $i = 1$ or $i = n-1$, then B_i is the midpoint of the side A_iA_{i+1}.
(ii) If $i \neq 1$, $i \neq n-1$, and S is the intersection point of A_1A_{i+1} and A_nA_i, then B_i is the intersection point of the bisector of the angle A_iSA_{i+1} with A_iA_{i+1}.
Prove the equality

$$\angle A_1B_1A_n + \angle A_1B_2A_n + \cdots + \angle A_1B_{n-1}A_n = 180^\circ.$$

21. **G6 (GBR)** Let $\mathcal{P}$ be a convex polygon. Prove that there is a convex hexagon that is contained in $\mathcal{P}$ and that occupies at least 75 percent of the area of $\mathcal{P}$.

22. **G7 (RUS)** For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

23. **G8 (Dušan Djukić, SMN)** A cyclic quadrilateral $ABCD$ is given. The lines AD and BC intersect at E, with C between B and E; the diagonals AC and BD intersect at F. Let M be the midpoint of the side CD, and let $N \neq M$ be a point on the circumcircle of the triangle ABM such that $AN/BN = AM/BM$. Prove that the points E, F, and N are collinear.

24. **N1 (BLR)** Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation

$$\tau(an) = n$$

does not have a positive integer solution n.

25. **N2 (RUS)** The function ψ from the set $\mathbb{N}$ of positive integers into itself is defined by the equality

$$\psi(n) = \sum_{k=1}^{n} (k, n), \qquad n \in \mathbb{N},$$

where (k, n) denotes the greatest common divisor of k and n.
(a) Prove that $\psi(mn) = \psi(m)\psi(n)$ for every two relatively prime $m, n \in \mathbb{N}$.
(b) Prove that for each $a \in \mathbb{N}$ the equation $\psi(x) = ax$ has a solution.

(c) Find all $a \in \mathbb{N}$ such that the equation $\psi(x) = ax$ has a unique solution.

26. **N3 (IRN)** A function f from the set of positive integers $\mathbb{N}$ into itself is such that for all $m, n \in \mathbb{N}$ the number $(m^2 + n)^2$ is divisible by $f^2(m) + f(n)$. Prove that $f(n) = n$ for each $n \in \mathbb{N}$.

27. **N4 (POL)** Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by

$$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \dots$$

has all of its terms relatively prime to m.

28. **N5 (IRN)**[IMO6] We call a positive integer *alternate* if its decimal digits are alternately odd and even. Find all positive integers n such that n has an alternate multiple.

29. **N6 (IRE)** Given an integer $n > 1$, denote by P_n the product of all positive integers x less than n and such that n divides $x^2 - 1$. For each $n > 1$, find the remainder of P_n on division by n.

30. **N7 (BUL)** Let p be an odd prime and n a positive integer. In the coordinate plane, eight distinct points with integer coordinates lie on a circle with diameter of length p^n. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by p^{n+1}.

2

Solutions

2.1 Solutions to the Shortlisted Problems of IMO 2004

1. By symmetry, it is enough to prove that $t_1 + t_2 > t_3$. We have

$$\left(\sum_{i=1}^{n} t_i\right)\left(\sum_{i=1}^{n} \frac{1}{t\,i}\right) = n^2 + \sum_{i<j}\left(\frac{t_i}{t_j} + \frac{t_j}{t_i} - 2\right). \tag{1}$$

All the summands on the RHS are positive, and therefore the RHS is not smaller than $n^2 + T$, where $T = (t_1/t_3 + t_3/t_1 - 2) + (t_2/t_3 + t_3/t_2 - 2)$. We note that T is increasing as a function in t_3 for $t_3 \geq \max\{t_1, t_2\}$. If $t_1 + t_2 = t_3$, then $T = (t_1+t_2)(1/t_1+1/t_2) - 1 \geq 3$ by the Cauchy–Schwarz inequality. Hence, if $t_1 + t_2 \leq t_3$, we have $T \geq 1$, and consequently the RHS in (1) is greater than or equal to $n^2 + 1$, a contradiction.

Remark. In can be proved, for example using Lagrange multipliers, that if $n^2 + 1$ in the problem is replaced by $(n + \sqrt{10} - 3)^2$, then the statement remains true. This estimate is the best possible.

2. We claim that the sequence $\{a_n\}$ must be unbounded.
The condition of the sequence is equivalent to $a_n > 0$ and $a_{n+1} = a_n + a_{n-1}$ or $a_n - a_{n-1}$. In particular, if $a_n < a_{n-1}$, then $a_{n+1} > \max\{a_n, a_{n-1}\}$.
Let us remove all a_n such that $a_n < a_{n-1}$. The obtained sequence $(b_m)_{m\in\mathbb{N}}$ is strictly increasing. Thus the statement of the problem will follow if we prove that $b_{m+1} - b_m \geq b_m - b_{m-1}$ for all $m \geq 2$.
Let $b_{m+1} = a_{n+2}$ for some n. Then $a_{n+2} > a_{n+1}$. We distinguish two cases:
(i) If $a_{n+1} > a_n$, we have $b_m = a_{n+1}$ and $b_{m-1} \geq a_{n-1}$ (since b_{m-1} is either a_{n-1} or a_n). Then $b_{m+1} - b_m = a_{n+2} - a_{n+1} = a_n = a_{n+1} - a_{n-1} = b_m - a_{n-1} \geq b_m - b_{m-1}$.
(ii) If $a_{n+1} < a_n$, we have $b_m = a_n$ and $b_{m-1} \geq a_{n-1}$. Consequently, $b_{m+1} - b_m = a_{n+2} - a_n = a_{n+1} = a_n - a_{n-1} = b_m - a_{n-1} \geq b_m - b_{m-1}$.

3. The answer is *yes*. Every rational number $x > 0$ can be uniquely expressed as a continued fraction of the form $a_0 + 1/(a_1 + 1/(a_2 + 1/(\cdots + 1/a_n)))$ (where $a_0 \in \mathbb{N}_0$, $a_1, \dots, a_n \in \mathbb{N}$). Then we write $x = [a_0; a_1, a_2, \dots, a_n]$. Since n depends only on x, the function $s(x) = (-1)^n$ is well-defined. For $x < 0$ we define $s(x) = -s(-x)$, and set $s(0) = 1$. We claim that this $s(x)$ satisfies the requirements of the problem.
The equality $s(x)s(y) = -1$ trivially holds if $x + y = 0$.
Suppose that $xy = 1$. We may assume w.l.o.g. that $x > y > 0$. Then $x > 1$, so if $x = [a_0; a_1, a_2, \dots, a_n]$, then $a_0 \geq 1$ and $y = 0 + 1/x = [0; a_0, a_1, a_2, \dots, a_n]$. It follows that $s(x) = (-1)^n$, $s(y) = (-1)^{n+1}$, and hence $s(x)s(y) = -1$.
Finally, suppose that $x + y = 1$. We consider two cases:
   (i) Let $x, y > 0$. We may assume w.l.o.g. that $x > 1/2$. Then there exist natural numbers $a_2, \dots, a_n$ such that $x = [0; 1, a_2, \dots, a_n] = 1/(1 + 1/t)$, where $t = [a_2, \dots, a_n]$. Since $y = 1 - x = 1/(1 + t) = [0; 1 + a_2, a_3, \dots, a_n]$, we have $s(x) = (-1)^n$ and $s(y) = (-1)^{n-1}$, giving us $s(x)s(y) = -1$.
   (ii) Let $x > 0 > y$. If $a_0, \dots, a_n \in \mathbb{N}$ are such that $-y = [a_0; a_1, \dots, a_n]$, then $x = [1 + a_0; a_1, \dots, a_n]$. Thus $s(y) = -s(-y) = -(-1)^n$ and $s(x) = (-1)^n$, so again $s(x)s(y) = -1$.

4. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$. For every $x \in \mathbb{R}$ the triple $(a, b, c) = (6x, 3x, -2x)$ satisfies the condition $ab + bc + ca = 0$. Then the condition on P gives us $P(3x) + P(5x) + P(-8x) = 2P(7x)$ for all x, implying that for all $i = 0, 1, 2, \dots, n$ the following equality holds:

$$\left(3^i + 5^i + (-8)^i - 2 \cdot 7^i\right) a_i = 0.$$

Suppose that $a_i \neq 0$. Then $K(i) = 3^i + 5^i + (-8)^i - 2 \cdot 7^i = 0$. But $K(i)$ is negative for i odd and positive for $i = 0$ or $i \geq 6$ even. Only for $i = 2$ and $i = 4$ do we have $K(i) = 0$. It follows that $P(x) = a_2 x^2 + a_4 x^4$ for some real numbers a_2, a_4.
It is easily verified that all such $P(x)$ satisfy the required condition.

5. By the general mean inequality ($M_1 \leq M_3$), the LHS of the inequality to be proved does not exceed

$$E = \frac{3}{\sqrt[3]{3}} \sqrt[3]{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 6(a + b + c)}.$$

From $ab + bc + ca = 1$ we obtain that $3abc(a + b + c) = 3(ab \cdot ac + ab \cdot bc + ac \cdot bc) \leq (ab + ac + bc)^2 = 1$; hence $6(a + b + c) \leq \frac{2}{abc}$. Since $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc} = \frac{1}{abc}$, it follows that

$$E \leq \frac{3}{\sqrt[3]{3}} \sqrt[3]{\frac{3}{abc}} \leq \frac{1}{abc},$$

where the last inequality follows from the AM–GM inequality $1 = ab+bc+ca \geq 3\sqrt[3]{(abc)^2}$, i.e., $abc \leq 1/(3\sqrt{3})$. The desired inequality now follows. Equality holds if and only if $a = b = c = 1/\sqrt{3}$.

6. Let us make the substitution $z = x + y$, $t = xy$. Given $z, t \in \mathbb{R}$, x, y are real if and only if $4t \leq z^2$. Define $g(x) = 2(f(x) - x)$. Now the given functional equation transforms into

$$f\left(z^2 + g(t)\right) = (f(z))^2 \quad \text{for all} \;\; t, z \in \mathbb{R} \;\; \text{with} \;\; z^2 \geq 4t. \qquad (1)$$

Let us set $c = g(0) = 2f(0)$. Substituting $t = 0$ into (1) gives us

$$f(z^2 + c) = (f(z))^2 \quad \text{for all} \;\; z \in \mathbb{R}. \qquad (2)$$

If $c < 0$, then taking z such that $z^2 + c = 0$, we obtain from (2) that $f(z)^2 = c/2$, which is impossible; hence $c \geq 0$. We also observe that

$$x > c \quad \text{implies} \quad f(x) \geq 0. \qquad (3)$$

If g is a constant function, we easily find that $c = 0$ and therefore $f(x) = x$, which is indeed a solution.
Suppose g is nonconstant, and let $a, b \in \mathbb{R}$ be such that $g(a) - g(b) = d > 0$. For some sufficiently large K and each $u, v \geq K$ with $v^2 - u^2 = d$ the equality $u^2 + g(a) = v^2 + g(b)$ by (1) and (3) implies $f(u) = f(v)$. This further leads to $g(u) - g(v) = 2(v - u) = \frac{d}{u + \sqrt{u^2 + d}}$. Therefore every value from some suitably chosen segment $[\delta, 2\delta]$ can be expressed as $g(u) - g(v)$, with u and v bounded from above by some M.
Consider any x, y with $y > x \geq 2\sqrt{M}$ and $\delta < y^2 - x^2 < 2\delta$. By the above considerations, there exist $u, v \leq M$ such that $g(u) - g(v) = y^2 - x^2$, i.e., $x^2 + g(u) = y^2 + g(v)$. Since $x^2 \geq 4u$ and $y^2 \geq 4v$, (1) leads to $f(x)^2 = f(y)^2$. Moreover, if we assume w.l.o.g. that $4M \geq c^2$, we conclude from (3) that $f(x) = f(y)$. Since this holds for any $x, y \geq 2\sqrt{M}$ with $y^2 - x^2 \in [\delta, 2\delta]$, it follows that $f(x)$ is eventually constant, say $f(x) = k$ for $x \geq N = 2\sqrt{M}$. Setting $x > N$ in (2) we obtain $k^2 = k$, so $k = 0$ or $k = 1$.
By (2) we have $f(-z) = \pm f(z)$, and thus $|f(z)| \leq 1$ for all $z \leq -N$. Hence $g(u) = 2f(u) - 2u \geq -2 - 2u$ for $u \leq -N$, which implies that g is unbounded. Hence for each z there exists t such that $z^2 + g(t) > N$, and consequently $f(z)^2 = f(z^2 + g(t)) = k = k^2$. Therefore $f(z) = \pm k$ for each z.
If $k = 0$, then $f(x) \equiv 0$, which is clearly a solution. Assume $k = 1$. Then $c = 2f(0) = 2$ (because $c \geq 0$), which together with (3) implies $f(x) = 1$ for all $x \geq 2$. Suppose that $f(t) = -1$ for some $t < 2$. Then $t - g(t) = 3t + 2 > 4t$. If also $t - g(t) \geq 0$, then for some $z \in \mathbb{R}$ we have $z^2 = t - g(t) > 4t$, which by (1) leads to $f(z)^2 = f(z^2 + g(t)) = f(t) = -1$, which is impossible. Hence $t - g(t) < 0$, giving us $t < -2/3$. On the other hand, if X is any subset of $(-\infty, -2/3)$, the function f defined by

$f(x) = -1$ for $x \in X$ and $f(x) = 1$ satisfies the requirements of the problem.
To sum up, the solutions are $f(x) = x$, $f(x) = 0$ and all functions of the form

$$f(x) = \begin{cases} 1, & x \notin X, \\ -1, & x \in X, \end{cases}$$

where $X \subset (-\infty, -2/3)$.

7. Let us set $c_k = A_{k-1}/A_k$ for $k = 1, 2, \dots, n$, where we define $A_0 = 0$. We observe that $a_k/A_k = (kA_k - (k-1)A_{k-1})/A_k = k - (k-1)c_k$. Now we can write the LHS of the inequality to be proved in terms of c_k, as follows:

$$\sqrt[n]{\frac{G_n}{A_n}} = \sqrt[n^2]{c_2 c_3^2 \cdots c_n^{n-1}} \quad \text{and} \quad \frac{g_n}{G_n} = \sqrt[n]{\prod_{k=1}^{n}(k - (k-1)c_k)}.$$

By the $AM - GM$ inequality we have

$$\begin{aligned} n \sqrt[n^2]{1^{n(n+1)/2} c_2 c_3^2 \dots c_n^{n-1}} &\le \frac{1}{n}\left(\frac{n(n+1)}{2} + \sum_{k=2}^{n}(k-1)c_k\right) \\ &= \frac{n+1}{2} + \frac{1}{n}\sum_{k=1}^{n}(k-1)c_k. \end{aligned} \tag{1}$$

Also by the AM–GM inequality, we have

$$\sqrt[n]{\prod_{k=1}^{n}(k - (k-1)c_k)} \le \frac{n+1}{2} - \frac{1}{n}\sum_{k=1}^{n}(k-1)c_k. \tag{2}$$

Adding (1) and (2), we obtain the desired inequality. Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

8. Let us write $n = 10001$. Denote by $\mathcal{T}$ the set of ordered triples $(a, C, \mathcal{S})$, where a is a student, C a club, and $\mathcal{S}$ a society such that $a \in C$ and $C \in \mathcal{S}$. We shall count $|\mathcal{T}|$ in two different ways.
Fix a student a and a society $\mathcal{S}$. By (ii), there is a unique club C such that $(a, C, \mathcal{S}) \in \mathcal{T}$. Since the ordered pair $(a, \mathcal{S})$ can be chosen in nk ways, we have that $|\mathcal{T}| = nk$.
Now fix a club C. By (iii), C is in exactly $(|C| - 1)/2$ societies, so there are $|C|(|C| - 1)/2$ triples from $\mathcal{T}$ with second coordinate C. If $\mathcal{C}$ is the set of all clubs, we obtain $|\mathcal{T}| = \sum_{C \in \mathcal{C}} \frac{|C|(|C|-1)}{2}$. But we also conclude from (i) that

$$\sum_{C \in \mathcal{C}} \frac{|C|(|C| - 1)}{2} = \frac{n(n-1)}{2}.$$

Therefore $n(n-1)/2 = nk$, i.e., $k = (n-1)/2 = 5000$.

On the other hand, for $k = (n-1)/2$ there is a desired configuration with only one club C that contains all students and k identical societies with only one element (the club C). It is easy to verify that (i)–(iii) hold.

9. Obviously we must have $2 \le k \le n$. We shall prove that the possible values for k and n are $2 \le k \le n \le 3$ and $3 \le k \le n$. Denote all colors and circles by $1, \dots, n$. Let $F(i,j)$ be the set of colors of the common points of circles i and j.
   Suppose that $k = 2 < n$. Consider the ordered pairs (i,j) such that color j appears on the circle i. Since $k = 2$, clearly there are exactly $2n$ such pairs. On the other hand, each of the n colors appears on at least two circles, so there are at least $2n$ pairs (i,j), and equality holds only if each color appears on exactly 2 circles. But then at most two points receive each of the n colors and there are $n(n-1)$ points, implying that $n(n-1) = 2n$, i.e., $n = 3$. It is easy to find examples for $k = 2$ and $n = 2$ or 3.
   Next, let $k = 3$. An example for $n = 3$ is given by $F(i,j) = \{i,j\}$ for each $1 \le i < j \le 3$. Assume $n \ge 4$. Then an example is given by $F(1,2) = \{1,2\}$, $F(i,i+1) = \{i\}$ for $i = 2, \dots, n-2$, $F(n-1,n) = \{n-2, n-1\}$ and $F(i,j) = n$ for all other $i, j > i$.
   We now prove by induction on k that a desired coloring exists for each $n \ge k \ge 3$. Let there be given n circles. By the inductive hypothesis, circles $1, 2, \dots, n-1$ can be colored in $n-1$ colors, k of which appear on each circle, such that color i appears on circle i. Then we set $F(i,n) = \{i,n\}$ for $i = 1, \dots, k$ and $F(i,n) = \{n\}$ for $i > n$. We thus obtain a coloring of the n circles in n colors, such that $k+1$ colors (including color i) appear on each circle i.

10. The least number of edges of such a graph is n.
    We note that deleting edge AB of a 4-cycle $ABCD$ from a connected and nonbipartite graph G yields a connected and nonbipartite graph, say H. Indeed, the connectedness is obvious; also, if H were bipartite with partition of the set of vertices into P_1 and P_2, then w.l.o.g. $A, C \in P_1$ and $B, D \in P_2$, so $G = H \cup \{AB\}$ would also be bipartite with the same partition, a contradiction.
    Any graph that can be obtained from the complete n-graph in the described way is connected and has at least one cycle (otherwise it would be bipartite); hence it must have at least n edges.
    Now consider a complete graph with vertices $V_1, V_2, \dots, V_n$. Let us remove every edge V_iV_j with $3 \le i < j < n$ from the cycle $V_2V_iV_jV_n$. Then for $i = 3, \dots, n-1$ we remove edges V_2V_i and V_iV_n from the cycles $V_1V_iV_2V_n$ and $V_1V_iV_nV_2$ respectively, thus obtaining a graph with exactly n edges: V_1V_i $(i = 2, \dots, n)$ and V_2V_n.

11. Consider the matrix $A = (a_{ij})_{i,j=1}^n$ such that a_{ij} is equal to 1 if $i, j \le n/2$, -1 if $i, j > n/2$, and 0 otherwise. This matrix satisfies the conditions from

the problem and all row sums and column sums are equal to $\pm n/2$. Hence $C \geq n/2$.

Let us show that $C = n/2$. Assume to the contrary that there is a matrix $B = (b_{ij})_{i,j=1}^n$ all of whose row sums and column sums are either greater than $n/2$ or smaller than $-n/2$. We may assume w.l.o.g. that at least $n/2$ row sums are positive and, permuting rows if necessary, that the first $n/2$ rows have positive sums. The sum of entries in the $n/2 \times n$ submatrix B' consisting of first $n/2$ rows is greater than $n^2/4$, and since each column of B' has sum at most $n/2$, it follows that more than $n/2$ column sums of B', and therefore also of B, are positive. Again, suppose w.l.o.g. that the first $n/2$ column sums are positive. Thus the sums R^+ and C^+ of entries in the first $n/2$ rows and in the first $n/2$ columns respectively are greater than $n^2/4$. Now the sum of all entries of B can be written as

$$\sum a_{ij} = R^+ + C^+ + \sum_{\substack{i>n/2 \\ j>n/2}} a_{ij} - \sum_{\substack{i\leq n/2 \\ j\leq n/2}} a_{ij} > \frac{n^2}{2} - \frac{n^2}{4} - \frac{n^2}{4} = 0,$$

a contradiction. Hence $C = n/2$, as claimed.

12. We say that a number $n \in \{1, 2, \ldots, N\}$ is *winning* if the player who is on turn has a winning strategy, and *losing* otherwise. The game is of type A if and only if 1 is a losing number.

Let us define $n_0 = N$, $n_{i+1} = [n_i/2]$ for $i = 0, 1, \ldots$ and let k be such that $n_k = 1$. Consider the sets $A_i = \{n_{i+1} + 1, \ldots, n_i\}$. We call a set A_i *all-winning* if all numbers from A_i are winning, *even-winning* if even numbers are winning and odd are losing, and *odd-winning* if odd numbers are winning and even are losing.

(i) Suppose A_i is even-winning and consider A_{i+1}. Multiplying any number from A_{i+1} by 2 yields an even number from A_i, which is a losing number. Thus $x \in A_{i+1}$ is winning if and only if $x + 1$ is losing, i.e., if and only if it is even. Hence A_{i+1} is also even-winning.

(ii) Suppose A_i is odd-winning. Then each $k \in A_{i+1}$ is winning, since $2k$ is losing. Hence A_{i+1} is all-winning.

(iii) Suppose A_i is all-winning. Multiplying $x \in A_{i+1}$ by two is then a losing move, so x is winning if and only if $x+1$ is losing. Since n_{i+1} is losing, A_{i+1} is odd-winning if n_{i+1} is even and even-winning otherwise.

We observe that A_0 is even-winning if N is odd and odd-winning otherwise. Also, if some A_i is even-winning, then all $A_{i+1}, A_{i+2}, \ldots$ are even-winning and thus 1 is losing; i.e., the game is of type A. The game is of type B if and only if the sets $A_0, A_1, \ldots$ are alternately odd-winning and all-winning with A_0 odd-winning, which is equivalent to $N = n_0, n_2, n_4, \ldots$ all being even. Thus N is of type B if and only if all digits at the odd positions in the binary representation of N are zeros.

Since $2004 = \overline{11111010100}$ in the binary system, 2004 is of type A. The least $N > 2004$ that is of type B is $\overline{100000000000} = 2^{11} = 2048$. Thus the answer to part (b) is 2048.

13. Since X_i, Y_i, $i = 1, \dots, 2004$, are 4008 distinct subsets of the set $S_n = \{1, 2, \dots, n\}$, it follows that $2^n \geq 4008$, i.e. $n \geq 12$.
Suppose $n = 12$. Let $\mathcal{X} = \{X_1, \dots, X_{2004}\}$, $\mathcal{Y} = \{Y_1, \dots, Y_{2004}\}$, $\mathcal{A} = \mathcal{X} \cup \mathcal{Y}$. Exactly $2^{12} - 4008 = 88$ subsets of S_n do not occur in $\mathcal{A}$.
Since each row intersects each column, we have $X_i \cap Y_j \neq \emptyset$ for all i, j. Suppose $|X_i|, |Y_j| \leq 3$ for some indices i, j. Since then $|X_i \cup Y_j| \leq 5$, any of at least $2^7 > 88$ subsets of $S_n \setminus (X_i \cap Y_j)$ can occur in neither $\mathcal{X}$ nor $\mathcal{Y}$, which is impossible. Hence either in $\mathcal{X}$ or in $\mathcal{Y}$ all subsets are of size at least 4. Suppose w.l.o.g. that $k = |X_l| = \min_i |X_i| \geq 4$. There are

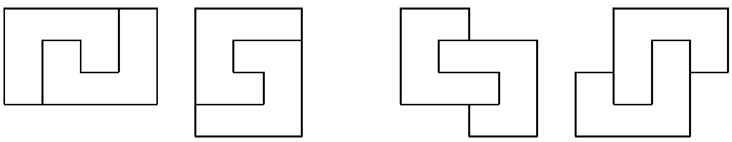
$$n_k = \binom{12-k}{0} + \binom{12-k}{1} + \cdots + \binom{12-k}{k-1}$$

subsets of $S \setminus X_l$ with fewer than k elements, and none of them can be either in $\mathcal{X}$ (because $|X_l|$ is minimal in $\mathcal{X}$) or in $\mathcal{Y}$. Hence we must have $n_k \leq 88$. Since $n_4 = 93$ and $n_5 = 99$, it follows that $k \geq 6$. But then none of the $\binom{12}{0} + \cdots + \binom{12}{5} = 1586$ subsets of S_n is in $\mathcal{X}$, hence at least $1586 - 88 = 1498$ of them are in $\mathcal{Y}$. The 1498 complements of these subsets also do not occur in $\mathcal{X}$, which adds to 3084 subsets of S_n not occurring in $\mathcal{X}$. This is clearly a contradiction.
Now we construct a golden matrix for $n = 13$. Let

$$A_1 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A_m = \begin{bmatrix} A_{m-1} & A_{m-1} \\ A_{m-1} & B_{m-1} \end{bmatrix} \text{ for } m = 2, 3, \dots,$$

where B_{m-1} is the $2^{m-1} \times 2^{m-1}$ matrix with all entries equal to $m+2$. It can be easily proved by induction that each of the matrices A_m is golden. Moreover, every upper-left square submatrix of A_m of size greater than 2^{m-1} is also golden. Since $2^{10} < 2004 < 2^{11}$, we thus obtain a golden matrix of size 2004 with entries in S_{13}.

14. Suppose that an $m \times n$ rectangle can be covered by "hooks". For any hook H there is a unique hook K that covers its " inside" square. Then also H covers the inside square of K, so the set of hooks can be partitioned into pairs of type $\{H, K\}$, each of which forms one of the following two figures consisting of 12 squares:



A_1 B_1 A_2 B_2

Thus the $m \times n$ rectangle is covered by these tiles. It immediately follows that $12 \mid mn$.
Suppose one of m, n is divisible by 4. Let w.l.o.g. $4 \mid m$. If $3 \mid n$, one can easily cover the rectangle by 3×4 rectangles and therefore by hooks. Also,

if $12 \mid m$ and $n \notin \{1, 2, 5\}$, then there exist $k, l \in \mathbb{N}_0$ such that $n = 3k+4l$, and thus the rectangle $m \times n$ can be partitioned into 3×12 and 4×12 rectangles all of which can be covered by hooks. If $12 \mid m$ and $n = 1, 2$, or 5, then it is easy to see that covering by hooks is not possible.
Now suppose that $4 \nmid m$ and $4 \nmid n$. Then m, n are even and the number of tiles is odd. Assume that the total number of tiles of types A_1 and B_1 is odd (otherwise the total number of tiles of types A_2 and B_2 is odd, which is analogous). If we color in black all columns whose indices are divisible by 4, we see that each tile of type A_1 or B_1 covers three black squares, which yields an odd number in total. Hence the total number of black squares covered by the tiles of types A_2 and B_2 must be odd. This is impossible, since each such tile covers two or four black squares.

15. Denote by $V_1, \dots, V_n$ the vertices of a graph G and by E the set of its edges. For each $i = 1, \dots, n$, let A_i be the set of vertices connected to V_i by an edge, G_i the subgraph of G whose set of vertices is A_i, and E_i the set of edges of G_i. Also, let v_i, e_i, and $t_i = f(G_i)$ be the numbers of vertices, edges, and triangles in G_i respectively.
The numbers of tetrahedra and triangles one of whose vertices is V_i are respectively equal to t_i and e_i. Hence

$$\sum_{i=1}^{n} v_i = 2|E|, \qquad \sum_{i=1}^{n} e_i = 3f(G) \quad \text{and} \quad \sum_{i=1}^{n} t_i = 4g(G).$$

Since $e_i \le v_i(v_i-1)/2 \le v_i^2/2$ and $e_i \le |E|$, we obtain $e_i^2 \le v_i^2|E|/2$, i.e., $e_i \le v_i\sqrt{|E|/2}$. Summing over all i yields $3f(G) \le 2|E|\sqrt{|E|/2}$, or equivalently $f(G)^2 \le 2|E|^3/9$. Since this relation holds for each graph G_i, it follows that

$$t_i = f(G_i) = f(G_i)^{1/3} f(G_i)^{2/3} \le \left(\frac{2}{9}\right)^{1/3} f(G)^{1/3} e_i.$$

Summing the last inequality for $i = 1, \dots, n$ gives us

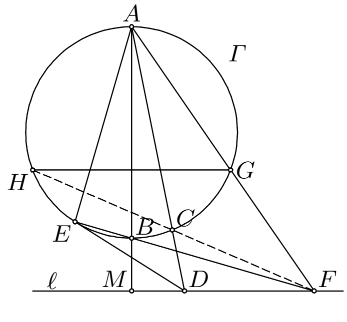
$$4g(G) \le 3\left(\frac{2}{9}\right)^{1/3} f(G)^{1/3} \cdot f(G), \qquad \text{i.e.} \qquad g(G)^3 \le \frac{3}{32} f(G)^4.$$

The constant $c = 3/32$ is the best possible. Indeed, in a complete graph C_n it holds that $g(K_n)^3/f(K_n)^4 = \binom{n}{4}^3\binom{n}{3}^{-4} \to \frac{3}{32}$ as $n \to \infty$.
Remark. Let N_k be the number of complete k-subgraphs in a finite graph G. Continuing inductively, one can prove that $N_{k+1}^k \le \frac{k!}{(k+1)^k} N_k^{k+1}$.

16. Note that $\triangle ANM \sim \triangle ABC$ and consequently $AM \ne AN$. Since $OM = ON$, it follows that OR is a perpendicular bisector of MN. Thus, R is the common point of the median of MN and the bisector of $\angle MAN$. Then it follows from a well-known fact that R lies on the circumcircle of $\triangle AMN$.

Let K be the intersection of AR and BC. We then have $\angle MRA = \angle MNA = \angle ABK$ and $\angle NRA = \angle NMA = \angle ACK$, from which we conclude that $RMBK$ and $RNCK$ are cyclic. Thus K is the desired intersection of the circumcircles of $\triangle BMR$ and $\triangle CNR$ and it indeed lies on BC.

17. Let H be the reflection of G about AB ($GH \parallel \ell$). Let M be the intersection of AB and ℓ. Since $\angle FEA = \angle FMA = 90°$, it follows that $AEMF$ is cyclic and hence $\angle DFE = \angle BAE = \angle DEF$. The last equality holds because DE is tangent to Γ. It follows that $DE = DF$ and hence $DF^2 = DE^2 = DC \cdot DA$ (the power of D with respect to Γ). It then follows that $\angle DCF = \angle DFA = \angle HGA = \angle HCA$. Thus it follows that H lies on CF as desired.



18. It is important to note that since $\beta < \gamma$, $\angle ADC = 90° - \gamma + \beta$ is acute. It is elementary that $\angle CAO = 90° - \beta$. Let X and Y respectively be the intersections of FE and GH with AD. We trivially get $X \in EF \perp AD$ and $\triangle AGH \cong \triangle ACB$. Consequently, $\angle GAY = \angle OAB = 90° - \gamma = 90° - \angle AGY$. Hence, $GH \perp AD$ and thus $GH \parallel FE$. That $EFGH$ is a rectangle is now equivalent to $FX = GY$ and $EX = HY$.
We have that $GY = AG \sin\gamma = AC \sin\gamma$ and $FX = AF \sin\gamma$ (since $\angle AFX = \gamma$). Thus,
$$FX = GY \Leftrightarrow CF = AF = AC \Leftrightarrow \angle AFC = 60° \Leftrightarrow \angle ADC = 30°.$$
Since $\angle ADC = 180° - \angle DCA - \angle DAC = 180° - \gamma - (90° - \beta)$, it immediately follows that $FX = GY \Leftrightarrow \gamma - \beta = 60°$. We similarly obtain $EX = HY \Leftrightarrow \gamma - \beta = 60°$, proving the statement of the problem.

19. Assume first that the points A, B, C, D are concyclic. Let the lines BP and DP meet the circumcircle of $ABCD$ again at E and F, respectively. Then it follows from the given conditions that $\widehat{AB} = \widehat{CF}$ and $\widehat{AD} = \widehat{CE}$; hence $BF \parallel AC$ and $DE \parallel AC$. Therefore $BFED$ and $BFAC$ are isosceles trapezoids and thus $P = BE \cap DF$ lies on the common bisector of segments BF, ED, AC. Hence $AP = CP$.
Assume in turn that $AP = CP$. Let P w.l.o.g. lie in the triangles ACD and BCD. Let BP and DP meet AC at K and L, respectively. The points A and C are isogonal conjugates with respect to $\triangle BDP$, which implies that $\angle APK = \angle CPL$. Since $AP = CP$, we infer that K and L are symmetric with respect to the perpendicular bisector p of AC. Let E be the reflection of D in p. Then E lies on the line BP, and the triangles APD and CPE are congruent. Thus $\angle BDC = \angle ADP = \angle BEC$, which

means that the points B, C, E, D are concyclic. Moreover, A, C, E, D are also concyclic. Hence, $ABCD$ is a cyclic quadrilateral.

20. We first establish the following lemma.

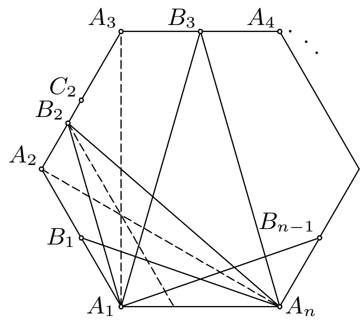
Lemma. Let $ABCD$ be an isosceles trapezoid with bases AB and CD. The diagonals AC and BD intersect at S. Let M be the midpoint of BC, and let the bisector of the angle BSC intersect BC at N. Then $\angle AMD = \angle AND$.

Proof. It suffices to show that the points A, D, M, N are concyclic. The statement is trivial for $AD \parallel BC$. Let us now assume that AD and BC meet at X, and let $XA = XB = a$, $XC = XD = b$. Since SN is the bisector of $\angle CSB$, we have

$$\frac{a - XN}{XN - b} = \frac{BN}{CN} = \frac{BS}{CS} = \frac{AB}{CD} = \frac{a}{b},$$

and an easy computation yields $XN = \frac{2ab}{a+b}$. We also have $XM = \frac{a+b}{2}$; hence $XM \cdot XN = XA \cdot XD$. Therefore A, D, M, N are concyclic, as needed.

Denote by C_i the midpoint of the side A_iA_{i+1}, $i = 1, \dots, n-1$. By definition $C_1 = B_1$ and $C_{n-1} = B_{n-1}$. Since $A_1A_iA_{i+1}A_n$ is an isosceles trapezoid with $A_1A_i \parallel A_{i+1}A_n$ for $i = 2, \dots, n-2$, it follows from the lemma that $\angle A_1B_iA_n = \angle A_1C_iA_n$ for all i.



The sum in consideration thus equals $\angle A_1C_1A_n + \angle A_1C_2A_n + \cdots + \angle A_1C_{n-1}A_n$. Moreover, the triangles $A_1C_iA_n$ and $A_{n+2-i}C_1A_{n+1-i}$ are congruent (a rotation about the center of the n-gon carries the first one to the second), and consequently

$$\angle A_1C_iA_n = \angle A_{n+2-i}C_1A_{n+1-i}$$

for $i = 2, \dots, n-1$.

Hence $\Sigma = \angle A_1C_1A_n + \angle A_nC_1A_{n-1} + \cdots + \angle A_3C_1A_2 = \angle A_1C_1A_2 = 180°$.

21. Let ABC be the triangle of maximum area S contained in $\mathcal{P}$ (it exists because of compactness of $\mathcal{P}$). Draw parallels to BC, CA, AB through A, B, C, respectively, and denote the triangle thus obtained by $A_1B_1C_1$ ($A \in B_1C_1$, etc.). Since each triangle with vertices in $\mathcal{P}$ has area at most S, the entire polygon $\mathcal{P}$ is contained in $A_1B_1C_1$.

Next, draw lines of support of $\mathcal{P}$ parallel to BC, CA, AB and not intersecting the triangle ABC. They determine a convex hexagon $U_aV_aU_bV_bU_cV_c$ containing $\mathcal{P}$, with $V_b, U_c \in B_1C_1$, $V_c, U_a \in C_1A_1$, $V_a, U_b \in A_1B_1$. Each of the line segments U_aV_a, U_bV_b, U_cV_c contains points of $\mathcal{P}$. Choose such points A_0, B_0, C_0 on U_aV_a, U_bV_b, U_cV_c, respectively. The convex hexagon

$AC_0BA_0CB_0$ is contained in $\mathcal{P}$, because the latter is convex. We prove that $AC_0BA_0CB_0$ has area at least 3/4 the area of $\mathcal{P}$.

Let x, y, z denote the areas of triangles U_aBC, U_bCA, and U_cAB. Then $S_1 = S_{AC_0BA_0CB_0} = S + x + y + z$. On the other hand, the triangle $A_1U_aV_a$ is similar to $\triangle A_1BC$ with similitude $\tau = (S - x)/S$, and hence its area is $\tau^2 S = (S - x)^2/S$. Thus the area of quadrilateral U_aV_aCB is $S - (S - x)^2/S = 2z - z^2/S$. Analogous formulas hold for quadrilaterals U_bV_bAC and U_cV_cBA. Therefore

$$\begin{aligned} S_{\mathcal{P}} &\le S_{U_aV_aU_bV_bU_cV_c} = S + S_{U_aV_aCB} + S_{U_bV_bAC} + S_{U_cV_cBA} \\ &= S + 2(x + y + z) - \frac{x^2 + y^2 + z^2}{S} \\ &\le S + 2(x + y + z) - \frac{(x + y + z)^2}{3S}. \end{aligned}$$

Now $4S_1 - 3S_{\mathcal{P}} \ge= S - 2(x+y+z) + (x+y+z)^2/S = (S-x-y-z)^2/S \ge 0$; i.e., $S_1 \ge 3S_{\mathcal{P}}/4$, as claimed.

22. The proof uses the following observation:

Lemma. In a triangle ABC, let K, L be the midpoints of the sides AC, AB, respectively, and let the incircle of the triangle touch BC, CA at D, E, respectively. Then the lines KL and DE intersect on the bisector of the angle ABC.

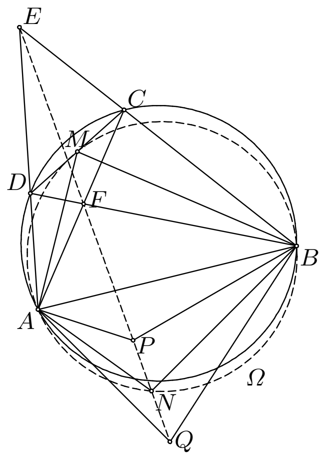
Proof. Let the bisector ℓ_b of $\angle ABC$ meet DE at T. One can assume that $AB \ne BC$, or else $T \equiv K \in KL$. Note that the incenter I of $\triangle ABC$ is between B and T, and also $T \ne E$. From the triangles BDT and DEC we obtain $\angle ITD = \alpha/2 = \angle IAE$, which implies that A, I, T, E are concyclic. Then $\angle ATB = \angle AEI = 90^\circ$. Thus L is the circumcenter of $\triangle ATB$ from which $\angle LTB = \angle LBT = \angle TBC \Rightarrow LT \parallel BC \Rightarrow T \in KL$, which is what we were supposed to prove.

Let the incircles of $\triangle ABX$ and $\triangle ACX$ touch BX at D and F, respectively, and let them touch AX at E and G, respectively. Clearly, DE and FG are parallel. If the line PQ intersects BX and AX at M and N, respectively, then $MD^2 = MP \cdot MQ = MF^2$, i.e., $MD = MF$ and analogously $NE = NG$. It follows that PQ is parallel to DE and FG and equidistant from them.

The midpoints of AB, AC, and AX lie on the same line m, parallel to BC. Applying the lemma to $\triangle ABX$, we conclude that DE passes through the common point U of m and the bisector of $\angle ABX$. Analogously, FG passes through the common point V of m and the bisector of $\angle ACX$. Therefore PQ passes through the midpoint W of the line segment UV. Since U, V do not depend on X, neither does W.

23. To start with, note that point N is uniquely determined by the imposed properties. Indeed, $f(X) = AX/BX$ is a monotone function on both arcs AB of the circumcircle of $\triangle ABM$.

Denote by P and Q respectively the second points of intersection of the line EF with the circumcircles of $\triangle ABE$ and $\triangle ABF$. The problem is equivalent to showing that $N \in PQ$. In fact, we shall prove that N coincides with the midpoint $\overline{N}$ of segment PQ.



The cyclic quadrilaterals $APBE$, $AQBF$, and $ABCD$ yield $\angle APQ = 180° - \angle APE = 180° - \angle ABE = \angle ADC$ and $\angle AQP = \angle AQF = \angle ABF = \angle ACD$. It follows that $\triangle APQ \sim \triangle ADC$, and consequently $\triangle A\overline{N}P \sim \triangle AMD$. Analogously $\triangle B\overline{N}P \sim \triangle BMC$. Therefore $A\overline{N}/AM = PQ/DC = B\overline{N}/BM$, i.e., $A\overline{N}/B\overline{N} = AM/BM$. Moreover, $\angle A\overline{N}B = \angle A\overline{N}P + \angle P\overline{N}B = \angle AMD + \angle BMC = 180° - \angle AMB$, which means that point $\overline{N}$ lies on the circumcircle of $\triangle AMB$. By the uniqueness of N, we conclude that $\overline{N} \equiv N$, which completes the solution.

24. Setting $m = an$ we reduce the given equation to $m/\tau(m) = a$.
Let us show that for $a = p^{p-1}$ the above equation has no solutions in $\mathbb{N}$ if $p > 3$ is a prime. Assume to the contrary that $m \in \mathbb{N}$ is such that $m = p^{p-1}\tau(m)$. Then $p^{p-1} \mid m$, so we may set $m = p^{\alpha}k$, where $\alpha, k \in \mathbb{N}$, $\alpha \geq p-1$, and $p \nmid k$. Let $k = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be the decomposition of k into primes. Then $\tau(k) = (\alpha_1+1)\cdots(\alpha_r+1)$ and $\tau(m) = (\alpha+1)\tau(k)$. Our equation becomes
$$p^{\alpha-p+1}k = (\alpha+1)\tau(k). \tag{1}$$
We observe that $\alpha \neq p-1$: otherwise the RHS would be divisible by p and the LHS would not be so. It follows that $\alpha \geq p$, which also easily implies that $p^{\alpha-p+1} \geq \frac{p}{p+1}(\alpha+1)$.
Furthermore, since $\alpha+1$ cannot be divisible by $p^{\alpha-p+1}$ for any $\alpha \geq p$, it follows that $p \mid \tau(k)$. Thus if $p \mid \tau(k)$, then at least one α_i+1 is divisible by p and consequently $\alpha_i \geq p-1$ for some i. Hence $k \geq \frac{p_i^{\alpha_i}}{\alpha_i+1}\tau(k) \geq \frac{2^{p-1}}{p}\tau(k)$. But then we have
$$p^{\alpha-p+1}k \geq \frac{p}{p+1}(\alpha+1)\cdot\frac{2^{p-1}}{p}\tau(k) > (\alpha+1)\tau(k),$$
contradicting (1). Therefore (1) has no solutions in $\mathbb{N}$.
Remark. There are many other values of a for which the considered equation has no solutions in $\mathbb{N}$: for example, $a = 6p$ for a prime $p \geq 5$.

25. Let n be a natural number. For each $k = 1, 2, \ldots, n$, the number (k, n) is a divisor of n. Consider any divisor d of n. If $(k, n) = n/d$, then $k = nl/d$

for some $l \in \mathbb{N}$, and $(k,n) = (l,d)n/d$, which implies that l is coprime to d and $l \le d$. It follows that (k,n) is equal to n/d for exactly $\varphi(d)$ natural numbers $k \le n$. Therefore

$$\psi(n) = \sum_{k=1}^{n}(k,n) = \sum_{d|n}\varphi(d)\frac{n}{d} = n\sum_{d|n}\frac{\varphi(d)}{d}. \tag{1}$$

(a) Let n, m be coprime. Then each divisor f of mn can be uniquely expressed as $f = de$, where $d \mid n$ and $e \mid m$. We now have by (1)

$$\begin{aligned}\psi(mn) &= mn\sum_{f|mn}\frac{\varphi(f)}{f} = mn\sum_{d|n,\ e|m}\frac{\varphi(de)}{de}\\ &= mn\sum_{d|n,\ e|m}\frac{\varphi(d)}{d}\frac{\varphi(e)}{e} = \left(n\sum_{d|n}\frac{\varphi(d)}{d}\right)\left(m\sum_{e|m}\frac{\varphi(e)}{e}\right)\\ &= \psi(m)\psi(n).\end{aligned}$$

(b) Let $n = p^k$, where p is a prime and k a positive integer. According to (1),

$$\frac{\psi(n)}{n} = \sum_{i=0}^{k}\frac{\varphi(p^i)}{p^i} = 1 + \frac{k(p-1)}{p}.$$

Setting $p = 2$ and $k = 2(a-1)$ we obtain $\psi(n) = an$ for $n = 2^{2(a-1)}$.

(c) We note that $\psi(p^p) = p^{p+1}$ if p is a prime. Hence, if a has an odd prime factor p and $a_1 = a/p$, then $x = p^p 2^{2a_1-2}$ is a solution of $\psi(x) = ax$ different from $x = 2^{2a-2}$.
Now assume that $a = 2^k$ for some $k \in \mathbb{N}$. Suppose $x = 2^\alpha y$ is a positive integer such that $\psi(x) = 2^k x$. Then $2^{\alpha+k}y = \psi(x) = \psi(2^\alpha)\psi(y) = (\alpha+2)2^{\alpha-1}\psi(y)$, i.e., $2^{k+1}y = (\alpha+2)\psi(y)$. We notice that for each odd y, $\psi(y)$ is (by definition) the sum of an odd number of odd summands and therefore odd. It follows that $\psi(y) \mid y$. On the other hand, $\psi(y) > y$ for $y > 1$, so we must have $y = 1$. Consequently $\alpha = 2^{k+1}-2 = 2a-2$, giving us the unique solution $x = 2^{2a-2}$.
Thus $\psi(x) = ax$ has a unique solution if and only if a is a power of 2.

26. For $m = n = 1$ we obtain that $f(1)^2 + f(1)$ divides $(1^2+1)^2 = 4$, from which we find that $f(1) = 1$.
Next, we show that $f(p-1) = p-1$ for each prime p. By the hypothesis for $m = 1$ and $n = p-1$, $f(p-1)+1$ divides p^2, so $f(p-1)$ equals either $p-1$ or p^2-1. If $f(p-1) = p^2-1$, then $f(1)+f(p-1)^2 = p^4-2p^2+2$ divides $(1+(p-1)^2)^2 < p^4-2p^2+2$, giving a contradiction. Hence $f(p-1) = p-1$.
Let us now consider an arbitrary $n \in \mathbb{N}$. By the hypothesis for $m = p-1$, $A = f(n)+(p-1)^2$ divides $(n+(p-1)^2)^2 \equiv (n-f(n))^2 \pmod{A}$, and hence A divides $(n-f(n))^2$ for any prime p. Taking p large enough, we can obtain A to be greater than $(n-f(n))^2$, which implies that $(n-f(n))^2 = 0$, i.e., $f(n) = n$ for every n.

27. Set $a = 1$ and assume that $b \in \mathbb{N}$ is such that $b^2 \equiv b + 1 \pmod{m}$. An easy induction gives us $x_n \equiv b^n \pmod{m}$ for all $n \in \mathbb{N}_0$. Moreover, b is obviously coprime to m, and hence each x_n is coprime to m.
It remains to show the existence of b. The congruence $b^2 \equiv b + 1$ (mod m) is equivalent to $(2b-1)^2 \equiv 5 \pmod{m}$. Taking $2b - 1 \equiv 2k$, i.e., $b \equiv 2k^2 + k - 2 \pmod{m}$, does the job.

Remark. A desired b exists whenever 5 is a quadratic residue modulo m, in particular, when m is a prime of the form $10k \pm 1$.

28. If n is divisible by 20, then every multiple of n has two last digits even and hence it is not alternate. We shall show that any other n has an alternate multiple.
   (i) Let n be coprime to 10. For each k there exists a number $A_k(n) = \overline{10\dots010\dots01\dots0\dots01} = \frac{10^{mk}-1}{10^k-1}$ ($m \in \mathbb{N}$) that is divisible by n (by Euler's theorem, choose $m = \varphi[n(10^k - 1)]$). In particular, $A_2(n)$ is alternate.
   (ii) Let $n = 2 \cdot 5^r \cdot n_1$, where $r \geq 1$ and $(n_1, 10) = 1$. We shall show by induction that, for each k, there exists an alternative k-digit odd number M_k that is divisible by 5^k. Choosing the number $10A_{2r}(n_1)M_{2r}$ will then solve this case, since it is clearly alternate and divisible by n.
   We can trivially choose $M_1 = 5$. Let there be given an alternate r-digit multiple M_r of 5^r, and let $c \in \{0, 1, 2, 3, 4\}$ be such that $M_r/5^r \equiv -c \cdot 2^r \pmod{5}$. Then the $(r+1)$digit numbers $M_r + c \cdot 10^r$ and $M_r + (5 + c) \cdot 10^r$ are respectively equal to $5^r(M_r/5^r + 2^r \cdot c)$ and $5^r(M_r/5^r + 2^r \cdot c + 5 \cdot 2^r)$, and hence they are divisible by 5^{r+1} and exactly one of them is alternate: we set it to be M_{r+1}.
   (iii) Let $n = 2^r \cdot n_1$, where $r \geq 1$ and $(n_1, 10) = 1$. We show that there exists an alternate $2r$-digit number N_r that is divisible by 2^{2r+1}. Choosing the number $A_{2r}(n_1)N_r$ will then solve this case.
   We choose $N_1 = 16$, and given N_r, we can prove that one of $N_r + m \cdot 10^{2r}$, for $m \in \{10, 12, 14, 16\}$, is divisible by 2^{2r+3} and therefore suitable for N_{r+1}. Indeed, for $N_r = 2^{2r+1}d$ we have $N_r + m \cdot 10^{2r} = 2^{2r+1}(d + 5^r m/2)$ and $d + 5^r m/2 \equiv 0 \pmod{4}$ has a solution $m/2 \in \{5, 6, 7, 8\}$ for each d and r.

Remark. The idea is essentially the same as in (SL94-24).

29. Let $S_n = \{x \in \mathbb{N} \mid x \leq n,\ n \mid x^2 - 1\}$. It is easy to check that $P_n \equiv 1 \pmod{n}$ for $n = 2$ and $P_n \equiv -1 \pmod{n}$ for $n \in \{3, 4\}$, so from now on we assume $n > 4$.
We note that if $x \in S_n$, then also $n - x \in S_n$ and $(x, n) = 1$. Thus S_n splits into pairs $\{x, n-x\}$, where $x \in S_n$ and $x \leq n/2$. In each of these pairs the product of elements gives remainder -1 upon division by n. Therefore $P_n \equiv (-1)^m$, where S_n has $2m$ elements. It remains to find the parity of m.

Suppose first that $n > 4$ is divisible by 4. Whenever $x \in S_n$, the numbers $|n/2-x|$, $n-x$, $n-|n/2-x|$ also belong to S_n (indeed, $n \mid (n/2-x)^2-1 = n^2/4 - nx + x^2 - 1$ because $n \mid n^2/4$, etc.). In this way the set S_n splits into four-element subsets $\{x, n/2-x, n/2+x, n-x\}$, where $x \in S_n$ and $x < n/4$ (elements of these subsets are different for $x \neq n/4$, and $n/4$ doesn't belong to S_n for $n > 4$). Therefore $m = |S_n|/2$ is even and $P_n \equiv 1 \pmod{m}$.

Now let n be odd. If $n \mid x^2 - 1 = (x-1)(x+1)$, then there exist natural numbers a, b such that $ab = n$, $a \mid x-1$, $b \mid x+1$. Obviously a and b are coprime. Conversely, given any odd $a, b \in \mathbb{N}$ such that $(a,b) = 1$ and $ab = n$, by the Chinese remainder theorem there exists $x \in \{1, 2, \dots, n-1\}$ such that $a \mid x-1$ and $b \mid x+1$. This gives a bijection between all ordered pairs (a,b) with $ab = n$ and $(a,b) = 1$ and the elements of S_n. Now if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the decomposition of n into primes, the number of pairs (a,b) is equal to 2^k (since for every i, either $p_i^{\alpha_i} \mid a$ or $p_i^{\alpha_i} \mid b$), and hence $m = 2^{k-1}$. Thus $P_n \equiv -1 \pmod{n}$ if n is a power of an odd prime, and $P_n \equiv 1$ otherwise.

Finally, let n be even but not divisible by 4. Then $x \in S_n$ if and only if x or $n-x$ belongs to $S_{n/2}$ and x is odd. Since $n/2$ is odd, for each $x \in S_{n/2}$ either x or $x + n/2$ belongs to S_n, and by the case of n odd we have $S_n \equiv \pm 1 \pmod{n/2}$, depending on whether or not $n/2$ is a power of a prime. Since S_n is odd, it follows that $P_n \equiv -1 \pmod{n}$ if $n/2$ is a power of a prime, and $P_n \equiv 1$ otherwise.

Second solution. Obviously S_n is closed under multiplication modulo n. This implies that S_n with multiplication modulo n is a subgroup of $\mathbb{Z}_n$, and therefore there exist elements $a_1 = -1, a_2, \dots, a_k \in S_n$ that generate S_n. In other words, since the a_i are of order two, S_n consists of products $\prod_{i \in A} a_i$, where A runs over all subsets of $\{1, 2, \dots, k\}$. Thus S_n has 2^k elements, and the product of these elements equals $P_n \equiv (a_1 a_2 \cdots a_k)^{2^{k-1}} \pmod{n}$. Since $a_i^2 \equiv 1 \pmod{n}$, it follows that $P_n \equiv 1$ if $k \geq 2$, i.e., if $|S_n| > 2$. Otherwise $P_n \equiv -1 \pmod{n}$.

We note that $|S_n| > 2$ is equivalent to the existence of $a \in S_n$ with $1 < a < n-1$. It is easy to find that such an a exists if and only if neither of n, $n/2$ is a power of an odd prime.

30. We shall denote by k the given circle with diameter p^n.
Let A, B be lattice points (i.e., points with integer coordinates). We shall denote by $\mu(AB)$ the exponent of the highest power of p that divides the integer AB^2. We observe that if S is the area of a triangle ABC where A, B, C are lattice points, then $2S$ is an integer. According to Heron's formula and the formula for the circumradius, a triangle ABC whose circumcenter has diameter p^n satisfies

$$2AB^2BC^2 + 2BC^2CA^2 + 2CA^2AB^2 - AB^4 - BC^4 - CA^4 = 16S^2 \quad (1)$$

and

$$AB^2 \cdot BC^2 \cdot CA^2 = (2S)^2 p^{2n}. \quad (2)$$

Lemma 1. Let A, B, and C be lattice points on k. If none of AB^2, BC^2, CA^2 is divisible by p^{n+1}, then $\mu(AB), \mu(BC), \mu(CA)$ are $0, n, n$ in some order.

Proof. Let $k = \min\{\mu(AB), \mu(BC), \mu(CA)\}$. By (1), $(2S)^2$ is divisible by p^{2k}. Together with (2), this gives us $\mu(AB) + \mu(BC) + \mu(CA) = 2k + 2n$. On the other hand, if none of AB^2, BC^2, CA^2 is divisible by p^{n+1}, then $\mu(AB) + \mu(BC) + \mu(CA) \leq k + 2n$. Therefore $k = 0$ and the remaining two of $\mu(AB), \mu(BC), \mu(CA)$ are equal to n.

Lemma 2. Among every four lattice points on k, there exist two, say M, N, such that $\mu(MN) \geq n + 1$.

Proof. Assume that this doesn't hold for some points A, B, C, D on k. By Lemma 1, μ for some of the segments $AB, AC, \dots, CD$ is 0, say $\mu(AC) = 0$. It easily follows by Lemma 1 that then $\mu(BD) = 0$ and $\mu(AB) = \mu(BC) = \mu(CD) = \mu(DA) = n$. Let $a, b, c, d, e, f \in \mathbb{N}$ be such that $AB^2 = p^n a$, $BC^2 = p^n b$, $CD^2 = p^n c$, $DA^2 = p^n d$, $AC^2 = e$, $BD^2 = f$. By Ptolemy's theorem we have $\sqrt{ef} = p^n\left(\sqrt{ac} + \sqrt{bd}\right)$. Taking squares, we get that $\frac{ef}{p^{2n}} = \left(\sqrt{ac} + \sqrt{bd}\right)^2 = ac + bd + 2\sqrt{abcd}$ is rational and hence an integer. It follows that ef is divisible by p^{2n}, a contradiction.

Now we consider eight lattice points $A_1, A_2, \dots, A_8$ on k. We color each segment A_iA_j red if $\mu(A_iA_j) > n$ and black otherwise, and thus obtain a graph G. The degree of a point X will be the number of red segments with an endpoint in X. We distinguish three cases:

(i) There is a point, say A_8, whose degree is at most 1. We may suppose w.l.o.g. that A_8A_7 is red and $A_8A_1, \dots, A_8A_6$ black. By a well-known fact, the segments joining vertices $A_1, A_2, \dots, A_6$ determine either a red triangle, in which case there is nothing to prove, or a black triangle, say $A_1A_2A_3$. But in the latter case the four points A_1, A_2, A_3, A_8 do not determine any red segment, a contradiction to Lemma 2.

(ii) All points have degree 2. Then the set of red segments partitions into cycles. If one of these cycles has length 3, then the proof is complete. If all the cycles have length at least 4, then we have two possibilities: two 4-cycles, say $A_1A_2A_3A_4$ and $A_5A_6A_7A_8$, or one 8-cycle, $A_1A_2 \dots A_8$. In both cases, the four points A_1, A_3, A_5, A_7 do not determine any red segment, a contradiction.

(iii) There is a point of degree at least 3, say A_1. Suppose that A_1A_2, A_1A_3, and A_1A_4 are red. We claim that A_2, A_3, A_4 determine at least one red segment, which will complete the solution. If not, by Lemma 1, $\mu(A_2A_3), \mu(A_3A_4), \mu(A_4A_2)$ are $n, n, 0$ in some order. Assuming w.l.o.g. that $\mu(A_2A_3) = 0$, denote by S the area of triangle $A_1A_2A_3$. Now by formula (1), $2S$ is not divisible by p. On the other hand, since $\mu(A_1A_2) \geq n + 1$ and $\mu(A_1A_3) \geq n + 1$, it follows from (2) that $2S$ is divisible by p, a contradiction.

A

Notation and Abbreviations

A.1 Notation

We assume familiarity with standard elementary notation of set theory, algebra, logic, geometry (including vectors), analysis, number theory (including divisibility and congruences), and combinatorics. We use this notation liberally.
We assume familiarity with the basic elements of the game of chess (the movement of pieces and the coloring of the board).
The following is notation that deserves additional clarification.

- $\mathcal{B}(A, B, C)$, $A - B - C$: indicates the relation of *betweenness*, i.e., that B is between A and C (this automatically means that A, B, C are different collinear points).
- $A = l_1 \cap l_2$: indicates that A is the intersection point of the lines l_1 and l_2.
- AB: line through A and B, segment AB, length of segment AB (depending on context).
- $[AB$: ray starting in A and containing B.
- $(AB$: ray starting in A and containing B, but without the point A.
- (AB): open interval AB, set of points between A and B.
- $[AB]$: closed interval AB, segment AB, $(AB) \cup \{A, B\}$.
- $(AB]$: semiopen interval AB, closed at B and open at A, $(AB) \cup \{B\}$. The same bracket notation is applied to real numbers, e.g., $[a, b) = \{x \mid a \leq x < b\}$.
- ABC: plane determined by points A, B, C, triangle ABC ($\triangle ABC$) (depending on context).
- $[AB, C$: half-plane consisting of line AB and all points in the plane on the same side of AB as C.

- $(AB, C$: $[AB, C$ without the line AB.
- $a, b, c, \alpha, \beta, \gamma$: the respective sides and angles of triangle ABC (unless otherwise indicated).
- $k(O, r)$: circle k with center O and radius r.
- $d(A, p)$: distance from point A to line p.
- $S_{A_1 A_2 \ldots A_n}$: area of n-gon $A_1 A_2 \ldots A_n$ (special case for $n = 3$, S_{ABC}: area of $\triangle ABC$).
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: the sets of natural, integer, rational, real, complex numbers (respectively).
- $\mathbb{Z}_n$: the ring of residues modulo n, $n \in \mathbb{N}$.
- $\mathbb{Z}_p$: the field of residues modulo p, p being prime.
- $\mathbb{Z}[x]$, $\mathbb{R}[x]$: the rings of polynomials in x with integer and real coefficients respectively.
- R^*: the set of nonzero elements of a ring R.
- $R[\alpha]$, $R(\alpha)$, where α is a root of a quadratic polynomial in $R[x]$: $\{a + b\alpha \mid a, b \in R\}$.
- X_0: $X \cup \{0\}$ for X such that $0 \notin X$.
- X^+, X^-, $aX + b$, $aX + bY$: $\{x \mid x \in X, x > 0\}$, $\{x \mid x \in X, x < 0\}$, $\{ax + b \mid x \in X\}$, $\{ax + by \mid x \in X, y \in Y\}$ (respectively) for $X, Y \subseteq \mathbb{R}$, $a, b \in \mathbb{R}$.
- $[x]$, $\lfloor x \rfloor$: the greatest integer smaller than or equal to x.
- $\lceil x \rceil$: the smallest integer greater than or equal to x.

The following is notation simultaneously used in different concepts (depending on context).

- $|AB|$, $|x|$, $|S|$: the distance between two points AB, the absolute value of the number x, the number of elements of the set S (respectively).
- (x, y), (m, n), (a, b): (ordered) pair x and y, the greatest common divisor of integers m and n, the open interval between real numbers a and b (respectively).

A.2 Abbreviations

We tried to avoid using nonstandard notations and abbreviations as much as possible. However, one nonstandard abbreviation stood out as particularly convenient:

- w.l.o.g.: without loss of generality.

Other abbreviations include:

- RHS: right-hand side (of a given equation).
- LHS: left-hand side (of a given equation).
- QM, AM, GM, HM: the quadratic mean, the arithmetic mean, the geometric mean, the harmonic mean (respectively).
- gcd, lcm: greatest common divisor, least common multiple (respectively).
- i.e.: in other words.
- e.g.: for example.

B

Codes of the Countries of Origin

ARG	Argentina	GRE	Greece	PHI	Philippines
ARM	Armenia	HKG	Hong Kong	POL	Poland
AUS	Australia	HUN	Hungary	POR	Portugal
AUT	Austria	ICE	Iceland	PRK	Korea, North
BEL	Belgium	INA	Indonesia	PUR	Puerto Rico
BLR	Belarus	IND	India	ROM	Romania
BRA	Brazil	IRE	Ireland	RUS	Russia
BUL	Bulgaria	IRN	Iran	SAF	South Africa
CAN	Canada	ISR	Israel	SIN	Singapore
CHN	China	ITA	Italy	SLO	Slovenia
COL	Colombia	JAP	Japan	SMN	Serbia and Montenegro
CUB	Cuba	KAZ	Kazakhstan	SPA	Spain
CYP	Cyprus	KOR	Korea, South	SWE	Sweden
CZE	Czech Republic	KUW	Kuwait	THA	Thailand
CZS	Czechoslovakia	LAT	Latvia	TUN	Tunisia
EST	Estonia	LIT	Lithuania	TUR	Turkey
FIN	Finland	LUX	Luxembourg	TWN	Taiwan
FRA	France	MCD	Macedonia	UKR	Ukraine
FRG	Germany, FR	MEX	Mexico	USA	United States
GBR	United Kingdom	MON	Mongolia	USS	Soviet Union
GDR	Germany, DR	MOR	Morocco	UZB	Uzbekistan
GEO	Georgia	NET	Netherlands	VIE	Vietnam
GER	Germany	NOR	Norway	YUG	Yugoslavia
		NZL	New Zealand		